11/08/16 Claire Amiot

Cluster Categorification & Application to Tilting theory

§1 Cluster Categorification
§2 Cluster categories assoc. with surfaces
§3 —— " —— from derived categories

talks 1 & 2

§4 Derived invariant for surface alt algebras

§3 jt Steffen Oppermann
§4 jt Grüneland, Labardini, ?

Motivation: quiver mutation (see Roberts talk)
($\to$ cluster algebras)

Find categories in which quiver mutation appears "naturally"
use cluster combinatorics to solve problems in representation theory.

$k = \overline{k}$

§1: Cluster categorification

<u>Def</u>: Let $\mathcal{C}$ k-lin. triang. cat w. fin dim Hom-spaces (Hom-finite)

$\mathcal{C}$ is <u>2-Calabi Yau</u> if

$\forall X, Y \in \mathrm{ob}\,\mathcal{C} \quad \mathrm{Ext}^1_{\mathcal{C}}(X,Y) \simeq \underset{\text{dual over } k}{D}\,\mathrm{Ext}^1_{\mathcal{C}}(Y,X)$

$T \in \mathcal{C}$ basic is <u>cluster tilting</u> (cto) if

$$\mathrm{add}(T) = \{ X \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}\begin{smallmatrix}(X,T)\\(T,X)\end{smallmatrix} = 0 \}$$

Note: $\mathrm{Hom}(-,-[1]) = \mathrm{Ext}^1(-,-)$

<u>Slogan</u>: A 2-CY triang. cat. with cto's is a good setting to "categorify" quiver mutation.

<u>Indeed</u>: (1) One can mutate cto's

(2) the combinatorics of this mutation is quiver mutation

<u>Theorem</u> (Iyama-Yoshino 2008)

Let $T = T_i \oplus T_0$ cto in $\mathcal{C}$ with T_i indec.

Then $\exists$ unique T_i^* indec. not isom to T_i s.t. $T' = T_i^* \oplus T_0$ is cto (unique up to isom.)

$\hookrightarrow$ <u>mutation</u> of T at T_i (Notation $T' = \mu_{T_i}(T)$)

$$T_i \longrightarrow U \xrightarrow{\ \mathrm{add}(T_0)\ } T_i^* \longrightarrow T_i[1]$$

minimal left add(T_0)-approximation

$$T_i^* \longrightarrow V \longrightarrow T_i \longrightarrow T_i^*[1]$$

minimal right add(T_0)-approximation

these are called <u>exchange triangles</u>

<u>Theorem</u> (Buan, Iyama, Reiten, Scott)

$T = T_i \oplus T_0$ as above. Denote by Q_T the Gabriel quiver of $\mathrm{End}_{\mathcal{C}}(T)$ $(\cong kQ_T/I)$

If Q_T is <u>cluster quiver</u> then

$$\begin{tikzcd}
T = T_i \oplus T_0 \arrow[r, leftrightarrow, "\text{IY mut.}", "\mu_{T_i}"'] \arrow[d] & T_i^* \oplus T_0 = T' \arrow[d] \\
Q_T \arrow[r, leftrightarrow, "\mu_i", "\text{FZ mut}"'] & Q_{T'}
\end{tikzcd}$$

where i is vertex corresp. to T_i in Q_T.

<u>Examples</u>:

1. [Buan Marsh Reineke Reiten Todorov 2006]
   Cluster cat. $\mathcal{C}_Q$ assoc. with acyclic quiver Q.
   (more in next talk)

2. [Geiss, Leclerc, Schröer 2006]
   Q Dynkin quiver, <u>mod</u> Π_Q (preproj algebra)
   (in Frobenius setting this also works $\longrightarrow$ Robert's next talk)

[Iyama Yoshino 2006] CM(R) for some Gorenstein ring

Question: What happens to relations?

Problem: I is not uniquely defined (Q_I is unique)

Def [Derksen, Weyman, Zelevinsky]

Q quiver, W potential on Q = (possibly infinite) linear comb. of cycles in Q.

$$\mathrm{Jac}(Q,W) := \widehat{kQ} \big/ \langle \partial_a W, a \in Q_1 \rangle \quad \text{Jacobian algebra}$$

$$\partial_a \underbrace{(a_1 \cdots a_n)}_{\text{cycle in } Q} = \sum_{a_i = a} a_{i+1} \cdots a_n a_1 \cdots a_{i-1} \quad \text{remove } a \text{ from cycle}$$

Ex

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd} \qquad W = cba$$

then $\mathrm{Jac}(Q,W) = kQ \big/ \langle \underset{\partial_a W}{\overset{\shortparallel}{cb}}, \underset{\partial_b W}{\overset{\shortparallel}{ac}}, \underset{\partial_c W}{\overset{\shortparallel}{ba}} \rangle$

Fact: "almost" all endom. alg. of cto's are Jacobian (Ladkani)

[DWZ 2008] Extend the definition of quiver mutation to mutation of quiver with potential (QP's)

↳ much more technical, uses notion of right equivalence

$$(Q,W) \underset{\text{right eq.}}{\sim} (Q',W') \Rightarrow \mathrm{Jac}(Q,W) \cong \mathrm{Jac}(Q',W')$$

reduction process (corresponds to removing 2-cycles)

Example: mutation at 1

(steps a&c)
$$\begin{tikzcd} & 2 \arrow[dl, "a^*"'] \arrow[dr, "b"] & \\ 1 \arrow[rr, "c^*"'] & & 3 \arrow[ul, bend right, "{[ac]}"'] \end{tikzcd}$$
$$W' = b[ac] + [ac]c^*a^*$$

Exercise: check that this quiver reduces to

$$\begin{tikzcd} & 2 \arrow[dl, "a^*"'] & \\ 1 \arrow[rr, "c^*"'] & & 3 \end{tikzcd}$$
$$W'' = 0$$

↳ reduction process fixes which 2-cycle has to be removed ⇝ not any max set any more

W is non-degen. if after reduction Q (and any mutation of Q) is cluster quiver.

(reduction not always deletes all 2-cycles)

Theorem [Buan, Iyama, Reiten, Scott 2010]
$T = T_i \oplus T_0$ cto. Assume $\mathrm{End}_{\mathcal{C}}(T) = \mathrm{Jac}(Q, W)$ with W non-degenerate. (+ tech. gluing cond.)
Then $\mathrm{End}_{\mathcal{C}}(T') = \mathrm{Jac}(Q', W')$ and

$$\begin{tikzcd} T \arrow[r, leftrightarrow, "\text{IY mut}", "\mu_{T_i}"'] \arrow[d] & T' \arrow[d] \\ (Q, W) \arrow[r, leftrightarrow, "\text{DWZ mut}", "\mu_i"'] & (Q', W') \end{tikzcd}$$

where i corresponds to T_i in Q.

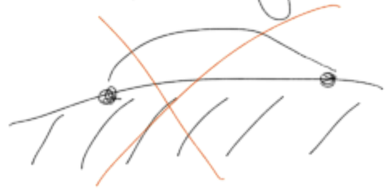
§2: Cluster categories assoc. to surfaces

[Fomin Shapiro Thurston 2006]

S = compact orientable surface with nonempty boundary

M = fin. set of marked pts on S (at least one marked pt on each boundary component)

γ <u>arc</u> is a simple curve on $S \setminus M$ with end-points in M, not contractable (not isotopic to pt), not isotopic to boundary segment



two arcs are <u>compatible</u> if they do not intersect (up to isotopy)

Δ <u>triangulation</u> is a maximal collection of pw compatible arcs.

case: no puncture in triangles

in case of puncture

self folded triangle

no unique way to replace γ when flipping

[FST08] In case of punctures, introduce notion of <u>tagged</u> arc & tagged triangulation.

They proved: flipping is always possible at any tagged arc in tagged triangulation.

<u>Theorem</u> [FST, Labardini 08]

One can assoc. <u>non-degen.</u> quiver QP $(Q(\Delta), S(\Delta))$ to any tagged triangulation Δ.

$$\begin{tikzcd} \Delta \arrow[r, leftrightarrow, "f_\gamma", "\text{flip}"'] \arrow[d] & \Delta' \arrow[d] \\ (Q(\Delta), S(\Delta)) \arrow[r, leftrightarrow, "\text{DWZ mut.}"'] & (Q(\Delta'), S(\Delta')) \end{tikzcd}$$

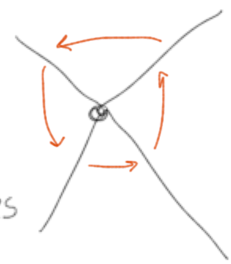
If Δ is valency ≥ 3-triangulation (any puncture has valency ≥ 3, marked bdy pts may have < 3)

Then $Q(\Delta)$ is the adjacency quiver

$Q(\Delta)_0 = \{\text{arcs in } \Delta\}$

arrows 

$Q(\Delta)_1$

$S(\Delta) = \sum_{\substack{\text{internal} \\ \text{triang.}}}$  $- \sum_{\text{punctures}}$ 

12/08/16

§2 Cluster categories from surfaces

Recall: (S,M) surface with marked pts

Δ tagged triangulation

$\rightsquigarrow (Q(\Delta), S(\Delta))$ non-deg QP

flip = [DWZ]-mutation

Theorem: Let (S,M) as above. Then there exists a 2-CY triang. cat $\mathcal{C}_{(S,M)}$ with cto's & bij.

$$\{\text{tagged arc}\}/\text{isot.} \overset{1:1}{\longleftrightarrow} \{\text{indec. summ. of cto}\}/\text{iso}$$

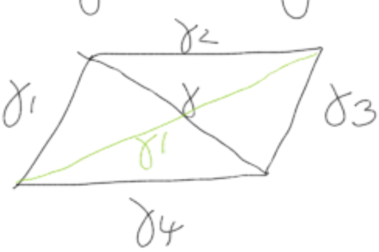
$$\gamma \longmapsto T_\gamma$$

$$\text{flip} \circlearrowright \{\text{tagged triang.}\} \overset{1:1}{\longleftrightarrow} \{\text{cto's}\}/\text{iso} \circlearrowleft \text{IY-mut.}$$

$$\Delta = \bigcup \gamma \longmapsto T_\Delta = \bigoplus T_\gamma$$

$$\mathrm{End}_{\mathcal{C}}(T_\Delta) \simeq \mathrm{Jac}(Q(\Delta), S(\Delta))$$

Exchange triangles:



$$T_\gamma \to T_{\gamma_1} \oplus T_{\gamma_3} \to T_{\gamma'} \to$$

$$T_{\gamma'} \to T_{\gamma_2} \oplus T_{\gamma_4} \to T_\gamma \to$$

[Caldero Chapoton Schiffler 06]: (S,M) = polygon (Keller)

[A 08] Construction of $\mathcal{C}_\Delta$ assoc with Δ
2-CY str $\mathrm{End}(T_\Delta) = \mathrm{Jac}(Q(\Delta), S(\Delta))$

[K Yang 09, Labardini 08] $\mathcal{C}_\Delta$ indep of Δ

[Brüstle Zhang 10, Qiu Zhou] bijections

[Canakci-Schroll 16] exchange triangles

§3 Cluster categories from derived categories

Λ fin. dim k-alg of gldim ≤ 2

$D^b\Lambda \circlearrowleft \mathbb{S}_2 = - \overset{L}{\underset{\Lambda}{\otimes}} D\Lambda[-2]$

Assume that $\mathrm{Hom}_{\mathcal{D}}(\Lambda, \mathbb{S}_2^p\Lambda) = 0$ for $|p| \gg 0$
(τ_2-finite)

<u>Theorem</u> [A 08]

There exists a 2-CY cat $\mathcal{C}_2(\Lambda)$ with triang fctor

$$\pi : \mathcal{D}^b\Lambda \longrightarrow \mathcal{C}_2(\Lambda) \quad \text{s.t.}$$

(i) $\pi\Lambda$ is cto

(ii) $\forall\ X, Y \in \mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}}(\pi X, \pi Y) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, \mathbb{S}_2^p Y)$$

Remark 1) $\Lambda = kQ$ Q acyclic then $\mathcal{C}_2(kQ) \cong \mathcal{C}_Q$, π is dense

2) τ_2-finite $\Leftrightarrow$ $\mathcal{C}_2(\Lambda)$ is Hom-finite

3) gldim ≤ 2 $\Leftrightarrow$ $\mathrm{Ext}^1_{\mathcal{C}}(\pi\Lambda, \pi\Lambda) = 0$

4) $\forall\, X \in \mathcal{D}^b\Lambda$ $\mathrm{End}_{\mathcal{C}}(\pi X)$ is naturally $\mathbb{Z}$-graded.

Theorem [Keller 10]

$\Lambda = kQ_\Lambda / \langle R \rangle$ R union of basis of $\mathrm{Ext}^2_\Lambda(S_i, S_j)$ $\forall i,j$

Then $\mathrm{End}_{\mathcal{C}}(\pi\Lambda) \cong \mathrm{Jac}(Q_{\pi\Lambda}, W_{\pi\Lambda})$ where

$$(Q_{\pi\Lambda})_0 = (Q_\Lambda)_0$$

$$(Q_{\pi\Lambda})_1 = (Q_\Lambda)_1 \cup \{ i \xrightarrow{a_r} j \mid r \in R \cap \mathrm{Ext}^2_\Lambda(S_i, S_j) \}$$

$$W_{\pi\Lambda} = \sum_{r \in R} a_r r$$

Example: Λ:
$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, dashed] & & 3 \end{tikzcd}$$
$ba = 0$

$$Q_{\pi\Lambda} = \begin{tikzcd} & 2 \arrow[dr, "0"] & \\ 1 \arrow[ur, "0"] & & 3 \arrow[ll, "1"] \end{tikzcd}$$

grading

$W = bac$

If we define $d : (Q_{\pi\Lambda})_1 \longrightarrow \mathbb{Z}$ as $d(a) = 0$
$\forall a \in (Q_\Lambda)_1$ and $d(a_r) = 1$ $\forall r \in R$

then $\omega_{\pi\Lambda}$ is homogeneous of degree 1

and $\mathrm{End}_{\mathcal{C}}(\pi\Lambda) \cong \bigoplus \mathrm{Hom}_{\mathcal{D}}(\Lambda, \mathbb{S}_2^{-p}\Lambda)$

$\underset{\text{graded}}{\cong} \mathrm{Jac}(Q_{\pi\Lambda}, \omega_{\pi\Lambda}, d)$

<u>Graded mutation:</u>

<u>Fact</u> [A.-Oppermann]

$T' \in \mathcal{C}_2(\Lambda)$ cto then $T' \cong \pi(T)$
for some $T \in \mathcal{D}^b\Lambda$

$\longrightarrow$ cluster combinatorics are encoded in $\mathcal{D}^b\Lambda$

Take $T \in \mathcal{D}^b\Lambda$ s.t. $\pi(T) \in \mathcal{C}_2(\Lambda)$ is cto
$T = T_i \oplus T_0$ with T_i indecomp.

$$T_i \xrightarrow[\text{min left add}(\pi^{-1}(T_0))\text{-approx.}]{} U \longrightarrow T_i^L \longrightarrow \quad \pi(T_i^L) = \pi(T_i)^* = \pi(T_i^R)$$

$$T_i^R \longrightarrow V \xrightarrow[\text{min. right}]{} T_i \longrightarrow$$

$\mu^L_{T_i}(T) = T_i^L \oplus T_0$ left mutation

$\pi(T_i^L \oplus T_0)$ is cto

Right
Left mutation of graded quivers:

(a) $k \xrightarrow{\alpha} i \xrightarrow{\beta} j$ $\quad \alpha, \beta \in \mathbb{Z}$ degrees

$$\Downarrow$$

$$k \xrightarrow{\alpha+\beta} j$$

(b) Remove 2-cycles of degree 1

(c) $i \xrightarrow{\alpha} j \rightsquigarrow i \xleftarrow[]{\overset{1-\alpha}{-\alpha}} j$

$k \xrightarrow{\beta} i \rightsquigarrow k \xleftarrow[\beta]{1-\beta} i$

Theorem [A. Oppermann]

(Q, W, d) non deg QP, graded with W homog of degree 1. Then one can mutate (Q, W, d) and $\mu_i^L(Q, W, d) = (\underbrace{\mu_i Q}_{\text{FZ mut}}, \underbrace{\mu_i W}_{\text{DWZ}}, \mu_i^L d)$

Prop [AO] $T \in \mathcal{D}^b\Lambda$ s.t. $\pi(T)$ is cto and $\mathrm{End}(\pi T) \cong \mathrm{Jac}(Q, W, d)$ with W homog. of degree 1 & non-deg.

$$\begin{tikzcd} T \arrow[r, "\mu_{T_i}^L"] \arrow[d] & T' = T_i^L \oplus T_0 \arrow[d] \\ (Q, W, d) \arrow[r, "\mu_i^L"] & (Q', W', d') \end{tikzcd}$$

then $\mathrm{End}(\pi(T_i^L \oplus T_0))$ is graded Jacobian

Application to tilting theory

Easy to see: if T is tilting complex in $\mathcal{D}^b\Lambda$ s.t. gldim $\mathrm{End}_{\mathcal{D}}(T) \leq 2$

$\pi(T)$ is cto in $\mathcal{C}$

Hope: use cluster tilting combinatorics to deduce derived equivalences.

Theorem [AO 11]

Λ, Λ' gldim ≤ 2 & τ_2-finite. Consider assoc. graded QP's (Q, W, d) & (Q', W', d') homdeg. If one can pass from (Q, W, d) to (Q', W', d') by sequence of graded (right or left) mutation

Then $$\mathcal{D}^b\Lambda \simeq \mathcal{D}^b\Lambda'$$

Remark: $\Lambda = kQ \rightsquigarrow (Q, 0, 0)$

i source in $Q \Rightarrow (S_i Q, 0, 0)$ assoc. to $\Lambda' = kS_iQ$

[reflection at i [BGP]]

Converse is not known (it is ok if $\Lambda = kQ$)

$\Lambda = kQ$

$$Q = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd} \quad \overset{}{\underset{\mu_2^L}{\rightsquigarrow}} \quad \begin{tikzcd} 1 \arrow[rr, bend left, "0"] & 2 \arrow[l, "0"] & 3 \arrow[l, "1"] \end{tikzcd} \qquad W = cba$$

$(Q, 0, 0)$

corr. to (derived eq.)

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \end{tikzcd} = \Lambda'$$

13/08/16

Last talk → beamer presentation

4 Derived invariants for surface cut algebras