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Noetherian properties in Rep. Theory

Example 1:

X "nice" manifold of dim ≤ 2

eg cpt, conn., orientable

For $n \geq 0$ let $PConf_n X = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \ i \neq j\}$

Theorem (Church, CEFN, Nagpal)

fix field $\mathbb{F}$

1) $\forall i \geq 0$ the function $n \to \dim_{\mathbb{F}} H^i(PConf_n X, \mathbb{F})$

is a polynomial fct for $n \gg 0$

2) If char $\mathbb{F} = 0$ $n \to \dim_{\mathbb{F}} H^i(PConf_n X / S_n, \mathbb{F})$

is eventually constant.

3) If char $\mathbb{F} > 0$ $n \to \dim_{\mathbb{F}} H^i(PConf_n X / S_n, \mathbb{F})$

is eventually periodic.

Example 2

X proj. var. $\subseteq \mathbb{P}^n$, over field of char 0.

two constructions

1) dth Veronese embedding $X \hookrightarrow \mathbb{P}^n \xhookrightarrow{\mathcal{O}(d)} \mathbb{P}^N$

$(x_1 : \dots : x_n) \mapsto$ [all monom of deg d]

2) Given $X, Y \subseteq \mathbb{P}^n$, $\mathrm{Join}(X,Y) = \{x+y \mid x \in X, y \in Y\}$

<u>secant varieties</u>: $\mathrm{Sec}^1 X = X$

$\mathrm{Sec}^r X = \mathrm{Join}(X, \mathrm{Sec}^{r-1} X)$

"<u>Meta-Theorem</u>": algebraic properties of varieties "improve" as you Veronese re-embed them. ($\rightsquigarrow V_d(X)$)

<u>Theorem</u> (Sam)

Fix X, $r>0$ there exists constant $C_X(r)$ s.t. the ideal defining $\mathrm{Sec}^r(V_d(X))$ is generated in degs $\leq C_X(r)$

(main pt: indep. of d)

Inspiration: prev. work with Draisma-Kuttler on secant varieties of Segre-embed.

<u>Common features</u>:

Possible framework: twisted commut alg.

$\rightsquigarrow$ tcals

<u>Def</u>. Let $A = \bigoplus_{n \geq 0} A_n$ graded assoc. unital alg. s.t. each A_n has an action of S_n sym. grp.

and $\mu: A_n \otimes A_m \longrightarrow A_{n+m}$ mult. is equivariant for $S_n \times S_m \subset S_{n+m}$

and $\tau_{n,m}(xy) = yx$ for $x \in A_n, y \in A_m$

where $\tau_{n,m} \in S_{n+m}$ is

$$\begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n+m \\ n+1 & n+2 & & n+m & 1 & 2 & \cdots & n \end{pmatrix}$$

<u>Example</u> k field $E \cong k^d$

$A_n = E^{\otimes n}$ action of S_n perm tensor factors

μ is concatenation of tensors. (usually)

$\rightsquigarrow$ twisted commut

denote by $\mathrm{Sym}(E\langle 1\rangle)$

("gen. in deg 1")

<u>Modules</u>: $M = \bigoplus_{n \geq 0} M_n$, $S_n \circlearrowright M_n$

ass. $\mu: A_n \otimes M_m \to M_{n+m}$ is $S_n \times S_m$ equiv.

<u>Def</u>: M is <u>fin. gen.</u> if $\exists$ $x_1, \ldots, x_r$ gen. M using A-module str., S_n-actions.

<u>Models for module cats</u>:

def cat. FI_d : obj fin. sets

morph $S \longrightarrow T$ consists of

1) injection $f: S \hookrightarrow T$

2) "coloring" $g: T \setminus f(S) \longrightarrow \{1, \dots, d\}$

Cat. of $\mathrm{Sym}(E\langle 1\rangle)$-modules is equiv. to cat. of functors $FI_d \longrightarrow k\text{-mod}$.

<u>Theorem</u> (CEFN, Snowden, SS)

k noeth.

1) Submodules of fin gen. FI_d-modules are again f.g.
(ie FI_d-modules are "locally noeth.")

2) k field, M fg FI_1-module then the function
$n \longrightarrow \dim_k M(\{1, \dots, n\})$ is eventually polynomial.

Back to Example 1:

X top. space, S fin set

$\mathrm{PConf}_n X$ = space of injections $S \hookrightarrow X$

gives contravar. functor $FI_1 \longrightarrow \mathrm{Top}$

$\rightsquigarrow$ for each i get FI_1-module

$$S \longrightarrow H^i(\mathrm{PConf}_S(X); k).$$

<u>Example 2</u>: more involved, uses cat. containing
FI_d for ∞-many d.

↳ extends to more gen. commut. algs

↳ get results on syzygies

Some open things:

Define $A_n = \langle \text{set of perfect matchings of } \{1 \cdots n\} \rangle_k$

$\circlearrowleft$

S_n can mult. by concatenating matchings

(tca gen. in deg 2)

Thm (Nagpal-Snowden)

A-modules are locally Noeth. if $\mathbb{Q} \subset k$.

Question: Is this true more generally?

Even $k = \mathbb{F}_p$?