Upper bound constructions for untangling planar geometric graphs

Javier Cano[1], Csaba D. Tóth[2], and Jorge Urrutia[3]

[1] Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México, D.F. México, `j_cano@uxmcc2.iimas.unam.mx`
[2] Department of Math., University of Calgary, Canada, `cdtoth@ucalgary.ca`
[3] Instituto de Matemáticas, Universidad Nacional Autónoma de México, D.F. México, `urrutia@matem.unam.mx`

Abstract. For every $n \in \mathbb{N}$, there is a straight-line drawing D_n of a planar graph on n vertices such that in any *crossing-free* straight-line drawing of the graph, at most $O(n^{.4982})$ vertices lie at the same position as in D_n. This improves on an earlier bound of $O(\sqrt{n})$ by Goaoc *et al.* [6].

1 Introduction

A *straight-line drawing* of a graph G is a representation of G in the plane where the vertices are mapped to distinct points in the plane, and each edge is represented by a line segment joining pairs of points representing adjacent vertices. A drawing is *crossing-free* if no two edges intersect, except perhaps at a common endpoint. A *geometric graph* is a graph given with a straight-line drawing. Every planar graph has a crossing-free straight-line drawing by Fary's Theorem [5], however, not all straight-line drawings are crossing-free. Suppose that we are given a *planar geometric graph* G. Since G is planar, it can be redrawn (by relocating some of its vertices) such that no two edges cross anymore. The process of redrawing G to obtain a crossing-free straight-line drawing, is called an *untangling* of G.

In this paper we study the following problem: For an integer $n \in \mathbb{N}$, what is the maximum number $f(n)$ such that every planar geometric graph with n vertices can be untangled such that at least $f(n)$ vertices remain in their original position.

The first question on untangling planar geometric graphs was posed by Mamoru Watanabe in 1998: Is it true that every polygon P with n vertices can be untangled in at most ϵn steps, for some absolute constant $\epsilon < 1$, where in each step, we move a vertex of G to a new location. Watanabe's question was proved to be false by Pach and Tardos [9], who also showed that every n-gon can be untangled in at most $n - \sqrt{n}$ moves. Recently, Cibulka [3] proved that every n-gon can be untangled while keeping $\Omega(n^{2/3})$ vertices fixed, and there are n-gons where no more than $O((n \log n)^{2/3})$ vertices can be fixed.

The problem of untangling planar geometric graphs was studied by Goaoc *et al.* [6]. They constructed planar geometric graphs showing that $f(n) \leq \sqrt{n} + 2$.

Kang *et al.* [8] explored several families of graphs in which no more than $O(\sqrt{n})$ of n vertices can be fixed. Bose *et al.* [2] devised an untangling algorithm that fixes at least $(n/3)^{1/4}$ of n vertices, which proves $f(n) \geq (n/3)^{1/4}$.

In this note, we improve the upper bound for $f(n)$ to $O(n^{1/(3-\log_{38} 37)}) \subset O(n^{.4982})$. We construct planar geometric graphs such that any untangling of them fixes $O(n^{1/(3-\log_{38} 37)})$ of n vertices. The framework of our construction leads to new problems in graph drawing, which we discuss in Section 5. Any improvement in these problems would immediately improve the upper bound for $f(n)$.

2 Preliminaries

Monotone subsequences. Erdős and Szekeres showed that every permutation of $[n] = \{0, 1, \ldots, n-1\}$ contains a monotonically increasing or degreasing subsequence of length at least $\lceil\sqrt{n}\rceil$, and this bound is the best possible. The lower bound is attained on many different permutations. The best known construction consists of $\lceil\sqrt{n}\rceil$ monotonically increasing subsequences of consecutive elements, where the minimum element of each subsequence is larger than the maximum element of the next. We will use permutations in which monotone subsequences "spread out" more evenly. In a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we define the *spread* of a subsequence $(\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k})$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, to be $j_k - j_1$.

Lemma 1. *For every $m \in \mathbb{N}$, there is a permutation π_n of $[n] = [4^m]$ such that*

- *the length of every monotone subsequence is at most $2^m = \sqrt{n}$; and*
- *the spread of every monotone subsequence of length $k \geq 2$ is at least $\frac{k^2+2}{6}$.*

Proof. We construct the permutation π_n by induction on m. For $m = 1$, let $\pi_4 = (2, 3, 0, 1)$ and observe that it has the desired properties. Assume that $\pi_n = (\sigma_1, \ldots, \sigma_n)$ is a permutation of $[n]$ with the desired properties. We construct a permutation π_{4n} of $[4n]$ by replacing each σ_i with the 4-tuple

$$(4\sigma_i + 2, 4\sigma_i + 3, 4\sigma_i + 0, 4\sigma_i + 1).$$

Let L be a monotone subsequence of length k in π_{4n}. Note that L has at most two elements from each 4-tuple. The sequence of these 4-tuples corresponds to a monotone subsequence of π_n, which we denote by L'. The length of L' is at least $k/2$, with equality iff L contains exactly two elements from each of the 4-tuples involved. By induction, the length of L' is $k/2 \leq 2^m$. Hence, we have $k \leq 2^{m+1}$, as required. If the length of L' is exactly $k/2$, then its spread is at least $\frac{(k/2)^2+2}{6}$ in π_n, and so the spread of L is at least $4(\frac{(k/2)^2+2}{6}) - 1 = \frac{k^2+2}{6}$. If the length of L' is more than $k/2$, then its spread is at least $\frac{(k/2+1)^2+2}{6}$, and the spread of L is at least $4(\frac{(k/2+1)^2+2}{6}) - 1 \geq \frac{k^2+2}{6}$, as required. □

A recursive construction. We say that a planar straight-line graph T is an (a, b, c)*-triangulation* for integers $a \geq b > c > 0$ if T is a 3-connected triangulation such that it has a total of a faces, b of which are marked, and any line intersects at most c marked faces in any plane straight-line drawing of T.

Note that, by Steiniz's theorem, a 3-connected triangulation is the 1-skeleton of a combinatorially unique 3-dimensional polytope. Hence an (a, b, c)-triangulation has a unique embedding in the plane up to homeomorphisms and the choice of the outer face. In the following lemma, we recursively construct a larger triangulation from an (a, b, c)-triangulation.

Lemma 2. *If there exists an (a, b, c)-triangulation for constants $a \geq b > c > 0$, then for every $n \in \mathbb{N}$, there is an (a', b', c')-triangulation with $a' = \Theta(n)$, $b' = \Theta(n)$, and $c' = \Theta(n^{\log_b c})$.*

Proof. Let $T_{a,b,c}$ be an (a, b, c)-triangulation. Plug in $T_{a,b,c}$ in all marked faces of $T_{a,b,c}$ recursively k times, where k is specified shortly. We obtain a 3-connected triangulation $T^k_{a,b,c}$ (that is, $T_{a,b,c} = T^0_{a,b,c}$), which has $b' = b^{k+1}$ marked faces, a line intersects at most $c' = c^{k+1}$ marked faces in any plane straight-line drawing, and the total number of faces is $a' = b^{k+1} + (a - b)(b^{k+2} - 1)/(b - 1)$. If we denote by v the number of vertices of $T^k_{a,b,c}$, then it has $2v - 4$ faces, $\Theta(v)$ of which are marked, and a line intersects at most $\Theta(v^{\log_b c})$ marked faces in any plane straight-line drawing of $T^k_{a,b,c}$. Choose k such that $a' = \Theta(v)$. □

3 Upper Bound Constructions

Theorem 1. *If there exists an (a, b, c)-triangulation for constants $a \geq b > c > 0$, then $f(n) \in O(n^\kappa)$ for $\kappa = 1/(3 - \log_b c)$.*

Note that $b > c$, and so we have $0 < \log_b c < 1$ and $0 < \kappa < 1/2$. That is, the existence of *any* (a, b, c)-triangulation implies an upper bound $f(n) \in O(n^{\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$. We discuss (a, b, c)-triangulations in Section 4.

Proof. For every $n \in \mathbb{N}$, we construct a drawing of a planar graph G_n with $\Theta(n)$ vertices such that in any untangling of G_n, at most $O(n^\kappa)$ vertices remain fixed.

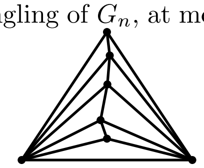

Fig. 1. Triangulation $S = P_2 * P_5$.

Construction. We first construct the planar graph G_n. By Lemma 2, there is a 3-connected triangulation T with $\Theta(n^\kappa)$ vertices and $\Theta(n^\kappa)$ marked faces such that any line intersects at most $\Theta(n^{\kappa \log_b c})$ marked faces in any plane straight-line drawing of T. Let S be the join $P_2 * P_{s+1}$ of two paths with 2 and $s + 1$ vertices, respectively, where $s = \Theta(n^{1-\kappa})$ and s is a power of 4 (see Fig. 1). Note that S has exactly s interior vertices, which have a natural order along an interior path. We construct G_n by plugging in a copy of S into each marked face of T. Denote the copies of S by S_i, for $i = 1, 2, \ldots, \Theta(n^\kappa)$. The total number of vertices of G_n is $\Theta(n^\kappa + n^\kappa \cdot n^{1-\kappa}) = \Theta(n)$.

Next, we describe a straight-line drawing of G_n. Embed the vertices of the triangulation T arbitrarily in general position above the x-axis. Embed the interior vertices of S_1 into integer points $\{0, 1, \ldots, s-1\} \times \{0\}$ on the x-axis such that their natural order is permuted by π_s from Lemma 1. The interior vertices of S_i, for each $i > 1$, are embedded into a translated copy of this permutation, translated along the x-axis by δi for some small $0 < \delta \ll n^{-\kappa}$.

Bounding the number of fixed vertices. Consider a crossing-free straight-line drawing of G_n. The $\Theta(n^\kappa)$ vertices of T may be fixed. It is sufficient to consider the interior vertices of S_i, $i = 1, 2, \ldots, \Theta(n^\kappa)$. Suppose that ℓ_i interior vertices of S_i are fixed, for $i = 1, 2, \ldots, \Theta(n^\kappa)$. Since the x-axis intersects at most $O(n^{\kappa \log_b c})$ triangles of T, all but at most $O(n^{\kappa \log_b c})$ values of ℓ_i are zero.

Consider now a triangulation S_i where $\ell_i > 0$. Note that S_i contains a sequence of $s + 1$ nested triangles that share a common edge (the horizontal edge in Fig. 1). In *any* straight-line drawing of S_i (independent of the choice of the outer face), at least $(s + 1)/2$ of these triangles form a nested sequence. Hence, at least $\ell_i/2$ fixed interior vertices of S_i are vertices in a sequence of nested triangles in the crossing-free straight-line drawing of G_n. The intersection of the x-axis with a sequence of nested triangles is a line segment. It can be partitioned into two directed segments, with opposite directions, such that each of them is directed towards the deepest point in the arrangement of nested triangles. At least $\ell_i/4$ fixed points of S_i lie on the same directed segment, and these points must form a monotone sequence along the x-axis. Furthermore, the elements of this monotone subsequence are all contained in the largest triangle from the nested sequence of triangles in S_i, therefore, their convex hull is disjoint from the convex hulls of similar sequences in any other S_j, $j \neq i$.

By Lemma 1, the spread of the monotone subsequence of length at least $\ell_i/4$ is at least $(\ell_i^2 + 32)/96$. Hence these fixed points "occupy" an interval of length $(\ell_i^2 + 32)/96$ on the x-axis. As noted above, the convex hulls of monotone sequences from distinct copies of S are disjoint, and so we have

$$\sum_{i=1}^{\Theta(n^\kappa)} \frac{\ell_i^2 + 32}{96} \leq 2s. \tag{1}$$

Recall that at most $O(n^{\kappa \log_b c})$ values of ℓ_i are nonzero. By Jensen's inequality, the sum $\sum_{i=1}^{\Theta(n^\kappa)} \ell_i$ is maximized if all nonzero values of ℓ_i are equal. Suppose, by relabeling the copies of S if necessary, that $\ell_i = \ell$ for $i = 1, 2, \ldots, \Theta(n^{\kappa \log_b c})$; and $\ell_i = 0$ for all other i. In this case, Inequality (1) becomes $\Theta(n^{\kappa \log_b c}) \cdot \ell^2 \leq \Theta(n^{1-\kappa})$, or $\ell \in O(n^{(1-\kappa(1+\log_b c))/2})$. Therefore, the number of fixed vertices is at most

$$\sum_{i=1}^{\Theta(n^\kappa)} \ell_i \leq \Theta(n^{\kappa \log_b c}) \cdot \ell = \Theta(n^{(1+\kappa(\log_b c - 1))/2}) = \Theta(n^\kappa),$$

as required. □

4 (a, b, c)-Triangulations.

Non-Hamiltonian triangulations. By Steinitz's theorem, every 3-connected cubic planar graph G is the 1-skeleton of a convex polytope. The dual graph G^*, corresponding to the dual polytope, is a 3-connected triangulation. Tait [10] conjectured in 1884 that every 3-connected cubic planar graph is Hamiltonian. Tutte [11] found a counterexample with 44 vertices in 1946. The smallest known counterexample, due to Bernette, Bosák, and Lenderberg, has 38 vertices, and it is known that there is no counterexample with 36 or fewer vertices [7].

A Hamiltonian cycle of G corresponds to a simple closed curve visiting every face exactly once in any plane drawing of G^*. In a straight-line drawing, every face of a triangulation is convex and thus it is visited by a line at most once. Therefore, if G is not Hamiltonian, then G^* has no plane straight-line drawing in which a line visits every face (including the outer face). The smallest known counterexample to Tait's conjecture implies that there is a $(38, 38, 37)$-triangulation. Combined with Theorem 1, we obtain a new upper bound for $f(n)$.

Corollary 1. $f(n) \in O(n^{1/(3-\log_{38} 37)}) \subset O(n^{.4982})$.

5 Conclusion

Our upper bounds for $f(n)$ depend on the value $\log_b c$ of an (a, b, c)-triangulation. The (a, b, c)-triangulations we considered are all derived from counterexamples for Tait's conjecture. Since these are counterexamples for Hamiltonicity, they all have $a = b > c > 0$. It is conceivable, though, that there are better constructions for (a, b, c)-triangulations in which $a > b$.

The best possible upper bound for $f(n)$ achievable with our framework would come from the minimum value of $\log_b c$, leading to the following problems.

Problem 1. What is the minimum value of $\log_b c$ over all (a, b, c)-triangulations?

Problem 2. What is the minimum value of $\log_b c$ over all 3-connected cubic planar graphs G, where G has b has marked vertices and any simple cycle visits at most c marked vertices?

The latter problem is purely graph theoretical. But the two problems are, in fact, equivalent. The dual of Problem 2 asks for the minimum value of $\log_b c$ over all 3-connected plane triangulations T with a faces, b of which are marked, such that any closed Jordan curve γ that visits every face at most once can visit at most c marked faces. One can show that every such Jordan curve γ is "stretchable." That is, T has a plane straight-line drawing T' in which a line L visits the exact same faces as γ visited in T (in the same cyclic order). See Fig. 2. Details are omitted, and will be given in the full version of this paper.

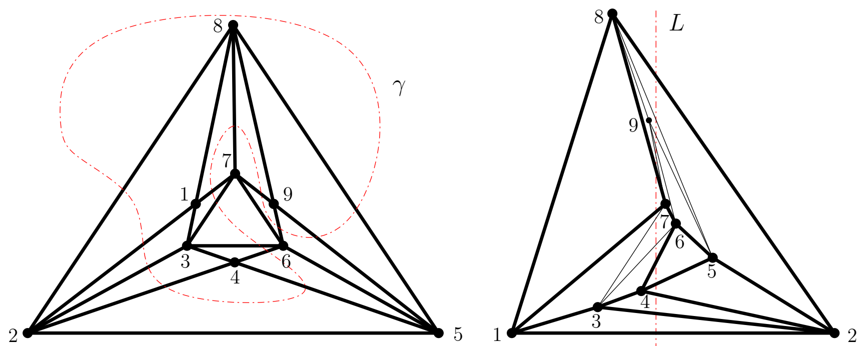

Fig. 2. Left: a plane 3-connected triangulation T, where curve γ visits every face exactly once. Right: a plane straight-line drawing T' of T, where line L stabs every face.

References

1. E. M. Arkin, M. Held, J. S. B. Mitchell, and S. Skiena, Hamiltonian triangulations for fast rendering, *The Visual Computer* **12** (9) (1996), 429–444.
2. P. Bose, V. Dujmovic, F. Hurtado, S. Langerman, P. Morin, D. R. Wood, A polynomial bound for untangling geometric planar graphs, *Discrete Comput. Geom.* **42** (4) (2009), 570–585.
3. J. Cibulka, Untangling polygons and graphs, *Discrete Comput. Geom.* **43** (2010), 402–411.
4. P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.
5. Fáry. I, On straight line representation of planar graphs, *Acta Univ. Szeged, Acta Sci. Math.* **11** (1948), 229–233.
6. X. Goaoc, J. Kratochvíl, Y. Okamoto, C. S. Shin, A. Spillner, and A. Wolff, Untangling a planar graph, *Discrete Comput. Geom.* **42** (4) (2009), 542–569.
7. D. A. Holton and B. D. McKay, The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices, *J. Combin. Theory Ser. B* **45** (3) (1988), 305–319.
8. M. Kang, O. Pikhurko, A. Ravsky, M. Schacht, and O. Verbitsky, Untangling planar graphs from a specified vertex position—Hard cases, *Discrete Appl. Math.* **159** (8) (2011), 789–799.
9. J. Pach and G. Tardos, Untangling a polygon, *Discrete Comput. Geom.* **28** (4) (2002), 585–592.
10. P. G. Tait, Listing's Topologie, *Philosophical Magazine* **17** (1884), 30–46.
11. W. T. Tutte, On Hamiltonian circuits, *J. LMS* **21** (2) (1946), 98–101.