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Newton-Okounkov bodies in Lie theory

1. Motivation

1) Toric degen. of flag varieties

- Gonciulea - Lakshmibai
- Caldero
- Feigin - Fourier - Littelmann

Newton-Okounkov bodies

- Anderson
- Kaveh
- Khovanskii

We want to produce valuations in a general way

2) Computations of points in NO body

3) Understand the work of GHKK

polytope P → H discription ⇝ facets / hyperpl.
→ V discription ⟷ conv. (vert.)

Rietsch-Williams : Grassmannians

A-model	B-model
total positivity	LG-model
quantum cohom.	W superpotential

↳ valuations

flow model = Trop (W_t) = 0

② Birational sequences and NO bodies

jt. Fourier, Littelmann

special case : G/B to simplify

G connected, simply-con., simple / $\mathbb{C}$

$$\begin{array}{ccccc} T & \subset & B & \subset & G \\ \shortparallel & & \shortparallel & & \shortparallel \\ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} & & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & & SL_n \end{array}$$

$$\mathfrak{g} = \text{Lie}\, G = \underbrace{\mathfrak{n}^+ \oplus \overset{\text{Lie}\,T}{\mathfrak{h}}}_{\text{Lie}\,B} \oplus \mathfrak{n}^-$$

$U^- = \exp(\mathfrak{n}^-)$, Δ_+ pos. roots, $\alpha \in \Delta_+$ $\mathfrak{g}_{-\alpha} = \langle f_\alpha \rangle$

$U_{-\alpha} = \{\exp(t f_\alpha) \mid t \in \mathbb{C}\}$ root subgroup

$X = G/B$ flag variety

λ dom. int. wt

$\mathcal{L}_\lambda$ line bundle on X, very ample

$$H^0(X, \mathcal{L}_\lambda) \cong V(\lambda)^*$$

$$X \hookrightarrow \mathbb{P}(V(\lambda))$$

homog. coord ring $R(\mathcal{L}_\lambda) = \bigoplus_{m \geq 0} H^0(X, \mathcal{L}_\lambda^m)$

$$= \bigoplus_{m \geq 0} V(\lambda)^*$$

$$R(\mathcal{L}_\lambda) \longrightarrow \mathbb{C}(G/B)$$

$s_0 \in H^0(X, \mathcal{L}_\lambda)$ highest wt. section

$$\begin{array}{ccc} s & \longmapsto & \dfrac{s}{s_0^m} \\ \cap & & \\ H^0(X, \mathcal{L}_\lambda^{\otimes m}) & & \end{array}$$

$N = \dim G/B$

<u>valuation</u>: $\quad v : \mathbb{C}(G/B) \longrightarrow \mathbb{Z}^N$

$$\begin{array}{ccc} U^- & \dashrightarrow & G/B \\ x & \longmapsto & xB \end{array} \quad \text{is birational}$$

hence $\quad v : \mathbb{C}(U^-) \longrightarrow \mathbb{Z}^N$

<u>Def</u>: A sequence $S = (\beta_1, \dots, \beta_N)$ is <u>birational</u> if $\quad m : U_{-\beta_1} \times \dots \times U_{-\beta_N} \longrightarrow U^-$ is birational.

$$\mathbb{A}^N \longrightarrow U_{-\beta_1} \times \cdots \times U_{-\beta_N} \longrightarrow U^-$$

$$(t_1, \dots, t_N) \longmapsto (\exp(t_1 f_{\beta_1}), \dots, \exp(t_N f_{\beta_N})) \longrightarrow \prod_k \exp(t_k f_{\beta_k})$$

Take $x_1, \dots, x_N$ coord. on $\mathbb{A}^N$

$$\mathbb{C}(U^-) \xrightarrow{\sim} \mathbb{C}(x_1, \dots, x_N)$$

$$v : \mathbb{C}(x_1, \dots, x_N) \longrightarrow \mathbb{Z}^N$$

Total orders:

(1) lex

(2) oplex

(3) rlex

(4) roplex

Example: $\underline{s}_1 = (a_1, b_1, c_1)$ $\quad \underline{s}_2 = (a_2, b_2, c_2)$

$\underline{s}_1 \geq_{rlex} \underline{s}_2 \iff (c_1 > c_2)$ or $(c_1 = c_2 \;\&\; b_1 > b_2)$
or $(c_1 = c_2, b_1 = b_2, a_1 \geq a_2)$

$\underline{s}_1 \geq_{oplex} \underline{s}_2 \iff \underline{s}_1 \leq_{lex} \underline{s}_2$

Valuation: $\quad v : \mathbb{C}(x_1, \dots, x_N) \setminus \{0\} \longrightarrow \mathbb{Z}^N$ fix total order

$f \in \mathbb{C}[x_1, \dots, x_N]$, $\quad f = \sum_{\underline{n} \in \mathbb{N}^N} c_{\underline{n}} x^{\underline{n}}$

then $v(f) = \min_{\mathbb{Z}^N} \{ \underline{n} \in \mathbb{N}^N \mid c_{\underline{n}} \neq 0 \}$

for $\frac{f}{g}$ $\quad v\left(\frac{f}{g}\right) = v(f) - v(g)$

Then: $\Gamma(S,>) := \{ (m, \nu(f)) \mid f \in V(m\lambda)^* \setminus \{0\} \}$

$\subseteq \mathbb{N} \times \mathbb{N}^N$ semigroup of valuation

$C(S,>) := \mathrm{conv}(\Gamma(S,>)) \subseteq \mathbb{R} \times \mathbb{R}^N$ cone

$\Delta(S,>) := C(S,>) \cap (\{1\} \times \mathbb{R}^N)$ NO body

Q: 1) When is $\Gamma(S,>)$ fin. gen. & saturated?

2) How to compute $\Gamma(S,>)$?

Examples

1) $\underline{w_0} = s_{i_1} \cdots s_{i_N}$ & $S = (\alpha_{i_1}, \dots, \alpha_{i_N})$ birational

fix oplex total order on $\mathbb{Z}^N$

Theorem: $C(S,>)$ is weighted string cone
$\Delta(S,>)$ is string polytope

↳ Berenstein–Zelevinski & Littelmann

↳ $\Gamma(S,>)$ is fin. gen. & saturated

Recover degenerations of Caldero & Kaveh

Example 2 $\underline{w_0} = s_{i_1} \cdots s_{i_N}$ $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$

$S = (\beta_1, \dots, \beta_N)$, roplex

Theorem: $\Gamma(S,>)$ fin. gen. & saturated

(Lusztig paramatrization of canonical bases)

Example 3: $G = SL_n$ or Sp_{2n} $\quad \Delta_+ = \{\gamma_1, \dots, \gamma_N\}$

where $i < j \Leftarrow \gamma_i - \gamma_j \in \Delta_+$

e.g. for SL_4 $(\underbrace{\alpha_1+\alpha_2+\alpha_3}, \underbrace{\alpha_1+\alpha_2}, \underbrace{\alpha_2+\alpha_3}, \underbrace{\alpha_1, \alpha_2, \alpha_3})$

$S = (\gamma_1, \dots, \gamma_N)$ good sequence & rlex

Theorem: $\Gamma(S,>)$ is fin. gen. & saturated

$\Delta(S,>)$ is FFLV polytope

Conjecture: for S birat seq. & "$<$" total order on $\mathbb{Z}^N$

$\Gamma(S,>)$ is fin. gen.

③ Computing points in $\Gamma(S,>)$ & applications

$S = (\beta_1, \dots, \beta_N)$ birat. seq. & "$>$" total order

$U(n^-)$ $\quad \underline{m} \in \mathbb{N}^N$

$U(n^-)_{\leq \underline{m}} = \{ f^{\underline{k}} = f_{\beta_1}^{k_1} \cdots f_{\beta_N}^{k_N} \mid \underline{k} \leq \underline{m} \}$ filtr.

Similarly $U(n^-)_{<m}$

fix $V(\lambda)$ rep., $V(\lambda) = U(n^-) . v_\lambda$ ← h. wt. vec.

$\rightsquigarrow$ induced filtration $V(\lambda)_{\leq m}$, $V(\lambda)_{<\underline{m}}$

Def: $\underline{m} \in \mathbb{N}^N$ essential, if $\dim \left(V(\lambda)_{\leq m} / V(\lambda)_{<m} \right) = 1$

$es(\lambda)$ set of ess. exponents

$\underline{m} \in es(\lambda)$ then $f^{\underline{m}}$ essential monomial

Properties : (1) $es(n\lambda) + es(m\lambda) \subset es((n+m)\lambda)$

$\longrightarrow \Gamma(\lambda) := \bigcup_{n \geq 0} \{n\} \times es(n\lambda) \subseteq \mathbb{N} \times \mathbb{N}^N$

is a semigroup

(2) $\Gamma(\lambda) = \Gamma(S, >)$

(3) if $\Gamma(\lambda)$ fin. gen. then $\mathrm{conv}(es(\lambda)) = \Delta(S, >)$

Example $G = SL_4$, $\lambda = \omega_2$, $V(\lambda) = \Lambda^2 \mathbb{C}^4$, $e_1, \dots, e_4$ basis

$X = Gr(2, \mathbb{C}^4)$

fix $S = (\alpha_2+\alpha_3, \alpha_3, \alpha_1, \alpha_2)$ & "$>$" is lex order

highest wt vector is $e_1 \wedge e_2$

$e_1 \wedge e_2 = f^0 \cdot e_1 \wedge e_2 \Rightarrow \underline{0} \in es(\omega_2)$

$e_1 \wedge e_3 = f_{\alpha_2} \cdot e_1 \wedge e_2 \Rightarrow (0,0,0,1) \in es(\omega_2)$

$e_1 \wedge e_4 = f_{\alpha_3} f_{\alpha_2} e_1 \wedge e_2$

$= f_{\alpha_2+\alpha_3} e_1 \wedge e_2$

$(0,1,0,1) < (1,0,0,0)$

$\cap$

$es(\omega_2)$

Applications on Gromov-width:

Gromov width: (M^{2n}, ω) sympl. mnfd

$B^{2n}(a)$ open ball w. radius $\sqrt{\frac{a}{\pi}}$

$\hookrightarrow (\mathbb{R}^{2n}, \omega_{std})$

$GW(M) := \sup\{a \mid B^{2n}(a) \hookrightarrow (M, \omega) \text{ sympl.}\}$

$M = \mathcal{O}_\lambda$ coadjoint orbit ass. to max cp. $K \subset G$

Theorem (Caviedes, Castro, F.-Littelmann-Pabiniak)

$$GW(\mathcal{O}_\lambda) = r_\lambda = \min\{ |\langle\lambda, \alpha^\vee\rangle| \mid \alpha \in \Delta_+, \langle\lambda, \alpha^\vee\rangle \neq 0\}$$

How to prove?

(1) $GW(\mathcal{O}_\lambda) \leq r_\lambda$ (C-C) Gromov-Witten theory

(2) $GW(\mathcal{O}_\lambda) \geq r_\lambda$ (F-L-P)

construct a ball

(2.1) (Kaveh-Pabiniak)

$GW(\mathcal{O}_\lambda) \geq R$ if up to $GL_n(\mathbb{Z})$ $\exists$ an open simplex of size R in a NObody assoc. to $\mathcal{O}_\lambda$

(2.2) How to find this simplex?

$S = (\beta_1, \dots, \beta_N)$ good sequence & "<" roplex

then the open simplex of size r_λ is cont. in $\Delta(S, >)$