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Indecomposable objects in the homotopy category of a derived-discrete algebra

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1 Introduction

$\mathrm{Hom}(C, \oplus X_i) \cong \oplus \mathrm{Hom}(C, X_i) \Leftrightarrow C$ compact

$\boxed{\text{Compactly gen. triang. cat. } \quad T^c \subseteq T}$

Ziegler spectrum $Zg(T)$ of T

Functor category $\text{mod-}T^c : ((T^c)^{op}, Ab)^{fp}$ — fin. pres.

relation: 1) closed subset in $Zg(T)$ $\longleftrightarrow$ Serre subcat

$\mathcal{D} \cap Zg(T)$

$\updownarrow$

def. subcat $\mathcal{D}$ in T^c

2) Isolated pt $\longleftrightarrow$ simple functor $\leftrightarrow$ LHS of AR triang in T^c

3) (IC) Cantor Bendixson $\longleftrightarrow$ (IC) Krull-Gabriel analysis

$\updownarrow$

(IC)' family of morph^h in T^c

Main result: (ALPP)

Consider $\mathcal{K} = K(\Lambda\text{-proj})$ for Λ DDA

$\boxed{A}$ $KG(\text{mod-}\mathcal{K}^c) = CB(Zg(\mathcal{K})) = 2$

$\boxed{B}$ $\{\text{indec. obj in } \mathcal{K}\} = \{\text{indec. pure injectives}\}$

2 Derived discrete algebras

Let A fd alg

Def: $D^b(A\text{-mod})$ discrete if $\forall n = (n_i)_{i\in\mathbb{Z}},\ n_i \in \mathbb{N}$

$\exists$ only fin. many indec. X st $n = (\underline{\dim}_k H^i(X))_{i\in\mathbb{Z}}$

(by Vossieck 2001)

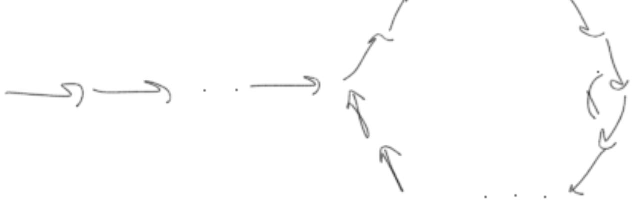
Ex: kQ, Q Dynkin

Theorem (Bobiński–Geiß–Skowroński 2004)

If A connected & not Dynkin then TFAE

1) $D^b(\text{mod} A)$ discrete
2) $D^b(\text{mod} A) \cong D^b(\text{mod} \Lambda)$ where

$\Lambda = \Lambda(r,n,m)$



m = length of tail
r = # relations
n = length of cycle
$m \geq 0$
$n \geq r \geq 1$

Moreover $\mathcal{D}^b(\Lambda(r,m,n)) \cong \mathcal{D}^b(\Lambda(r',m',n'))$

iff $(r,m,n) = (r',m',n')$

"<u>DDA</u>" means $\Lambda(r,m,n)$

<u>Remark</u>: a) DDAs are gentle

<u>String complexes</u> $\quad \mathcal{D}^b(\Lambda\text{-mod}) \cong K^b(\operatorname{proj}\Lambda)$

$$\cdots \quad -2 \qquad -1 \qquad 0 \qquad 1 \qquad 2 \quad \cdots$$

$$\begin{tikzcd}
P_0 \arrow[r] & P_1 \arrow[r] & P_3 \arrow[r] & P_5 \\
& \oplus & \oplus & \oplus \\
& P_2 \arrow[r] \arrow[ddr] & P_4 \arrow[uur] & P_6 \\
& & \oplus & \\
& & P_7 \arrow[uur] &
\end{tikzcd}$$

nonzero comp. are single paths in Q

$\{\text{indec.}\} = \{\text{string compl.} + \text{bdry cond.}\}$
in $\mathcal{D}^b(\Lambda\text{-mod})$

<u>Joergensen 05</u>

$K(\Lambda\text{-proj})$ compactly gen. $\quad K^c \cong \mathcal{D}^b(\Lambda\text{-mod})$

$$\underbrace{\qquad\qquad\qquad\qquad K^c = K^{b,+}(\Lambda\text{-proj})}_{\text{string complexes + boundary cond.}}$$

• (Broomhead ?P) $r \neq 1$

$\underline{\dim}$ $\mathrm{Hom}(X,Y) \leq 1$ for X, Y indec.

$r = 1$ — " — ≤ 2 — " —

• In $K(X\text{-proj})$ can consider all string cx's

Theorem (ALPP)

string cx's are indec. & pure inj

③ Purity in compactly gen. triang. cat

Functor cat: $\mathrm{Mod}\text{-}T^c := ((T^c)^{op}, Ab)$

$\mathrm{mod}\text{-}T^c := ((T^c)^{op}, Ab)^{fp}$

Restricted Yoneda functor:

$X \in T$ $\quad (-,X) := \mathrm{Hom}_T(-,X)|_{T^c}$

$f \in T$ $\quad (-,f) := \mathrm{Hom}_T(-,f)|_{T^c}$

Def: $X \in T$ pure inj if $(-,X)$ is injective

The Ziegler spectrum is top. sp. with points

pure inj indec.

basis/open sets: $F \in \mathrm{mod}\text{-}T^c$

$(F) := \{X \in T \mid (F, (-,X)) \neq 0\}$

Rmk: here $(\mathrm{mod}\text{-}T^c)^{op} \cong \mathrm{Coh}(T)$

[Krause 02] $\quad F \longmapsto F^\vee \quad F^\vee(X) = (F, (-,X))$

Cantor Bendixson rank of Z

$CB(Z) = n \in \mathbb{N}$ if $\exists$ closed subsets

$Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{n+1} = \emptyset$

where $\forall\, 0 \leq i \leq n$

Z_i / Z_{i+1} = isolated pts in Z_i

Krull-Gabriel dimension of $\mathcal{A}$

$KG\dim \mathcal{A} = n$ if $\exists$ Serre subcats

$0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subseteq \mathcal{A}_n \subseteq \mathcal{A}_{n+1} = \mathcal{A}$

s.t. $\forall i$ $\quad \mathcal{A}_{i+1}/\mathcal{A}_i$ = Serre subcat of $\mathcal{A}/\mathcal{A}_i$ gen. by simples.

(4) KG-dim + CB-rank for a DDA

Remark (Bobinski) Λ DDA, $\mathcal{C}$ cat of fp functors

$D^b(\Lambda\text{-proj}) \longrightarrow Ab$

$$KG(\mathcal{C}) = \begin{cases} 1 & \text{when } \mathrm{gldim} = \infty \\ 2 & \text{when } \mathrm{gldim} < \infty \end{cases}$$

Proposition: [ALPP]

$KG(A)$ is defined $\Rightarrow \exists$ bij for each i

$$\left\{ \begin{matrix} \text{isolated pts in} \\ Z_i \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{simple functors} \\ A/A_i \end{matrix} \right\}$$

Theorem (ALPP) $KG(A) = CB(2) = 2$

1. Rank zero points = { finite strings }
2. Rank one pts = { 1-sided ∞ strings }
3. Rank two pts = { 2-sided ∞ strings }

Corollary:

Points in $Zg(K)$ are exactly string complexes.

5. Indecomposables

Remark (Han '13)

no tail

$n = r$

$K(\Lambda\text{-inj})$ $\Lambda = \Lambda(n, n, 0)$ Result holds

Theorem (ALPP)

All indec. are pure inj and hence string.

$M \in T$

$$\langle M \rangle := \left\{ X \in T \;\middle|\; \begin{array}{l} \forall F \in \operatorname{mod} T^c \;\; (F, (-, M)) \neq 0 \\ \quad \Rightarrow (F, (-, X)) \neq 0 \end{array} \right\}$$

definable subcat gen. by M.

$\operatorname{supp}(M) := \langle M \rangle \cap Zg(T)$

Idea of pf:

M indec.

What is in $\mathrm{Supp}(M)$?

1. If $C \in \mathrm{Supp}(M)$ cpt then $C = M$.
2. ~~Ass.~~ no cpt in $\mathrm{Supp}(M)$

If $\mathrm{gldim}\,\Lambda = \infty$

we can prove that $\langle M \rangle \subseteq \langle Z \rangle$

with Z is Σ-pure inj.

$\Rightarrow M$ (Σ-) pure inj.

If $\mathrm{gldim}\,\Lambda < \infty$

- prove that $\exists\, N$ with $CB(N) = 1$

$\rightsquigarrow$ argue that $M = N$

$\square$