Joaquim Roé: Functions on Newton-Okounkov bodies

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Newton-Okounkov bodies:

Kaveh-Khovanskii

Lazarsfeld-Mustata '09

Boucksom Bourbaki '12

Functions on NO-bodies:

Boucksom-Chen '11

Witt-Nyström '10

Boucksom-Küronya-Maclean-Szemberg

1. Filtered Algebras:

R A-algebra (mostly $A=k$), $A \xrightarrow{m} R$

A $\underbrace{\text{multiplicative } A\text{-filtration}}_{\text{filtration}}$ on R: indexed by $\mu \in G$ ordered abelian group

$\mathcal{F}_\cdot$ $\quad \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_\mu \subset \cdots$

s.t.

1) $\mathcal{F}_\mu \subset \mathcal{F}_\nu$ if $\mu < \nu$

2) $\mathcal{F}_\mu \cdot \mathcal{F}_\nu \subseteq \mathcal{F}_{\mu+\nu}$

3) $m(A) \subseteq \mathcal{F}_0$

4) $\bigcup \mathcal{F}_\mu = R$

$\mathcal{F}_\mu \in A$-mod

$\mathcal{F}_0 \subset R$ subring

Examples:

1) $R = \bigoplus_{d \geq 0} R_d$ graded algebra

then $F_\mu = \bigoplus_{d=0}^{\mu} R_d$ $\quad d \geq 0 \quad G = \mathbb{Z}$

$F_\mu = 0$ for $d = 0$ increasing filtration

2) An R-filtration of R is a decreasing filtration by ideals (F_μ are R-modules)

2i) Given $I \subset R$, $F_\mu = I^{-\mu}$ $\quad \forall \mu < 0$

I-adic filtration

2ii) If R is a domain and $v : R \setminus \{0\} \to G$ valuation, then (non-archimedean)

$$F_\mu = \{ s \in R \mid v(s) \geq -\mu \}$$

$v(st) = v(s) + v(t)$

$v(s+t) \geq \min\{v(s), v(t)\}$

3) Given $\varphi : S \to R$ morphism and $F_\cdot$ a filtration on R, $\varphi^{-1}(F)$ is a filtration on S

Notation: $F_\mu^- = \bigcup_{\mu' < \mu} F_{\mu'} \subset F_\mu$

$\mathrm{supp}(F_\cdot) = \{ \mu \in G \mid F_\mu^- \neq F_\mu \} \subset G$

a subsemigroup $\subset \mathbb{R}^N$

$\boxed{\text{G will be ordered abelian grp, subgroup of } \mathbb{R}^N \text{ (sometimes discrete)}}$

$\mathrm{cone}(\mathcal{F}_\cdot) = \{ \sum a_i \mu_i \mid a_i \in \mathbb{R}_{\geq 0},\ \mu_i \in \mathrm{Supp}(\mathcal{F}_\cdot) \}$

4) Refining a filtration $\mathcal{F}_\cdot$ indexed by G by a second $\mathcal{F}_\cdot'$ indexed by G'

$$\mathcal{F}_\mu^- \subset \mathcal{F}_{(\mu,\nu)} = \mathcal{F}_\mu^- + (\mathcal{F}_\mu \cap \mathcal{F}_\nu') \subset \mathcal{F}_\mu$$

is a filtration indexed by $G \times G'_{lex}$

def. $$\mathrm{Gr}(\mathcal{F}_\cdot) = \bigoplus_{\mu \in \mathrm{Supp}(\mathcal{F}_\cdot)} \frac{\mathcal{F}_\mu}{\mathcal{F}_\mu'} \cdot t^\mu$$

If $\mathcal{F}$ is a decreasing filtration by ideals,

$$\mathrm{Rees}(\mathcal{F}_\cdot) = \bigoplus_{\mu \in \mathrm{Supp}(\mathcal{F}_\cdot)} \mathcal{F}_\mu \cdot t^\mu$$

with maps

$$\begin{tikzcd} & & R \\ \mathrm{Rees}(\mathcal{F}_\cdot) \arrow[urr, "t=1"] \arrow[d] & & \\ \mathrm{Gr}(\mathcal{F}_\cdot) & & k[t_1, \dots, t_N] \arrow[ull] \arrow[uu, phantom, "\cup\!\!\cup"] \end{tikzcd}$$

$G \subset \mathbb{R}^N$, $\mathrm{rk}\, G = N$

and an action of $(k^*)^N$: $a \in (k^*)^n$, $s \in \mathcal{F}_\mu$

$\Rightarrow \quad a(s) = a^t s$

Remark: A "test configuration" (Donaldson)

Given (X, L) over $\mathbb{C}$ $\rightsquigarrow$ $\mathcal{X}$ $\mathcal{L}$ relative line bundle

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{array}$$

$X_t \cong X$, $t \neq 0$

$\mathcal{L}_t \cong L$

$\mathbb{C}^*$ action on $(\mathcal{X}, \mathcal{L})$

$$H^0(\mathcal{X}, k\mathcal{L}) = \bigoplus_\mu H^0(\mathcal{X}, k\mathcal{L})_\mu$$

$a(s) = a^\mu s$ for $s \in H^0(\mathcal{X}, \mathcal{L})$

where $H^0(\mathcal{X}, k\mathcal{L})_\mu \cong \{s \in H^0(X,L) \mid \bar{s}\, t^{-\mu} \text{ holom.}\}$

$t \in \mathbb{C}$

filtration on $\bigoplus_k H^0(X,kL)$ indexed by $\mathbb{Z}$

2 Newton-Okounkov bodies

X normal proj variety / k, $n = \dim X$

D (big) divisor

$R = \bigoplus_{k \geq 0} H^0(X, kD)$ graded ring filtered by degree

$\underbrace{p \in X \text{ smooth}, \quad z_1, \dots, z_n \in \mathcal{O}_{X,p}}_{\text{choice}}$ system of param. (local coord.)

$\hat{\mathcal{O}}_{X,p} = k[[z_1, \dots, z_n]]$, for $f \in \mathcal{O}_{X,p}$ write $f = \sum_{m \in \mathbb{Z}^n_{\geq 0}} a_m z^m$

and define $v(f) = \min_{lex}\{m \mid a_m \neq 0\} \in \mathbb{Z}^n_{\geq 0}$

$$= (v_1(f), \dots, v_n(f))$$

v is a valuation of maximal rank on $\mathcal{O}_{X,p}$
($v = v_{p, z_1, \dots, z_n}$)

Consider $F_\cdot$ the refinement of the filtration by degree using the filtration by v.

Study $F_{k,m} = \{s \in H^0(X, kD) \mid v(s) \geq -m\}$
asymptotically for $k \gg 0$.

i.e. $\overline{\text{cone}(F_{k,m}) \cap \{k=1\}} = \Delta_v(D) \subset \mathbb{R}^n$ <u>Newton-Okounkov body</u>

Theorem (Okounkov, Lazarfeld-Mustata, Kaveh-Khovanskii) If D is big

1) $\Delta(D) = \overline{\left\{ \frac{\nu(s)}{k} \mid s \in H^0(X, kD) \right\}}$

2) $\Delta(D)$ convex and compact

3) $\Delta(D)$ has nonempty interior and

$$\mathrm{vol}(\Delta(D)) = \frac{1}{n!} \underbrace{\mathrm{vol}(D)}$$

$$= \lim_{k \to \infty} \frac{\dim H^0(X, kD)}{k^n / n!}$$

4) If $D \equiv_{num.} D'$ then $\Delta_\nu(D) = \Delta_\nu(D')$ $\forall \nu$

5) For every $S \subset R$ "big enough" graded

$\rightsquigarrow \Delta_\nu(S)$ satisfies (1) - (3)

Rmk: converse of (4) is also true

Rmk: $Gr(F)$ fin. gen. $\rightsquigarrow$ toric degen.
& $\Delta_\nu(D)$ polyhedron (Anderson)
converse not true: $\Delta_\nu(D)$ can be polyhedral &
$Gr(F)$ not fin. gen.

For (2):

Lemma: A rank n valuation is "linearly bounded" on R, i.e. $\exists$ $c', c \in \mathbb{R}$ s.t. $\forall s \in H^0(X, kD)$

$$ck \le \nu_i(s) \le c'k, \text{ in fact } c = 0.$$

For (3): For rank n valuations $\mathcal{O}_v/m_v = k$

$$\frac{F_{(k,m)}}{F^{-}_{(k,m)}} = \begin{cases} 0 \\ k \end{cases}$$

3. Functions & maps on NO bodies

<u>Thm</u> (Boucksom-Chen)
If $F^{'}_{\cdot}$ refines the filtration $F_{(k,m)}$ and is linearly bounded, then

$$\Delta(F^{'}_{\cdot}) = \overline{\mathrm{cone}(F^{'}_{\cdot})} \cap \{k=1\} \subset \mathbb{R}^{n+r}$$

$n+r = \mathrm{rank}(F^{'}_{\cdot})$

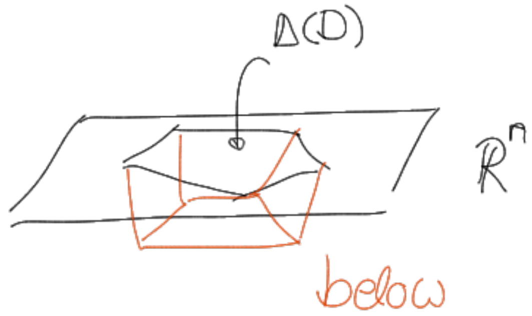
satisfies (1), (2)

3) $\mathrm{vol}(\Delta(F^{'}_{\cdot}))$ does not depend on choice of v

Most examples deal with $r=1$

Take $G_{\cdot}$ rank 1 filtration & refine $F_{(k,m)}$ using $G_{\cdot}$ $\rightsquigarrow$ $F^{'}_{\cdot}$.

then $\Delta(F^{'}_{\cdot}) \subseteq \mathbb{R}^{n+1}$, its* boundary is given by a convex function continuous on the interior of $\Delta(D)$



*lower

Examples:

* Boucksom-Chen: (Arithmetic NO bodies) X/k numb. field

  Arakelov theory $\rightsquigarrow$ $\|s\|$ for $s \in H^0(X, L)$

  $$G_t = \langle \{ s \in H^0(X, L) \mid \|s\| \leq e^{-t} \} \rangle$$

* Witt-Nyström: G_t from test configuration

* Boucksom-Küronya-Maclean-Szemberg:

  G_t given by order of vanishing at $q \neq p$

  $\to$ -ming related to Seshadri const.

  $\varepsilon(L, p)$

* McKinnon-Roth: diophantine approx.