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# On quiver Grassmannians & orbit closures for representation-finite algebras

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$k$ field, $A$ fd alg. of fin. rep. type

the Auslander algebra of $A$ is $\Gamma_A$

$= \mathrm{End}_A(E)^{op}$

where $E = \bigoplus$ of one copy of each indec. proj. $A$-module

$e\Gamma_A e = A \quad \exists e \in A$

Aim:

1) Define a variant of Auslander algebra, the projective quotient algebra $B_A$

If $A = kQ$, $Q$ Dynkin quiver due to Cerulli-Irelli, Feigin, Reineke 2013

Hernandez-Leclerc 2015

2) Use to get desingularizations of quiver Grassmannians

$$Gr_A\binom{M}{\underline{d}}$$

3) Use to realize orbit closures $\overline{O}_M$ in $R_A(\underline{d})$ as quotient varieties

unifies — CFR 2014 $A = kQ$, $Q$ Dynkin

— Kraft & Procesi 1979 $A = k[x]/(x^n)$

<u>Def</u>: The projective quotient algebra of $A$ is

$$B_A = \mathrm{End}_{\mathcal{H}}(G)^{op} \quad \text{where}$$

$$\mathcal{H} = \frac{\{\text{cat. of surj. maps } P \twoheadrightarrow X,\ P, X \text{ A-modules},\ P \text{ proj.}\}}{\text{morph. factoring through subcat. } \{P \xrightarrow{1} P\}}$$

= Homotopy cat. of 2-term complexes $P \twoheadrightarrow X$

$G$ = direct sum of one copy of each indec. in $\mathcal{H}$

$$\begin{cases} P_X \twoheadrightarrow X = (X),\ X \text{ indec. non-proj A-module} \\ P_i \twoheadrightarrow 0 = [i],\ P_i \text{ indec. proj. A-module} \end{cases}$$

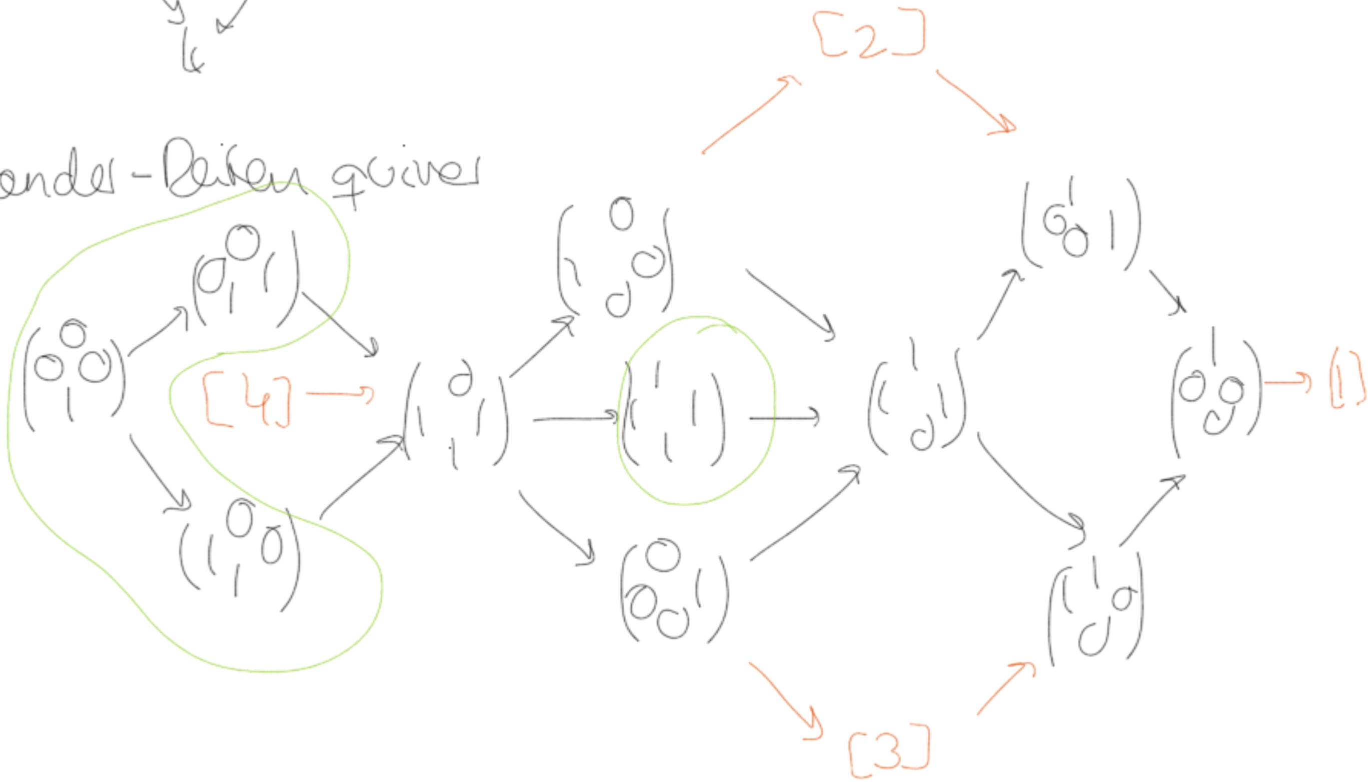
<u>Properties</u>:

1) $\exists\, e \in B_A$, $eB_Ae = A$, $e$ = proj. onto

$$A \twoheadrightarrow 0 = \bigoplus_i P_i \twoheadrightarrow 0$$

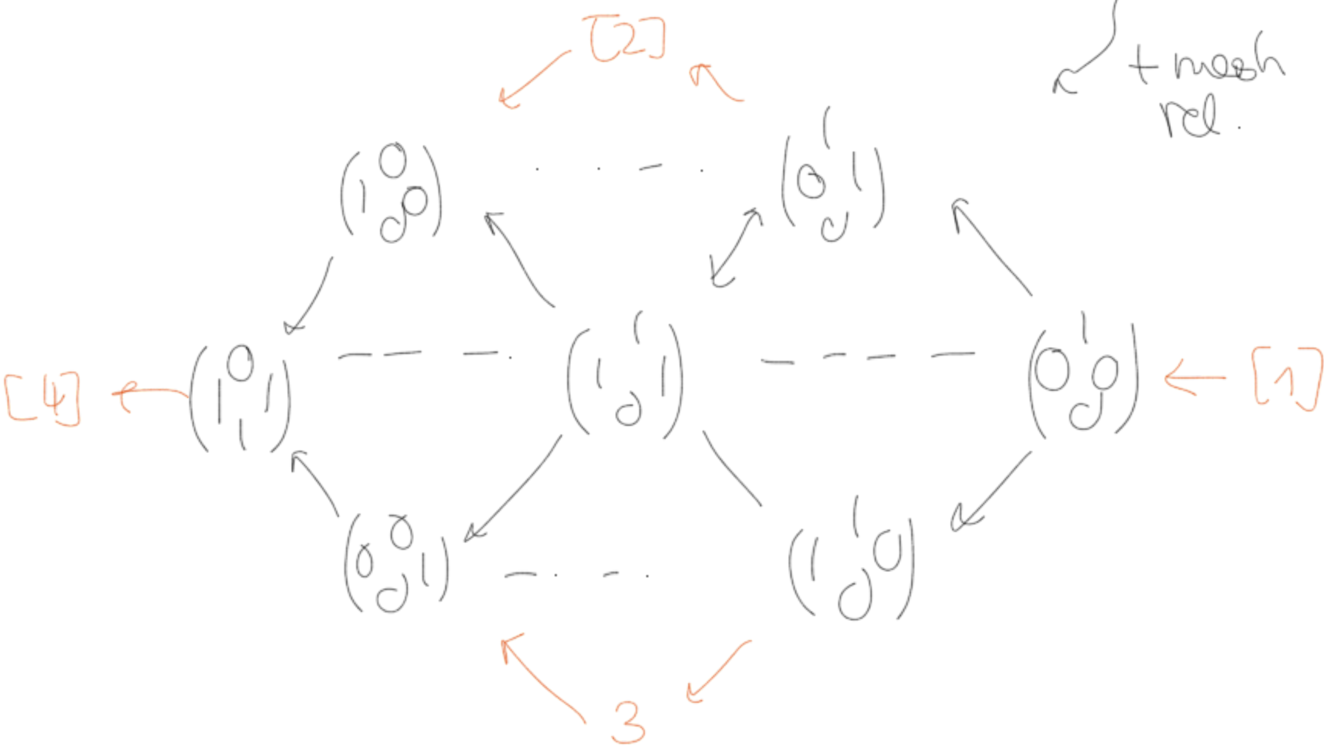
$$A = \begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 4 & \end{tikzcd}$$

Auslander-Reiten quiver




delete green ones and reverse red arrows

+ mesh rel.




Given $e \in B$ idemp. have

functor

there is a recollement of ab. cat. $e(N) = eN$

$$\begin{tikzcd}
B/BeB\text{-Mod} \arrow[r, "i"] & B\text{-Mod} \arrow[l, bend right=50, "q"'] \arrow[l, bend left=50, "p"] \arrow[r, "e"] & eBe\text{-Mod} \arrow[l, bend right=50, "l"'] \arrow[l, bend left=40, "r"] \arrow[l, bend left=70]
\end{tikzcd}$$

$c$ = intermediate ext. functor

where $C(M) = \mathrm{Im}(l(M) \longrightarrow r(M))$

<u>Properties</u> [Kuhn]

$ec(M) \cong M$

$c(eN)$ is a subquot. of $N$

$c$ is fully faithful

<u>Exp</u>:

$$B = \begin{matrix} \circ\, 1 \\ \downarrow \\ \circ\, 2 \\ \downarrow \\ \circ\, 3 \end{matrix} \qquad e = e_1 + e_3 \qquad eBe = \begin{matrix} \circ\, 1 \\ \downarrow \\ \circ\, 3 \end{matrix}$$

$$e\begin{pmatrix} V_1 \\ \downarrow f \\ V_3 \end{pmatrix} = \qquad \begin{array}{ccc} l & c & r \\ V_1 & V_1 & V_1 \\ \downarrow 1 & \downarrow & \downarrow f \\ V_1 & \mathrm{Im}(f) & V_3 \\ \downarrow f & \downarrow & \downarrow 1 \\ V_3 & V_3 & V_1 \end{array}$$

For the proj quot alg (pqa) $B_A$ of $A$

$$(\mathcal{H}^{op}, Ab) \xrightarrow{\sim} B_A\text{-Mod}$$
$$F \longmapsto F(G)$$

$$c(M)(P \twoheadrightarrow X) = \operatorname{coker}(\operatorname{Hom}(X, M) \hookrightarrow \operatorname{Hom}(P, M))$$

$\operatorname{pd}_{B_A} c(M) \leq 1$ $\quad$ $\operatorname{i.d.}_{B_A} c(M) \leq 1$

$$0 \to \operatorname{Hom}(-, P_M \to M) \to \operatorname{Hom}(-, P_M \to 0) \to c(M) \to 0$$

$C = c(E)$ is a (classical) tilting, cotilting module for $B_A$.

Theorem

A basic algebra $B$ is pqa

$\Leftrightarrow$ it has gldim $B \leq 2$ and it has a tilting cotilting module $C$ with

$$\operatorname{gen} C \cap \operatorname{cogen} C = \operatorname{add} C$$

$$\operatorname{Ker} \operatorname{Hom}(C, -) = \operatorname{Ker} \operatorname{Hom}(-, C)$$

Theorem A basic alg $\Gamma$ is Ausl. alg $\Leftrightarrow$ gldim $\leq 2$ and there is a tilting cotilting $\Gamma$-module $T$ which is gen. & cogen. by projective-injectives.

Thm ct'd

Moreover in this case $T$, if basic, is unique.

$$0 \to \Gamma_A \to \Pi' \xrightarrow{f} \Pi'' \qquad T = \mathrm{Im}(f) \oplus \Pi.$$

Theorem:

$$\mathrm{End}_{\Gamma_A}(T)^{op} \cong B_A.$$

Some Geometry:

$B = kQ_B / I_B$ $\quad e =$ sum of some of the vertices of $Q_B$

$A = eBe = kQ_A / I_A$ $\quad Q_A$ has vertices

dimension vector for $B$ is of the form $(\underline{d}, \underline{r})$, $\underline{d}$ a dim. vector for $A$.

$$Gr_B\begin{pmatrix} c(M) \\ \underline{d}, \underline{r} \end{pmatrix} \longrightarrow Gr_A\begin{pmatrix} M \\ \underline{d} \end{pmatrix} = \dot{\bigcup_{\underline{\dim} N = \underline{d}}} S_N$$

stratum

smooth $\qquad$ $M$ an $A$-module

$$= \bar{S}_{N_1} \cup \cdots \cup \bar{S}_{N_\ell}$$

$$\bar{S}_{c(N_i)} \subseteq Gr_{B_A}\begin{pmatrix} c(M) \\ c(N_i) \end{pmatrix}$$

$e$ induces a map

$$R_B(\underline{d},\underline{r})/\!/GL_{\underline{r}} \longrightarrow R_A(\underline{d}) \quad \text{closed embedding}$$

$$K[R_B(\underline{d},\underline{r})]^{GL_{\underline{r}}} \longleftarrow K[R_A(\underline{d})]$$

<u>Theorem</u>: If $M$ an $A$-module of dim $\underline{d}$, $B_A$ the pqa, then $\overline{\mathcal{O}_M} \cong R_{B_A}(\underline{d},\underline{r})/\!/GL_{\underline{r}}$

$\underline{d},\underline{r} = \underline{\dim}\, C(M)$.