18/08/16

Hall polynomials for tame type

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I Motivation

k fin. field, $|k| = q$

$\mathcal{A}$ fin. hereditary abelian cat. over k

$\mathcal{X}$ $\quad Ob(\mathcal{A})/\cong$

Ringel-Hall algebra $H(\mathcal{A}) = \bigoplus_{M \in \mathcal{X}} \mathbb{C}\, u_{[M]}$

$$u_{[M]} u_{[N]} = q^{\frac{1}{2}\langle M, N\rangle} \sum_{R \in \mathcal{X}} F_{M,N}^{R}\, u_{[R]}$$

$$F_{MN}^{R} = \#\{0 \to N \to R \to M \to 0\} / |\mathrm{Aut}\, M \times \mathrm{Aut}\, N|$$

If $\mathcal{A} = \mathrm{mod}\, kQ$, Q quiver

n^+ = pos. part of Lie alg. assoc. to Q

$U_v(n^+) \xrightarrow[\text{Green}]{\text{Ringel}} H(\mathcal{A})$

specialized at $v^2 = q$

If Q Dynkin then the Hall polynomial exists.

ie for any $M, N, R \in \mathcal{X}$ $\exists$ $\varphi_{M,N}^{R} \in \mathbb{Z}[t]$

s.t. $F_{M,N}^{R} = \varphi_{MN}^{R}(q)$

$\Rightarrow$ can def. generic Ringel-Hall alg.

$$\underline{H}(\mathcal{A}) = \bigoplus_{M \in \mathcal{X}} \mathbb{C}[t, t^{-1}]\, u_{[M]}$$

$$\Rightarrow \quad U_v(n^+) \cong \underline{H}(A)$$

$$\xrightarrow[t=1]{\text{degen.}} \quad U(n^+) \cong \underline{H}(A)_{t=1}$$

$$\Rightarrow n^+ = (\{u_{[M]} \cdot M \in \operatorname{ind} A\}, [\,,\,]) \subseteq \underline{H}(A)_{t=1}.$$

Question:

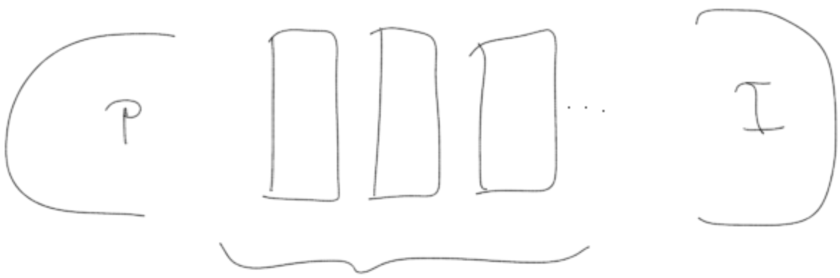
1) Can we extend above result to tame quivers?
2) Can we give general realization of $U_v(n^+)$ or n^+ for affine case?
3) Can we show that Hall polys $\exists$ for tame quivers?

II Main ideas

	reps of Dynkin quiver	reps of tame quivers
iso classes of obj	$X \overset{1:1}{\longleftrightarrow} \{f : \Phi^+ \to \mathbb{N}_0\}$	Segre seq. / decomp. seq
ext. of module	indep. of field	

Recall: Q tame quiver

mod kQ



regular

Q isoclasses of modules

P & I	$\rightsquigarrow$	real Schur root
nonhomo. tubes	$\rightsquigarrow$	multipartition
homo. tubes	$\rightsquigarrow$	partition + closed pt
		$(\lambda, z) \longrightarrow (\lambda, d)$
		dep. on k

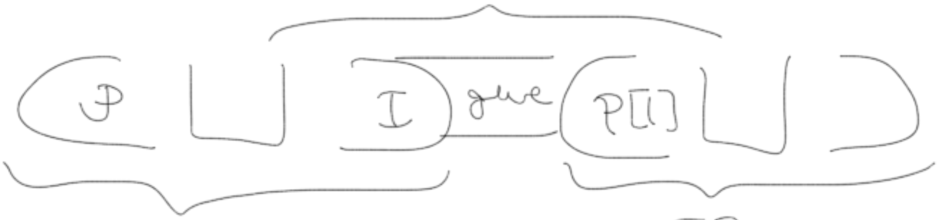
Segre seq. / decomp seq.

Extensions:

$$0 \to P \to \boxed{///} \longrightarrow I \to 0$$

dep. on k!

$\mathcal{A}' = \mathrm{coh}\,\mathbb{X}$



$\mathcal{A} = kQ\text{-mod}$ $\qquad$ $\mathcal{A}[1]$

$$\rightsquigarrow \quad D^b(\mathrm{mod}\,kQ) \cong D^b(\mathrm{coh}\,\mathbb{X})$$

$H(\mathrm{mod}\,kQ) \qquad H(\mathrm{coh}\,\mathbb{X})$

preserve generic form

$DH(\mathrm{mod}\, kQ)$ $\qquad$ $DH(\mathrm{coh}\, \mathbb{X})$

Drinfeld doubles

II Main results

(1) Weighted proj. line $\mathbb{X}$ of domestic (Fano) type

$p = (p_1, p_2, p_3)$ $\quad p_i \geq 1$

$$\mathbb{L} = \mathbb{L}(p) = \mathbb{Z}(\vec{x}_1, \vec{x}_2, \vec{x}_3) / (p_1\vec{x}_1 = p_2\vec{x}_2 = p_3 x_3)$$

$$S = k[X_1, X_2, X_3] / (X_1^{k} + X_2^{u} + X_3^{k})$$ $\mathbb{L}$-graded $\deg X_i = \vec{x}_i$

$$\mathrm{coh}\, \mathbb{X} = \mathrm{mod}^{\mathbb{L}} S / \mathrm{mod}_0^{\mathbb{L}} S$$

where $\mathbb{X} = \mathrm{Spec}^{\mathbb{L}} S$ weighted proj. line.

(2) Hall poly for $\mathrm{coh}\, \mathbb{X}$

<u>Def</u>: • A <u>Segre seq</u> is a seq. $\lambda = ((\lambda_1, d_1), \dots, (\lambda_n, d_n))$
λ_i partition, $d_i \in \mathbb{Z}$ $\quad 1 \leq d_1 \leq d_2 \leq \dots \leq d_n$

• A <u>decomp. seq</u> is $\tilde{\lambda} = (\alpha, \lambda)$ where α specifies coh. sheaf without homog. summands.
λ is Segre seq.

Remarks:

(1) Such seq. $\alpha, \tilde{\alpha}$ are called of type $\underline{d} = (d_1, d_2, \dots, d_n)$

(2) Denote by $X_k(\underline{d}) = \left\{ \begin{array}{l} \underline{z} = (z_1, \dots, z_n) \text{ of pw dist. "ord."} \\ \text{pts with } \deg(z_i) = d_i \end{array} \right\}$

Define $S_k(\alpha, \underline{z}) = \bigoplus_{1 \le i \le n} S_k(\alpha_i, z_i)$

$S_k(\tilde{\alpha}, \underline{z}) = S_k(\alpha) \oplus S_k(\alpha, \underline{z})$

(3) Hall polys exist wrt decomp. seq. in coh $\mathbb{X}$ if
$\forall$ decomp. seq $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ of type $\underline{d}$
$\exists$ poly $\varphi^{\tilde{\gamma}}_{\tilde{\alpha}\tilde{\beta}} \in \mathbb{Z}[t]$ s.t. $\forall$ k field, $|k| = q >> 0$

$$F^{S_k(\tilde{\gamma}, \underline{z})}_{S_k(\tilde{\alpha}, \underline{z}), S_k(\tilde{\beta}, \underline{z})} = \varphi^{\tilde{\gamma}}_{\tilde{\alpha}, \tilde{\beta}}(q) \quad \forall \underline{z} \in X_k(\underline{d}).$$

Theorem Hall poly's exist wrt decomp. seq. in coh $\mathbb{X}$.

proof: 1) associativity
2) Green's formula
3) By induction on rank & degree

(3) Hall poly's for mod kQ
can def. decomp. seq. $\tilde{\alpha} = (\alpha, \alpha)$ by replacing α
to specify a mod. with homo summands.
& Hall polys.

Theorem: Hall poly's exist wrt decomp seq. for
tame quivers

pf: generic hall alg. of coh $\mathbb{X}$.
& PBW basis

formulate Drinfeld double version
& Cramer's isom. $DH(\mathrm{coh}\, X) \cong DH(\mathrm{mod}\, kQ)$

Corollary:

For $M, N, R \in \mathrm{mod}\, R$ there exists a polynomial $\varphi_{MN}^{R} \in \mathbb{Z}[t]$ s.t. $\forall$ conservative field ext. K of k (wrt M, N, R)

$$F_{M^K N^K}^{R^K} = \varphi_{MN}^{R}(|K|).$$

IV Compare with Hubery's result

[BDR] [Hubery]

Segre seq.	Segre symbol
$\lambda = ((\lambda_1, d_1), \dots, (\lambda_n, d_n))$	$\lambda' = \{(\lambda_1, d_1), \dots, (\lambda_n, d_n)\}$
decomp. seq.	decomp. symbol
$\tilde{\alpha} = (\alpha, \lambda)$	$\tilde{\alpha}' = (\alpha, \lambda')$
rep. $S_K(\tilde{\alpha}, \underline{z})$ "mod"	decomp. class
	$S(\tilde{\alpha}, k) = \{S(\tilde{\alpha}, \underline{z}) \mid \underline{z} \in X_k(\underline{d})\}$

Theorem (Hubery)

Hall poly's exist wrt decomp. classes

i.e. $\forall$ decomp. classes $\tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}'$ $\exists\, \varphi_{\tilde{\alpha}' \tilde{\beta}'}^{\tilde{\gamma}'} \in \mathbb{Q}[t]$

s.t. $\forall$ field k with $|k| = q$

$$\varphi_{\tilde{\alpha}' \tilde{\beta}'}^{\tilde{\gamma}'}(q) = \sum_{\substack{A \in S(\tilde{\alpha}', k) \\ B \in S(\tilde{\beta}', k)}} F_{AB}^{C} \quad \forall\, C \in S(\tilde{\gamma}', k)$$

Remark: Thm 2 $\Rightarrow$ Thm(Hubery) "refined"

In fact
$$\varphi^{\tilde{\gamma}'}_{\tilde{\alpha}',\tilde{\beta}'}(t) = \sum_{\substack{\alpha \sim \tilde{\alpha} \\ \beta \sim \tilde{\beta}}} \varphi^{\tilde{\gamma}}_{\tilde{\alpha}\tilde{\beta}} \underset{\mathbb{Q}(t)}{\underset{\cap}{n_{\tilde{\alpha}\hat{\beta}}}}$$

Exp: $\deg d = 1$, $\lambda = ((\lambda_1, d_1), \ldots, (\lambda_n, d_n))$
$= (\lambda_1, \lambda_2 \cdots)$

Fix Segre symbols

$\rho = \{(1,1), (1,1,1), (2,1)\}$

$\sigma = \{(1), (1)\}$

$\tau = \{(1,1,1), (2,1,1), (2,1)\}$

Segre seq.

$\rho_1 = ((1,1), (1,1,1), (2,1))$

$\sigma_1 = ((1), (1), \emptyset\}$

$\tau_1 = ((1,1,1), (2,1,1), (2,1))$

$$\varphi^{\tau_1}_{\rho_1,\sigma_1} = \varphi^{(1,1,1)}_{(1)(1)} \varphi^{(2,1,1)}_{(1,1,1),(1)} \varphi^{(2,1)}_{(2,1)\emptyset}$$

$$= q^2 + q + 1$$