Sean Paul: Test configurations, K-stability and Kähler-Einstein metrics

(X^n, ω) cpt Kähler mnfd

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{ij} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

$(g_{i\bar{j}}) > 0$

$$Ric(\omega) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\det(g_{i\bar{j}})$$

$$= \sum R_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \in C^\infty(\Lambda^{1,1}_M)$$

$$[Ric(\omega)] = C_1(M^n)$$

$$Scal(\omega) := \sum g^{i\bar{j}} R_{i\bar{j}} \in C^\infty(M)$$

$$V := \int_M \omega^n$$

$$\mu := \frac{\wedge}{V} \int_M Scal(\omega)\, \omega^n$$

$$\mu = \mu([\omega])$$

$$\mathcal{H}_\omega = \{ \varphi \in C^\infty(M) \mid \omega_\varphi = \omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0 \}$$

$$g_{i\bar{j}} + \varphi_{i\bar{j}} > 0$$

$\mathcal{H}_\omega$ = all Kähler metrics in $[\omega]$

$M^n_f \subset \mathbb{P}^{n+1}$ $\quad C_1(M^n_f) = (n+2-d)(H) \quad d = \deg(f)$

$\omega_{FS}|_{M^n_f}$ $\quad Ric(\omega_{FS}|_{M^n_f}) = (n+2-d)\omega_{FS}$ ...

<u>Basic problem:</u>

Given Kähler mfd (M^n, ω)

$\exists\ \varphi \in \mathcal{H}_\omega$ s.t. $Scal(\omega_\varphi) \equiv \mu$ + constant.

Let $\lambda \in \mathbb{R}$ $\quad \lambda[\omega] = c_1(X)$ then $Scal(\omega) \equiv \mu$

$\Rightarrow$ ω is Kähler Einstein i.e. $Ric(\omega) = \lambda\omega$

$\Longrightarrow$ K-stability (Test config.)

<u>Known</u> $\lambda = -1$ Yau & Aubin 70's show

$[-\omega] = c_1(M)$

$\exists!\ \varphi \in \mathcal{H}_\omega$ s.t. $Ric(\omega_\varphi) = -\omega_\varphi$

$(\lambda = 0)$ Yau showed that $\exists!\ \varphi \in \mathcal{H}_\omega$

s.t. $Ric(\omega_\varphi) = 0$

P.D.E's

However when $\lambda > 0$ $(\lambda = +1)$ known that

one cannot in gen. $Ric(\omega_\varphi) = \omega_\varphi$

1950's

If M (Fano) is K.E. $\Rightarrow$ $\eta(M)$ red. Lie alg

e.g. $\mathbb{P}^2([1:0:0])$ $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$

1980's Akito Futaki ($[\omega] = c_1(X)$)

$$F_{[\omega]} : \eta(M) \longrightarrow \mathbb{C} \qquad Ric(\omega) = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial h\omega$$

$$X \longrightarrow \int_M X(h\omega)\omega^n$$

Thm (Futaki)

- $F_{[\omega]}$ depends only on $[\omega]$
- $F_{[\omega]}$ is a Lie alg. char.
- $F_{[\omega]} \equiv 0$ if $\exists\ \omega_\varphi \in [\omega]$ which is K.E.

$$M = \mathbb{P}(P_1^*\mathcal{O}_{\mathbb{P}}(1) \oplus P_2^*\mathcal{O}_{\mathbb{P}^2}(1)) \qquad \exists\ X \in \eta(M)$$

st. $F_{[\omega]}(X) \neq 0$

$$\downarrow$$

$$\mathbb{P}^1 \times \mathbb{P}^2$$

Folklore conj: Assume $\eta(M) = \{0\} \Rightarrow$ M admits K.E. metric

Tian '97 Folklore is false

Mukai-Umemura 3-fold

$$SL_2(\mathbb{C})[(f_d,1)] \subset \mathbb{P}(S^d \oplus \mathbb{C})$$
$$\| \; M^3_6$$

counterexample:

$$Gr(3, H^0(Gr(4,\mathbb{C}^7)), \Lambda^2 Q)$$
$$\Psi$$
$$L = (S_0, S_1, S_2)$$
$$Z(L) \subset Gr(4,\mathbb{C}^7)$$

(eventually exploit that M is algebraic $M^n \xrightarrow{-2K_{M^n}} \mathbb{P}^{N_2}$)
$$Aut(M^n) \subseteq SL(N_2+1, \mathbb{C})$$

- $F_{[\omega]}$ is Lie alg. character
  - Q: can be extended to group char.?
  - want to 'integrate' $F_{[\omega]}$

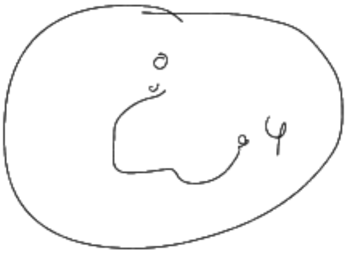
1986: Mabuchi introduced K-energy map (Fano)

$$\nu_\omega : \mathcal{H}_\omega \longrightarrow \mathbb{R}$$
$$\nu_\omega(\varphi) = \frac{(-n)}{V}\int_0^1 \int_M \dot\varphi_t \left(Scal(\omega_{\varphi_t}) - \mu\right)\omega^n_{\varphi_t} \, dt$$

φ_t is pw C^1 path in $\mathcal{H}_\omega$

$\varphi_0 = 0$

$\varphi_1 = \varphi$



$\Rightarrow$ (φ is critical $\Leftrightarrow$ $\left.\frac{\partial \nu_\omega(\varphi_s)}{\partial s}\right|_{s=0} = 0$ $\;Scal(\omega_\varphi) \equiv \mu$)

$$\nu_\omega(\varphi) = \frac{1}{V}\underbrace{\int_M \log\left(\frac{\omega_\varphi^n}{\omega^n}\varphi\right)\omega_\varphi^n}_{\geq -\frac{1}{e}} - \underbrace{\frac{\mu}{n}}_{=1}\underbrace{\left(I_\omega(\varphi) - J_\omega(\varphi)\right)}_{(n=1)\ =\ \frac{1}{2V}\int_M |\bar\partial\varphi|^2\omega} + O(1)$$

scale

Bando-Mabuchi

If M^n is K.E. $\Rightarrow$ $\exists$ const $C \geq 0$ s.t. $\nu_\omega(\varphi) \geq -C$

ν_ω integrates $F_{[\omega]}$ as follows:

- $X \in$
- $\mathrm{Re}(X)$ - rel pred
- $\Phi_{\mathrm{Re}(X)}(t)$ pg of diffeo $\in$