10/08/16

Linckelmann

Hochschild cohomology and modular representation theory

G fin grp, k field, char $p > 0$

Which algebras arise as indec. factors ("blocks") of kG?

Very small class of algebras

(1) They have semisimple lifts to char 0

$\mathcal{O}$ complete discrete val ring $\mathcal{O}/J(\mathcal{O}) = k$

K FoF

B block alg of kG $\Rightarrow$ unique block $\hat{B}$ of $\mathcal{O}G$ s.t. $k \otimes_{\mathcal{O}} \hat{B} \cong B$

char $K = 0$ $\Rightarrow$ $K \otimes_{\mathcal{O}} \hat{B}$ semisimple

(2) C Cartan matrix of B, $C = (c_{ij})_{ij}$

$c_{ij} = \dim_k \mathrm{Hom}_B(P_i, P_j)$ $\quad P_i, P_j$ proj. indec. obj

$\det C$ power of p (Brauer)

C positive definite.

(3) Block algebras are symmetric: $B \cong B^\vee = \mathrm{Hom}_k(B,k)$ as B-B-bimodules

(kG is symmetric)

(4) B block alg of $kG \Rightarrow \exists$ fin. subfield $\mathbb{F}$ of k s.t. $B \cong k \otimes_{\mathbb{F}} B'$ for some $\mathbb{F}$-alg B'

Cliff-Plesken-Weiss (1987):

$Z(B)$ is defined over $\mathbb{F}_p$

(5) B satisfies <u>property (Fg)</u>

(Erdmann HSST, 2004)

$HH^*(B) = \mathrm{Ext}_{B\otimes B^{op}}(B,B)$ Hochschild cohom.

is fin. gen. and $\forall$ fg B-mod U

$\mathrm{Ext}^*_B(U,U)$ is fin gen as $HH^*(B)$-mod

via $HH^*(B) \xrightarrow[-\otimes_B U]{} \mathrm{Ext}_B^*(U,U)$

(by Evans-Venkov)

<u>II Block Invariants</u>

$\mathcal{O}, k, p, G$ as above

<u>block</u> of $\mathcal{O}G$ $B = \mathcal{O}G \cdot b$ for some

b primitive idemp. in $Z(\mathcal{O}G)$ <u>block idemp.</u>

$b = 1_B$

B is an indec. $\mathcal{O}G$-$\mathcal{O}G$-bimod summand of $\mathcal{O}G$

P finite p-group $\Rightarrow$ $\mathcal{O}P$ indec., local

<u>Principal block</u> unique block $B_0 = B_0(\mathcal{O}G)$
not in kernel of augmentation

$$\left\{ \mathcal{O}G \xrightarrow{n} \mathcal{O} \right.$$

<u>Def.</u> (Brauer 1940s)
A <u>defect group</u> of a block B of $\mathcal{O}G$ is a maximal p-subgroup P s.t. $\mathcal{O}P \mid_{\mathcal{O}P} B_{\mathcal{O}P}$ (isom to dir. sum of)
as $\mathcal{O}P$-$\mathcal{O}P$-bimod

$\Longleftrightarrow$ a min subgroup P s.t. $B \mid B \otimes_{\mathcal{O}P} B$
as B-B-bimod

<u>Basic properties</u>:

- Def. grp form a G-conj class of p-subgrp
- Def. grp of princ. block are Sylow p-subgrp
- If $Q \trianglelefteq G$ p-subgrp $\Rightarrow$ Q contained in any def. grp of any block of $\mathcal{O}G$

<u>Ex.</u> B block of $\mathcal{O}G$ defect grp P

- $P = \{1\}$ $\Longleftrightarrow$ B separable $\mathcal{O}$-alg, (proj as $B \otimes_{\mathcal{O}} B^{op}$-mod)

($\Leftrightarrow$ $k\otimes_{\mathcal{O}} B$ matrix alg. $/k$ if k "large enough")

- P cyclic $k\otimes_{\mathcal{O}} B$ is <u>Brauer tree algebra</u>
  (exactly those of finite rep. type)

- char 2 $P = C_2 \times C_2 \Rightarrow$ B Morita equiv to
  $\mathcal{O}P$, $\mathcal{O}A_4$ or $B_0(\mathcal{O}A_5)$ ($\mathcal{O}=k$ Erdmann '80s)
  alt. grp prine. block alg

- $P = Q_8$ quaternion grp order 8
  $\Rightarrow$ $B \underset{\text{Morita}}{\sim} \mathcal{O}P$, $\mathcal{O}\tilde{A}_4$ or $B_0(\mathcal{O}\tilde{A}_5)$
  non triv. centr. ext. of A_4

$\rightsquigarrow$ "what happens locally, happens globally"

Application: Brauer-Suzuki Thm

<u>Conjecture</u> (Donovan 1970s) $\mathcal{O}=k$

For a fixed finite p-group P there are only fin. many Morita equiv. classes of algebras of block alg. of fin grps with defect grps isom. to P.

true if P cyclic (Janusz, Kupisch '70s)
char $p=2$ (Erdmann '80s)

P elem. abelian 2-group (Eaton Kessar Külshammer Sambale 2014)

uses class. of fin. simple groups

Donovan's conj makes sense over $\mathcal{O} \neq k$

Q: Not known wether Morita equiv class of B of $\mathcal{O}G$ is det. by that of $k \otimes_{\mathcal{O}} B$ of kG

Q: Not known wether $\mathrm{mod}(B)$ determines defect groups

Q: —— " —— " —— " ——
if defect grps are abelian

Dichotomy "global vs local"

Global: B alg str., $\mathrm{mod}(B)$, $\ell(B) = \#$ iso cl of simple $k \otimes_{\mathcal{O}} B$-mod
$D^b(B)$, $\underline{\mathrm{mod}}(B)$
$HH^*(B)$

Local: Defect grp P, fusion system
cohom. information

III Basics on Hochschild cohom HH^*

A fg proj over commut ring $\mathcal{O}$

Gerstenhaber (1960)

$HH^*(A)$ is graded commutative

$HH^{>0}(A)$ is graded Lie algebra of degree -1

Zimmermann (2007)

A over k char $p>0 \Rightarrow HH^{odd}(A)$ p-restricted Lie algebra (p odd)

if $p=2$ $HH^{>0}(A)$

Functoriality:

A, B sym. $\mathcal{O}$-alg, M A–B–bimod fg. proj as left & right mod (but not as bimod)

$M \otimes_B -$, $M^\vee \otimes_A -$ btw. mod A & mod B are biadjoint (left & right adjoint)

Fixing isom btw. $A \cong A^\vee$ and $B \cong B^\vee$ amounts to fixing adjunction isom. determines transfer map

$$tr_M : HH^*(B) \longrightarrow HH^*(A)$$

$$B \xrightarrow{\tau} B[n] \text{ in } D^b(B \otimes B^{op})$$ goes to

$$\begin{tikzcd} A \arrow[r, "\text{adj}", "\lim"'] & M \otimes_B M^\vee \arrow[d, "\mathrm{Id}_M \otimes \mathrm{Id}_{M^\vee}"] \\ & M \otimes_B M^\vee[n] \arrow[d, "\text{adj: colim}"] \\ & A[n] \end{tikzcd}$$

$\mathrm{tr}_M(\tau)$

graded k-lin. but in general neither algebra nor Lie alg. hom.

$$\Delta G = \{(x,x) \mid x \in G\} \subseteq G \times G$$

$\mathrm{Ind}_{\Delta(G)}^{G \times G}(k) = kG$ induces $\delta_G : H^*(G,k) \to HH^*(kG)$

split inj alg. hom., retraction induced by $- \otimes_{kG} k$

$H \leq G$

$$\begin{tikzcd} H^*(H,k) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, hook] & H^*(G,k) \arrow[r, "\mathrm{res}_H^G"] \arrow[d, hook] & H^*(H,k) \arrow[d, hook] \\ HH^*(kH) \arrow[r, "\mathrm{tr}_{kG_{kH}}"'] & HH^*(kG) \arrow[r, "\mathrm{tr}_{{}_{kH}kG}"'] & HH^*(kH) \end{tikzcd}$$

$$\mathrm{tr}_{M \otimes_B N} = \mathrm{tr}_M \circ \mathrm{tr}_N$$

IV HH^0

$HH^0(B) \cong \mathrm{End}_{B \otimes B^{op}}(B) \cong Z(B)$

$K \supseteq \mathcal{O} \longrightarrow k$, char $K = 0$

G fin. grp $\quad KG = \prod_{\chi \in \mathrm{Irr}(G)} KG\,e(\chi)$

$e(\chi)$ primitive idemp. in $Z(KG)$

B block of $\mathcal{O}G$: $\quad K \otimes_{\mathcal{O}} B = \prod KG e(\chi)$

$\chi' \in \mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$

$|\mathrm{Irr}(B)| = \mathrm{rk}_{\mathcal{O}}(Z(B)) = \dim_K(Z(K \otimes_{\mathcal{O}} B))$

<u>Theorem</u> (Brauer-Feit 1959)

B block with P defect grp, order p^d, $d \in \mathbb{N}^0$

Then $|\mathrm{Irr}(B)| \leq \frac{1}{4} p^{2d} + 1$

<u>Theorem</u> (Kessar, Brauer-Feit, Cliff-Plesken-Weiss)

There are only fin. many commut. k-algebras up to isom. arising as $Z(k \otimes_{\mathcal{O}} B)$ of block B with a fixed defect group.

Can classify $Z(k \otimes_{\mathcal{O}} B)$ if

- P cyclic
- Klein four, dihedral, quaternion
- $P = C_2 \times C_2 \times C_2$ class. of fin. simple grps
- $P = C_2 \times C_2 \times C_2 \times C_2$ partial result

$\overline{V}$ HH^1

A algebra, <u>derivation</u> on A is lin. map $f : A \to A$
s.t. $f(ab) = f(a)b + af(b)$

If $c \in A$, then $[c, -]$, $[c,a] = ca - ac$ $(a \in A)$
is a derivation, called <u>inner</u>

$\mathrm{End}_k(A)$ is algebra & Lie algebra with
$\cup\!\shortmid$ $\quad [f,g] = f \circ g - g \circ f$
$\mathrm{Der}(A)$ set of derivations, Lie algebra
$\cup\!\shortmid$
$\mathrm{IDer}(A)$ inner derivations, Lie ideal

$$HH^1(A) \cong \mathrm{Der}(A) / \mathrm{IDer}(A) \quad \text{Lie algebra}$$

$HH^1(A)$ is a module over $Z(A)$.

If k field, $\mathrm{char}\, k = p > 0$, then $HH^1(A)$ is
<u>restricted Lie algebra</u>

via $f^{[p]} = \underbrace{f \circ f \circ \dots \circ f}_{p \text{ times}}$ $\qquad (f \in \mathrm{Der}(A))$

Witt 1930s

$\underset{\text{Witt algebra}}{W_1} = HH^1(k\overset{\text{cyclic p-group}}{C_p})$ $\qquad kC_p \cong k[x]/(x^p)$

$f_i(x) = x^i \qquad 0 \le i \le p-1$

Theorem (Jacobson 1943)

P elementary abelian p-group, $P \cong (C_p)^n$ $n>0$

then $W_n = HH^1(kP)$ simple Lie algebra

Q: Can any other simple Lie algebras in char p show up as $HH^1(B)$ of some block B?

Conjecture: Supp. B block of kG with defect grp $\neq 1$ then $HH^1(B) \neq 0$.

Remark: FJL 1993 $\Rightarrow$ $HH^1(kG) \neq 0$ if $p \mid |G|$

Gerstenhaber, Green (1994) Donald Flanigan conj. if true $\Rightarrow$ $HH^1(kG) \neq 0$

Prop. (Benson-Kessar-L.)

A split fd sym. k-alg.

$$\sum_{S \text{ simple } \mathrm{Mod}} \dim \mathrm{Ext}^1_A(S,S) \leq \dim(\mathrm{soc}_{ZA}(HH^1(A)))$$

Broué's abelian defect conj $\Rightarrow$ $HH^1(B) \neq 0$ if B has nontriv. ab. defect group.

V Integrable Derivations

Gerstenhaber 1964

fd k-alg, Φ autom of $A[[t]] = k[[t]] \otimes_k A$

as $k[[t]]$-alg. s.t. Φ induces Id on A via

$A[[t]] \to A$, $t \to 0$

$\Phi(a) = a + d_1(a)t + d_2(a)t^2 + \dots$

The d_i are endom. of A. Comparing

$\Phi(ab) = \Phi(a)\Phi(b)$ in deg 1 shows

d_i derivation on A; these are called <u>integrable</u>.

$HH^1_{int}(A)$ integrable classes in $HH^1(A)$

<u>Theorem</u> (Farkas-Geiss-Marcos 2002)

$HH^1_{int}(A)$ is invar. under Morita equiv.

<u>Remark</u> If $k = \mathbb{C}$ every derivation is integrable

for $d \in Der(A)$ construct $\Phi(a) = \sum_{n \geq 0} \frac{d^n}{n!} t^n$

$HH^1(A)$ tangent space of $Out(A)$ as alg. grp

not true in char p

$\mathcal{O}$ compl. dvr, $J(\mathcal{O}) = \pi\mathcal{O}$, $k = \mathcal{O}/J(\mathcal{O})$ char p

A alg $/\mathcal{O}$ free of fin. rank $/\mathcal{O}$

$A/\pi A \cong k \otimes_{\mathcal{O}} A = \bar{A}$ fd k-alg

$A/\pi^n A$, $n>0$ free $\mathcal{O}/\pi^n\mathcal{O}$

$\mathrm{Aut}_n(A) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha$ induces Id on $A/\pi^n A\}$

$\mathrm{Out}_n(A) =$ image of $\mathrm{Aut}_n(A)$ in $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}A$

<u>Prop</u>: A, supp. $Z(A) \to Z(A/\pi^n A)$ surj.

Let $\alpha \in \mathrm{Aut}_n(A)$. Then $\exists$ $d: A \to A$ s.t.

$\alpha(a) = a + \pi^n d(a)$

(1) The map on $A/\pi^n A$ induced by d is derivation

(2) corresp. $\alpha \leftrightarrow d$ induces grp hom $\mathrm{Out}_n(A) \to HH^1(A/\pi^n A)$

with kernel $\mathrm{Out}_{2n}(A)$

<u>Def</u>: derivation on $A/\pi^n A$ or its class in $HH^1(A/\pi^n A)$

<u>integrable</u> if image of grp hom as in Prop.

$HH^1_A(A/\pi^n A)$ set of A-int. classes.

The canonical map $A/\pi^n A \to A/\pi A$ induces

$HH^1(A/\pi^n A) \to HH^1(\bar{A})$

$HH^1_n(\bar{A}) =$ image of $HH^1_A(A/\pi^n A)$ n-integrable.

<u>Def</u> (Broué ~1990) A, B $\mathcal{O}$-alg, M A-B-bimod, N B-A-bimod

fg proj as left & right mod

M, N induce stable equiv. of Morita type if

$M \otimes_B N \cong A \oplus$ proj $A \otimes_{\mathcal{O}} A^{op}$-mod

$N \otimes_A M \cong B \oplus$ proj $B \otimes_{\mathcal{O}} B^{op}$-mod.

Examples:

- $p=2$ $\quad \mathcal{O}A_5 \supseteq \mathcal{O}A_4 \quad M = \mathcal{O}A_5|_{\mathcal{O}A_4}$

$$N = {}_{\mathcal{O}A_4}|\mathcal{O}A_5$$

- $P \in Syl_p(G)$ cyclic, H normalizes unique subgp of order p then $\mathcal{O}G \supseteq \mathcal{O}H$, $M = \mathcal{O}G_{\mathcal{O}H}$, $N = {}_{\mathcal{O}H}\mathcal{O}G$

Theorem (L, 2015)

A, B $\mathcal{O}$-alg, $\bar{A}$ and $\bar{B}$ split self inj

Supp. $Z(A) \to Z(\bar{A})$, $Z(B) \to Z(\bar{B})$ surj

and M, N bimod ind. st. equiv. Mor. type.

Then $HH^1_A(A/\pi^n A) \cong HH^1_B(B/\pi^n B)$ ind. $N \otimes_A - \otimes_A M$

$$\begin{tikzcd}
HH^1_A(A/\pi^n A) \arrow[r, phantom, "\cong"] & HH^1_B(B/\pi^n B) \\
Out_n(A) \arrow[u] \arrow[r, phantom, "\cong"] & Out_n(B) \arrow[u]
\end{tikzcd}$$

Theorem (Rubio, 2015)

A, B fd self inj, M, N as above. Then $[p]$ sends

$HH^1_n(A)$ to $HH^1_{np}(A)$ and

$$\begin{tikzcd}
HH^1_n(A) \arrow[r, phantom, "\cong"] \arrow[d, "{[p]}"'] & HH^1_n(B) \arrow[d, "{[p]}"] \\
HH^1_{np}(A) \arrow[r, phantom, "\cong"] & HH^1_{np}(B)
\end{tikzcd}$$