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Ryo Takahashi (Nagoya University): Thick tensor ideals of right bounded derived categories of commutative rings

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§1 Tensor triangulated cats & Balmer spectra

§2 (Co) compactly gen. thick tensor ideals of $D^-(\text{mod } R)$

§3 Balmer spectrum of $D^-(\text{mod } R)$ and classification of thick tensor ideals

§4 The case of discrete valuation rings

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§1

Def. 1.1 A tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1})$ is a triangulated category $\mathcal{T}$ with sym. tensor product $\otimes$ and unit $\mathbb{1}$.

Example 1.2:

1. $(D^{perf}(X), \overset{L}{\underset{\mathcal{O}_X}{\otimes}}, \mathcal{O}_X)$ for scheme X is t.t. cat
2. $(K^b(\text{proj } R), \otimes_R, R)$ for commut. ring R is tt cat

③ $(\underline{\mathrm{mod}}\, kG, \otimes_k, k)$ for field k, G finite group (scheme) is tt cat.

④ $(D^b(\mathrm{mod}\, G), \otimes_k, k)$ —— " ——

⑤ $(D^-(\mathrm{mod}\, R), \otimes^L_R, R)$ for commut. Noeth. R is tt cat

these are all essentially small

<u>Def. 1.3</u>: Let $\mathcal{T}$ be an essentially small tt cat then ① A thick subcat $\mathcal{I}$ of $\mathcal{T}$ is a <u>(tensor) ideal</u> if $a \in \mathcal{T}, x \in \mathcal{I}$ then $a \otimes x \in \mathcal{I}$

② An ideal $\mathcal{I}$ of $\mathcal{T}$ is <u>radical</u> if $\mathcal{I} = \sqrt{\mathcal{I}}$, here

$$\sqrt{\mathcal{I}} = \{ a \in \mathcal{T} \mid \underbrace{a \otimes \cdots \otimes a}_{\exists n > 0} \in \mathcal{I} \}$$

③ A proper ideal $\mathcal{P}$ of $\mathcal{T}$ is <u>prime</u> if $x \otimes y \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$

④ The <u>Balmer spectrum</u> of $\mathcal{T}$ is

$\mathrm{Spc}\,\mathcal{T} = \{$ prime ideals of $\mathcal{T} \}$

$(\longleftrightarrow \mathrm{Spec}\, R$, R commut. ring$)$

⑤ The <u>Balmer support</u> of $x \in \mathcal{T}$ is

$$\operatorname{supp}(x) = \{ \mathcal{P} \in \operatorname{Spc} \mathcal{T} \mid x \notin \mathcal{P} \}$$

($\leftrightarrow$ $V(x) = \{ p \in \operatorname{Spec} R \mid x \in p \}$ analogue for commut. rings)

Put $U(x) = \operatorname{supp}(x)^c = \{ \mathcal{P} \in \operatorname{Spc} \mathcal{T} \mid x \in \mathcal{P} \}$

($\leftrightarrow$ analogue $D(x) = \{ p \in \operatorname{Spec} R \mid p \not\ni x \}$)

<u>Prop. 1.4:</u>

① $\operatorname{Spc} \mathcal{T}$ is a topological space with open basis $\{ U(x) \}_{x \in \mathcal{T}}$

② Every proper ideal of $\mathcal{T}$ is contained in a maximal ideal

③ All maximal ideals are prime

④ Every prime ideal contains a minimal prime

⑤ For each $\mathcal{P} \in \operatorname{Spc} \mathcal{T}$, $\overline{\{ \mathcal{P} \}} = \{ \mathcal{Q} \in \operatorname{Spc} \mathcal{T} \mid \mathcal{Q} \subseteq \mathcal{P} \}$ irred. ($\leftrightarrow$ $\overline{\{p\}} = V(p)$ analogue)

Any nonempty irred. closed subset has to be of this form

⑥ Each $U(x)$ is quasi-compact for $x \in \mathcal{T}$
Any quasi compact open subset is of this form

⑦ For each ideal $\mathcal{I} \subseteq \mathcal{T}$

$$\sqrt{\mathcal{I}} = \bigcap_{\substack{\mathcal{P} \supseteq \mathcal{I} \\ \mathcal{P} \in \operatorname{Spc} \mathcal{T}}} \mathcal{P}$$

For $\mathcal{X} \subseteq \mathcal{T}$ and $S \subseteq \operatorname{Spc} \mathcal{T}$, set

$$\operatorname{Spp} \mathcal{X} = \bigcup_{x \in \mathcal{X}} \operatorname{Spp}(x)$$

$$\operatorname{Spp}^{-1} S = \{x \in \mathcal{T} \mid \operatorname{Spp}(x) \subseteq S\} \text{ full subcat.}$$

<u>Theorem 1.5</u> (Balmer 2005)

$$\left\{ \begin{matrix} \text{Radical ideals} \\ \text{of } \mathcal{T} \end{matrix} \right\} \underset{\operatorname{Spp}}{\overset{\operatorname{Spp}^{-1}}{\overset{1:1}{\longleftrightarrow}}} \left\{ \begin{matrix} \text{Thomason subset} \\ \text{of } \operatorname{Spc} \mathcal{T} \end{matrix} \right\}$$

Here, a subset A of a topological space X is <u>Thomason</u> if $A = \bigcup_i B_i$ where

$B_i^c = X \setminus B_i$ is quasi-compact open subset.

In part. A is specialization closed.

Theorem 1.6 (Balmer 2005 & 2010)

1. X (quasi-cpt quasi-sep.) scheme, then
$$\mathrm{Spc}\ \mathcal{D}^{perf}(X) \cong X$$

2. k field, G fin. group (scheme), then
$$\mathrm{Spc}\ \mathcal{D}^b(\mathrm{mod}\ kG) \cong \mathrm{Spec}^h\ H^\bullet(G,k)$$
$$\mathrm{Spc}(\underline{\mathrm{mod}}\ kG) \cong \mathrm{Proj}\ H^\bullet(G,k)$$

$(\mathcal{T}, \otimes, \mathbb{1})$ tensor triang.

Balmer constr. a continuous map
$$\rho^\bullet_{\mathcal{T}} : \mathrm{Spc}\ \mathcal{T} \longrightarrow \mathrm{Spec}^h R^\bullet_{\mathcal{T}}$$
$$R^\bullet_{\mathcal{T}} = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^\bullet \mathbb{1}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^n \mathbb{1})$$

The isom in Thm 1.6 are given by $\rho^\bullet_{\mathcal{T}}$

Conjecture 1.7 (Balmer, ICM 2010)

$\rho^\bullet_{\mathcal{T}}$ is locally injective if $\mathcal{T}$ is algebraic triangulated cat.

<u>Notation 1.8</u> • R is commut. Noeth. ring

• $\begin{cases} \operatorname{mod} R = \text{cat of fin. gen. } R\text{-modules} \\ \operatorname{Proj} R = \text{(full) subcat of } \operatorname{mod} R \text{ of proj. modules} \end{cases}$

• $\begin{cases} D^*(R) = D^*(\operatorname{mod} R) \quad \text{for } * \in \{-, b\} \\ K^*(R) = K^*(\operatorname{proj} R) \end{cases}$

<u>Difficulties for $D^-(R)$</u>

1. $D^-(R)$ does not have arbitrary $\bigoplus$, $\prod$.
2. $D^-(R)$ not closed under duals.
3. $D^-(R)$ is never rigid.

$\mathcal{T}$ tt cat is <u>rigid</u>, if $\exists$ $D : \mathcal{T} \to \mathcal{T}^{op}$ exact and $\exists$ $\operatorname{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \operatorname{Hom}_{\mathcal{T}}(a, D(b) \otimes c)$

4. $\operatorname{thick}_{D^-(R)} R \neq D^-(R)$ (thick closure)

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§2 (co)compactly gen. ideals of $D^-(R)$

Correction

$\mathrm{Spc}\, D^{perf}(X) \cong X$

- Not nec. given by $\rho^{\circ}_{D^{perf}(X)}$
- Given by $\rho^{\circ}_{D^{perf}(X)}$ if X is aff or proj

<u>Notation</u>: R commut. noeth. ring

$D^*(R) = D^*(\mathrm{mod}\, R)$

$K^*(R) = K^*(\mathrm{proj}\, R)$ for $* \in \{-, b\}$

<u>Def 2.1</u> $\mathcal{T}$ a triangulated cat

(1) $M \in \mathcal{T}$ <u>compact</u> (resp. cocompact) if

$$\bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{T}}(M, N_\lambda) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(M, \bigoplus_{\lambda \in \Lambda} N_\lambda)$$

for all $\{N_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}$ with $\bigoplus_{\lambda \in \Lambda} N_\lambda \in \mathcal{T}$

(resp. $\bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{T}}(N_\lambda, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(\prod_{\lambda \in \Lambda} N_\lambda, M)$

with $\prod_{\lambda \in \Lambda} N_\lambda \in \mathcal{T}$)

(2) $\mathcal{T}^c = \{$compact obj of $\mathcal{T}\}$

$\mathcal{T}^{cc} = \{$cocompact obj. of $\mathcal{T}\}$

(3) I ideal of $\mathcal{T}$ is <u>compactly gen.</u> (resp. <u>cocomp'ly gen.</u>) if gen. by compact resp. cocomp. objects

$\rightsquigarrow$ $I = \mathrm{thick}^{\otimes}_{\mathcal{T}} \mathcal{C}$ for some $\mathcal{C} \subseteq \mathcal{T}^c$
resp. $\mathcal{C} \subseteq \mathcal{T}^{cc}$

<u>Fact 2.2:</u>

(1) $D^-(R)^c = K^b(R)$ [who?]

(2) $D^-(R)^{cc} = D^b(R)$ [Oppermann-Stovicek 2012]

<u>Def. 2.3</u>

(1) The <u>support</u> of $X \in D^-(R)$ is

$$\mathrm{Supp}\, X = \bigcup_{i \in \mathbb{Z}} \mathrm{Supp}_R H^i(X)$$

$$= \{ p \in \mathrm{Spec}\, R \mid X_p \neq 0 \}$$

$$\boxed{=} \{ p \in \mathrm{Spec}\, R \mid \kappa(p) \otimes^L_R X \neq 0 \}$$

For $\mathcal{X} \subseteq D^-(R)$ set

$$\operatorname{Supp} \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp} X$$

(2) For $S \subseteq \operatorname{Spec} R$ set

$$\langle S \rangle = \operatorname{thick}^{\otimes}_{D^-(R)} \{ R/p \mid p \in S \}$$

<u>Theorem 2.4</u>

$$\left\{ \begin{matrix} \text{cocompactly gen.} \\ \text{ideals of } D^-(R) \end{matrix} \right\} \underset{\operatorname{Supp}}{\overset{\langle - \rangle}{\overset{1:1}{\longleftrightarrow}}} \left\{ \begin{matrix} \text{Specialization-closed} \\ \text{subsets of } \operatorname{Spec} R \end{matrix} \right\}$$

<u>Lemma 2.5</u> (Generalized Smash Nilpotence)

$f : X \longrightarrow Y$ in $K^-(R)$ with $Y \in K^b(R)$, if $f \otimes_R K(p) = 0$ $\forall\, p \in \operatorname{Spec} R$, then $\exists\, t \in \mathbb{Z}$ s.t. $f^{\otimes t} = 0$.

<u>Remark 2.6</u>

(1) Hopkins-Neeman show this for $X \in K^b(R)$, where one can reduce to the case $X = R$ which plays a key role.

(2) Show and use:

(a) $f \otimes \mathcal{X} = 0$ and $g \otimes \mathcal{Y} = 0$

$$\Rightarrow (f \otimes g) \otimes (\mathcal{X} * \mathcal{Y}) = 0$$

where $\mathcal{X} * \mathcal{Y}$ are extensions of $\mathcal{X}$ and $\mathcal{Y}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \boxed{E} \arrow[r] & Y \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \Sigma X \\ \mathcal{X} & & \mathcal{Y} & \end{tikzcd}$$

(b) $\underline{x} = x_1, \ldots, x_n \subseteq R$, $f \otimes R/(\underline{x}) = 0$

$$\Rightarrow f^{\otimes 2n} \otimes K(\underline{x}) = 0$$

$\hookrightarrow$ Kozul complex

③ Need $Y \in K^b(R)$ to have $\operatorname{ann}_{R_p}(f_p) = (\operatorname{ann}_R(f))_p$ for $p \in \operatorname{Spec} R$

<u>Proposition 2.7</u>

Let $X, Y \in D^-(R)$ $(\cong K^-(R))$. If $V(\operatorname{ann} X) \subseteq \operatorname{Supp} Y$ Then $X \in \operatorname{thick}^{\otimes} Y$.

<u>Remark 2.8</u>

① If $X \in D^b(R)$, then $V(\operatorname{ann} X) = \operatorname{Supp} X$

② Original: $X, Y \in K^b(R)$ with $\operatorname{Supp} X \subseteq \operatorname{Supp} Y$ then $X \in \operatorname{thick}^{\otimes} Y$.

③ Prop. 2.7 does not hold if $V(\operatorname{ann} X)$ is replaced by $\operatorname{Supp} X$.

④ Proof: $\exists\, Y' \in K^b(R)$ a truncation of Y s.t. $V(\operatorname{ann} X) \subseteq \operatorname{Supp} Y'$

Consider $R \longrightarrow \mathrm{Hom}_R(Y', Y)$
$\quad\ \ \rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$
$\quad\ 1 \longmapsto \text{inc.} \qquad$ (inclusion)

5 Can replace $Y \in D^-(R)$ by $\mathcal{Y} \subseteq D^-(R)$ subset.

Corollary 2.9

1 For $X \in D^-(R)$
$\mathrm{Supp}\, X = \mathrm{Spec}\, R \iff \mathrm{thick}^{\otimes} X = D^-(R)$

2 Let $I = (x) \subseteq R$ and $\mathcal{X} \subseteq D^-(R)$ be ideals
TFAE
(i) $V(I) \subseteq \mathrm{Supp}\, \mathcal{X}$
(ii) $R/I \in \mathcal{X}$
(iii) $K(x) \in \mathcal{X}$

Proof of Thm 2.4

Let $\mathcal{X}$ be cocompactly gen. ideal of $D^-(R)$
$\hookrightarrow \mathcal{X} = \mathrm{thick}^{\otimes} \mathcal{C}$ for some $\mathcal{C} \subseteq D^b(R)$

Wts: $\mathcal{X} = \langle \mathrm{Supp}\, \mathcal{X} \rangle$
"$\supseteq$" corollary 2.9. implies $R/p \in \mathcal{X}$ $\forall\, p \in \mathrm{Supp}\, \mathcal{X}$
"$\subseteq$" ets: $\mathcal{C} \subseteq \langle \mathrm{Supp}\, \mathcal{X} \rangle = \langle \mathrm{Supp}\, \mathcal{C} \rangle$
$M \in \mathcal{C} \Rightarrow M \in \mathrm{thick}\{R/p \mid p \in \mathrm{Supp}\, M\}$

$\square$

Corollary 2.10

TFAE for an ideal $\mathcal{X} \subseteq D^-(R)$

① $\mathcal{X}$ is compactly gen.

② $\mathcal{X}$ is cocompactly gen.

When this is the case, we call $\mathcal{X}$ compact.

Proof:

① $\Rightarrow$ ② ✓

② $\Rightarrow$ ① $W \subseteq \operatorname{Spec} R$ specialization-closed

$$\begin{cases} \mathcal{A} = \operatorname{thick}^{\otimes}\{R/p \mid p \in W\} & \text{cocomp.'ly gen.} \\ \mathcal{B} = \operatorname{thick}^{\otimes}\{k(p) \mid p \in W\} & \text{comp. gen.} \end{cases}$$

$\overset{\text{Thm 2.4}}{\Longrightarrow}$ $\mathcal{A} = \mathcal{B}$ as $\operatorname{Supp} \mathcal{A} = \operatorname{Supp} \mathcal{B}$ $\square$

Corollary 2.11

If R artinian. Then all ideals of $D^-(R)$ are compact

$$\therefore \left\{ \text{Ideals of } D^-(R) \right\} \underset{\longleftarrow}{\overset{\operatorname{Supp}}{\underset{1:1}{\longrightarrow}}} \left\{ \text{subsets of } \operatorname{Spec} R \right\}$$

§ 3 Spc $D^-(R)$ & classification

3.1 Structure of Spc $D^-(R)$

Def 3.1

① For $S \subseteq \operatorname{Spec} R$ set

$$\operatorname{Supp}^{-1} S = \{X \in D^-(R) \mid \operatorname{Supp} X \subseteq S\}$$

② An ideal $\mathcal{X}$ of $D^-(R)$ is tame if

$$\mathcal{X} = \operatorname{Supp}^{-1} S \text{ for some } S \subseteq \operatorname{Spec} R$$

Set $\ {}^t\operatorname{Spc} D^-(R) = \{\text{tame primes of } D^-(R)\}$

Prop. 3.2

① For $p \in \operatorname{Spec} R$

$$\mathcal{S}(p) = \{X \in D^-(R) \mid X_p = 0\}$$

is a prime ideal of $D^-(R)$

② For $P \in \operatorname{Spc} D^-(R)$

$$\{I \subseteq R \mid R/I \notin P\}$$

has a unique max. element $\mathfrak{s}(P)$, which is a prime ideal of R

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, shift left, "\mathcal{S}"] & \operatorname{Spc} D^-(R) \arrow[l, shift left, "\mathfrak{s}"] \end{tikzcd}$$

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<u>Recall</u>

$$\operatorname{Spec} R \underset{\rho}{\overset{\mathcal{S}}{\rightleftarrows}} \operatorname{Spc} D^-(R)$$

$$\mathfrak{p} \longmapsto \mathcal{S}(\mathfrak{p}) = \{ x \in D^-(R) \mid x_{\mathfrak{p}} = 0 \}$$

$$\max\{ I \subseteq R \mid R/I \notin \mathcal{P} \} \longleftarrow \mathcal{P}$$

<u>Theorem</u>

(1) $\mathcal{S}$ and ρ are order-reversing maps s.t.

$$\begin{cases} \rho \circ \mathcal{S} = 1 \\ \mathcal{S} \circ \rho = \operatorname{Supp}^{-1}\operatorname{Supp} \end{cases}$$

$\therefore \dim(\operatorname{Spc} D^-(R)) \geq \dim R$

(2)
$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\mathcal{S}"] \arrow[dr, "\mathcal{S}'"'] & \operatorname{Spc} D^-(R) \arrow[r, "\rho"] & \operatorname{Spec} R \\
& {}^t\operatorname{Spc} D^-(R) \arrow[u, phantom, "\cup"] \arrow[ur, "\rho'"'] &
\end{tikzcd}$$

contin.

cont.

dense

open bijection

contin. bijection

(3)
$$\begin{tikzcd}
\min(\operatorname{Spec} R) \arrow[r, "\text{homeo.}"] \arrow[d, phantom, "\cap"] & \max(\operatorname{Spc} D^-(R)) \arrow[d, phantom, "\cap"] \\
\operatorname{Spec} R \arrow[r, "\mathcal{S}"] & \operatorname{Spc} D^-(R) \\
\max(\operatorname{Spec} R) \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & \min(\operatorname{Spc} D^-(R)) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

homeo if R semilocal

④ TFAE

- δ is contin.
- δ' is homeo.
- $\mathcal{S}'$ is homeo.
- $\# \operatorname{Spec} R < \infty$

3.2. Classification of ideals of $\mathcal{D}(R)$

Prop.: $\mathcal{X} \subseteq \mathcal{D}(R)$ an ideal

① $\begin{cases} \mathcal{X}_{cpt} = \langle \operatorname{Supp} \mathcal{X} \rangle \\ \mathcal{X}^{rad} = \sqrt{\mathcal{X}} \\ \mathcal{X}^{tame} = \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X} \end{cases}$

Here, for a property $\mathbb{P}$ denote
$\mathcal{X}^{\mathbb{P}} = \mathbb{P}$-closure
= smallest $\mathbb{P}$-ideal containing $\mathcal{X}$
$\mathcal{X}_{\mathbb{P}} = \mathbb{P}$-interior
= largest $\mathbb{P}$-ideal contained in $\mathcal{X}$

② $\underbrace{\mathcal{X}_{cpt} \subseteq \mathcal{X} \subseteq \mathcal{X}^{rad} \subseteq \mathcal{X}^{tame}}$

these have all the same support

In particular: tame $\Rightarrow$ radical

Notation: Rad = {radical ideals of $\mathcal{D}(R)$}

Tame = { tame ideals of $\mathcal{D}(R)$}

Cpt = { compact ideals of $\mathcal{D}(R)$}

$\mathrm{Spcl}(\mathrm{Spec}) = \{\text{specialization-closed in } \mathrm{Spec}(R)\}$

$\mathrm{Spcl}({}^t\mathrm{Spc}) = \{\text{———— " ————} \text{ in } {}^t\mathrm{Spc}\,D^-(R)\}$

$\mathrm{Thom} = \{\text{Thomason subset of } \mathrm{Spc}\,D^-(R)\}$

<u>Theorem</u>

commutes

$$\begin{tikzcd}
\mathrm{Rad} \arrow[r, shift left, "\mathrm{Spp}"] \arrow[d, shift left, "(\ )_{cpt}"] \arrow[d, phantom, "\dashv"] & \mathrm{Thom} \arrow[l, shift left, "1-1"] \arrow[d, shift right, "\mathcal{S}^{-1}"'] \arrow[d, phantom, "\vdash"] \arrow[dr, bend left, "(\ )_{spcl}"] \arrow[dr, phantom, "\top"] & \\
\mathrm{Cpt} \arrow[u, shift left, "(\ )^{rad}"] \arrow[r, shift left, "\mathrm{supp}"] \arrow[dr, bend right, "(\ )^{tame}"] & \mathrm{Spcl}(\mathrm{Spec}) \arrow[l, shift left, "1-1", "\langle - \rangle"'] \arrow[u, shift right, "\overline{\mathcal{S}}"'] \arrow[r, shift left, "\mathcal{S}"] \arrow[d, shift left] \arrow[d, phantom, "1-1"] & \mathrm{Spcl}({}^t\mathrm{Spc}) \arrow[l, shift left, "1-1", "\wr"'] \arrow[ul, bend right, "(\ )^{spcl}"'] \arrow[dl, bend left, "\mathrm{Sp}^{-1}"'] \\
& \mathrm{Tame} \arrow[ul, bend left, "(\ )_{cpt}", "1-1"'] \arrow[u, shift left] \arrow[ur, bend right, "\mathrm{Sp}"', "1-1"] &
\end{tikzcd}$$

where $f \dashv g \overset{\text{def}}{\iff} gf = 1$

$$\mathcal{S}(W) = \bigcup_{p \in W} \overline{\{\mathcal{S}(p)\}}$$

A_{spcl} = largest specialization-closed subset of ${}^t\mathrm{Spc}\,D^-(R)$ contained in A

B^{spcl} = smallest spl cl. subset of $\mathrm{Spc}\,D^-(R)$ containing B

$\mathrm{sp}(\ ) = \mathrm{Spp}(\ ) \cap {}^t\mathrm{Spc}\, D^-(R)$ ....

Moreover, TFAE

1. $\mathrm{Spec}\, R \underset{1-1}{\overset{\mathcal{S}}{\rightleftarrows}} \mathrm{Spc}\, D^-(R)$
2. $((\ )^{rad}, (\ )_{cpt})$ is 1-1
3. $(\overline{\mathcal{S}}, \mathcal{S}^{-1})$ is 1-1
4. $((\ )^{spcl}, (\ )_{spcl})$ is 1-1
5. Rad = Tame

<u>Cor</u>: Suppose R is artinian, then

(a) 1 to 5 in Thm hold

(b) Every ideal of $D^-(R)$ is compact, tame and radical.

3.3 On Balmer's conjecture for $D^-(R)$

$$\begin{cases} \mathcal{X}^{rad} \subseteq \mathcal{X}^{tame} \\ \mathrm{Rad} \supseteq \mathrm{Tame} \end{cases}$$

<u>Theorem</u>: Let $W \in \mathrm{Spcl}(\mathrm{Spec})$ and $\mathcal{X} = \langle W \rangle$

Assume R either integral domain or local and $\emptyset \neq W \neq \mathrm{Spec}\, R$

Then $\mathcal{X}^{\mathrm{rad}} \subsetneq \mathcal{X}^{\mathrm{tame}}$.

<u>Idea of proof:</u>

$\exists\ \underset{\substack{\parallel \\ (x_1, \dots, x_r) = \underline{x}}}{\mathfrak{p} \in W}$ and consider $C = \bigoplus_{i \geq 0} K(\underline{x}^i)[i] \in D(R)$

$\underline{x}^i = (x_1^i, \dots, x_n^i)$

$C \in \mathcal{X}^{\mathrm{tame}}$ but $C \notin \mathcal{X}^{\mathrm{rad}}$.

Key: $\mathrm{ann}_R C = \bigcap_{i \geq 0} \underline{x}^i R = 0$

Krull's intersection Theorem

$\square$

<u>Conj 1.7</u> (Balmer, ICM 2010)

$\rho^{\bullet}_{\mathcal{T}} : \mathrm{Spc}\, \mathcal{T} \longrightarrow \mathrm{Spec}^h R^{\bullet}_{\mathcal{T}}$ is locally injective, if $\mathcal{T}$ is algebraic. $R^{\bullet}_{\mathcal{T}} = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^{\bullet}\mathbb{1})$

$$\rho^{\bullet}_{D(R)} : \mathrm{Spc}\, D(R) \longrightarrow \mathrm{Spec}^h R^{\bullet}_{D(R)}$$

where $R^{\bullet}_{D(R)} = \mathrm{Hom}_{D(R)}(R, \Sigma^{\bullet} R) = R$

$\therefore\ \mathrm{Spec}^h R^{\bullet}_{D(R)} = \mathrm{Spec}\, R$

$$\boxed{\rho^{\bullet}_{D(R)} = \rho}$$

Cor: assume $\dim R > 0$ and R either a domain or local. Then ρ is not locally injective.
Hence, Balmer's conjecture 1.7. does not hold for $D^-(R)$.

§4 The case of discrete valuation rings

Theorem:
Let (R, xR) be DVR. For $n \geq 0$ set

$$\mathcal{P}_n = \{ X \in D^-(R) \mid \exists t \geq 0 \text{ s.t. } \ell\ell(H^{-i}X) \leq t \cdot i^n \ \forall i \in \mathbb{Z} \}$$

Then

1. $\mathcal{P}_n = \mathrm{thick}^{\otimes}\{(\cdots \longrightarrow R/x^{3n}R \xrightarrow{0} R/x^{2n}R \xrightarrow{0} R/x^n R \longrightarrow 0)\}$
2. $\mathcal{P}_n$ is prime
3. $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots \longrightarrow \dim(\mathrm{Spc}\, D^-(R)) = \infty$