Cluster duality for Grassmannians

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Overview:

§1 Motivation: mirror symmetry

§2 Cluster structures

§3 Relating two superpotentials.

§1 Motivation from mirror symmetry

1990s: Hori-Vafa, Batyrev, Givental, Yau, ...

$D \subset X$ toric Fano variety with toric divisor

$\Rightarrow$ $X \setminus D = T$ complex torus

Def: The mirror of X is a pair $(T^\vee, W)$:

1. $T^\vee$ dual torus of T
2. $W : T^\vee \to \mathbb{C}$ superpotential

s.t.

$$\mathrm{Jac}(W) \cong QH(X)$$

Jacobian ring — B-model | A-model — quantum cohom. ring

Idea: "W encodes the geometry of D"
↳ Gromov-Witten invariants

Note ① many versions of mirror symmetry, later we will see another one (Gross-Siebert)

② many generalizations exist, we are interested in

$$Gr_{n-k}(\mathbb{C}^n) = \{ V \subset \mathbb{C}^n : \dim_{\mathbb{C}} V = n-k \}$$

Plücker embedding: $Gr_{n-k}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\Lambda^{n-k}\mathbb{C}^n)$

$\langle v_1, \dots, v_{n-k} \rangle \mapsto [v_1 \wedge \dots \wedge v_{n-k}]$

coordinates given by basis of $(\Lambda^k \mathbb{C}^n)^*$

Plücker coord. $p_{i_1 \dots i_{n-k}} := (e_{i_1} \wedge \dots \wedge e_{i_{n-k}})^*$, $1 \leq i_1 < \dots < i_{n-k} \leq n$

Q: What's the replacement of the $\boxed{\text{toric divisor}}$?

def. $D_{n-k,n} := \bigcup_{i=1}^{n} \underbrace{\{ P_{i+1,\dots,i+n-k} = 0 \}}_{=: D_i} \subset Gr_{n-k}(\mathbb{C}^n)$

called the boundary divisor.

$Gr_{n-k}(\mathbb{C}^n) \setminus D_{n-k,n} =: \mathbb{X}^\circ$ open positroid variety.

Q: What's the replacement of the $\boxed{\text{dual torus}}$?

$\rightsquigarrow$ $\check{\mathbb{X}}^\circ := Gr_k((\mathbb{C}^n)^*) \setminus D_{k,n}$ dual open positroid variety

Langlands dual Grassmannian

(conjectured)

Many B-models for Grassmannians, e.g. by Witten '95, Eguchi-Hori-Xiong '97, Batyrev-Ciocan Fontanine-van Straten '98, ..., Rietsch '08 and

Def [Marsh-Rietsch '20]

$$W_q : \check{X}^\circ \longrightarrow \mathbb{C}$$ Marsh-Rietsch superpotential

$$W_q = q \cdot W_{n-k} + \sum_{i \in \{1,\dots,n\} \setminus \{n-k\}} W_i$$

where

$$W_i = \frac{P[i+1, i+k-1] \cup \{i+k+1\}}{P[i+1, i+k]}$$ corresponds to D_i

Theorem [MR20]

$$\operatorname{Jac}(W_q) \cong QH(Gr_k(\mathbb{C}^n))$$

Remark: MR use Lie theory &
they show that W_q specializes to EHX-
& BCFvS- superpotentials.

§2 Cluster structures: 1) Cluster varieties

The following is called *fixed data*:

1) $N = \mathbb{Z}^r$ a lattice and $M = N^*$ its dual,
2) $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ the associated alg. torus and
$T_M = M \otimes_{\mathbb{Z}} \mathbb{C}^*$ its dual torus,
3) $\{-,-\} : N \times N \longrightarrow \mathbb{Z}$ skew-sym. bilinear form.

Def: For $n \in N$ and $m \in M$ define <u>mutation</u>:

$$\mu_{(n,m)} : T_N \dashrightarrow T_N$$

by pull-back: $\mu_{(n,m)}^*(z^{m'}) = z^{m'}(1+z^m)^{m'(n)}$

can swap m & n $\rightsquigarrow$ $\mu_{(m,n)} : T_M \dashrightarrow T_M$

Remark: $\mu_{(n,m)}$ preseves the volume form $dx_1 \wedge \dots \wedge dx_n$.

Fix $s_0 = \{e_i\}$ basis for N & $v_i = \{e_i, -\} \in M$

Def: <u>$\mathcal{A}$-</u> resp. <u>$\mathcal{X}$-mutation in direction k</u>

$\mu_{(-e_k, v_k)} : T_{N, s_0} \dashrightarrow T_{N,s}$ resp. $\mu_{(v_k, e_k)} : T_{M, s_0} \dashrightarrow T_{M,s}$

where $s = \{e_i'\}$ basis for N obtained from s_0 by certain pseudo-reflections.

Def: The $\mathcal{A}$- resp. $\mathcal{X}$- cluster variety associated to the fixed data above is

$$\mathcal{A} := \bigcup_{s \sim s_0} T_{N,s} \quad \text{resp.} \quad \mathcal{X} = \bigcup_{s \sim s_0} T_{M,s}$$

glued along $\mathcal{A}$ resp. $\mathcal{X}$- mutation.

↳ glue wherever μ is defined

$\mathcal{A}$ & $\mathcal{X}$ have global volume form $\Rightarrow$ log Calabi-Yau

Remark: $s \sim s_0$ refers to all bases s of N obtained from s_0 by (a sequence of) certain pseudo-reflexions.

$\rightsquigarrow$ the atlas for $\mathcal{A}$ & for $\mathcal{X}$ are indexed by the same bases s of N.

§2.2 Cluster structure for $Gr_k(\mathbb{C}^n)$

$$\left[\begin{array}{c}\text{skew-sym. bilin. form}\\ \{-,-\} \text{ \& basis } \{e_i\}\\ \text{for } N\end{array}\right] \overset{1:1}{\longleftrightarrow} \left[\begin{array}{c}\text{quiver } Q=(V,E) \text{ with}\\ V=\{1,\dots,n\} \text{ and}\\ \#\{i\to j\}-\#\{j\to i\}=\{e_i,e_j\}\end{array}\right]$$

Exp: $\{-,-\}:\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ defined for (e_1,e_2) by $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \Rightarrow Q = 1\to 2$

Def: $\widetilde{Gr_k(\mathbb{C}^n)} := \mathrm{Spec}(\mathbb{C}[Gr_k(\mathbb{C}^n)])$ the affine cone over $Gr_k(\mathbb{C}^n)$

homogeneous coordinate ring of $Gr_k(\mathbb{C}^n)$ w.r.t. its Plücker embedding

Def:

$$
Q = \begin{tikzcd}
\emptyset & & & & \\
& 1\times1 \arrow[ul] \arrow[dr] & 1\times2 \arrow[l] \arrow[dr] & \cdots \arrow[l] \arrow[dr] & 1\times k \arrow[l] \\
& 2\times1 \arrow[u] \arrow[dr] & 2\times2 \arrow[l] \arrow[u] \arrow[dr] & \cdots \arrow[l] & 2\times k \arrow[l] \\
& \vdots & \vdots & & \vdots \\
& (n-k-1)\times1 \arrow[dr] & (n-k-1)\times2 \arrow[l] \arrow[dr] & \cdots \arrow[l] \arrow[dr] & (n-k-1)\times k \arrow[l] \\
& (n-k)\times1 \arrow[u] & (n-k)\times2 \arrow[u] & \cdots & (n-k)\times k
\end{tikzcd}
$$

blue vertices are frozen, ie never mutate in this direction

Notice, Q has $\underbrace{(n-k)k+1}_{=\dim_{\mathbb{C}} \widetilde{Gr}_k(\mathbb{C}^n)}$ many vertices

Let $S_0 = \{e_{i\times j}\}$ be the standard basis of $N := \mathbb{Z}^{(n-k)k+1}$

Theorem [Scott '08]

(Q, s_0) defines the $\mathcal{A}$-cluster variety $\mathcal{A}_Q \subset \widetilde{Gr_k}(\mathbb{C}^n)$ and (up to codim 2)

$$\mathcal{A}_Q \cong \widetilde{X}^\circ \longrightarrow \widetilde{Gr_k}(\mathbb{C}^n) \setminus \widetilde{D}_{k,n}$$

For example, the initial torus of $\mathcal{A}_Q$ is

$$T_{N,Q} = N \otimes \mathbb{C}^* = \mathrm{Spec}\big(\mathbb{C}[\underbrace{z^{\pm f_{ixj}}}_{}]_{ij}\big)$$

where $\{f_{ixj}\}_{i,j}$ dual basis for M

monomial in $\mathbb{C}[z_1, \dots, z_{(n-k)(k+1)}]$ with exp. vector $\pm f_{ixj}$

$$\boxed{z^{f_{ixj}} = p_{ixj} = P_{[1,k-j] \cup [k-j+i+1,\, k+i]}}$$

Plücker coordinate

§2.3 Partial compactifications and superpotentials

Slogan: The role of the torus & its dual in toric mirror symmetry are played by $\mathcal{A}$- and $\mathcal{X}$- cluster varieties.

Given Q a quiver with "frozen vertices" can partially compactify $\mathcal{A}_Q$:

Def: $D := \bigcup_{i \times j \text{ frozen}} \{ z^{f_{i \times j}} = 0 \}$ and $\overline{\mathcal{A}_Q}$

(partial) compactification of $\mathcal{A}_Q$ with

$$\boxed{\overline{\mathcal{A}_Q} \setminus D = \mathcal{A}_Q}$$

Recall frozens in $\dot{Q}$ are

$\emptyset$, $k \times j$ for $1 \le j \le n-k$, $i \times (n-k)$ for $1 \le i \le k$

$$\Rightarrow \quad \tilde{D} := \bigcup_{i=1,\dots,n} \{ P_{[i+1,\, i+k]} = 0 \}$$

$$\Rightarrow \quad \begin{tikzcd}[column sep=small] \widetilde{Gr_k(\mathbb{C}^n)} \arrow[d] & \cong & \overline{\mathcal{A}_Q} & \supset & \mathcal{A}_Q & \cong & \tilde{\check{X}}^{\circ} \arrow[d] \\ & & Gr_k(\mathbb{C}^n) & \supset & \check{X}^{\circ} & & \end{tikzcd}$$

Q: Is there a potential on X_Q for $\overline{\mathcal{A}_Q} \supset \mathcal{A}_Q$?

Construction [Gross-Hacking-Keel-Kontsevich]

$$W = \sum_{i \times j \text{ frozen}} \vartheta_{i \times j} \quad \text{where}$$

all arrows incoming

$$\vartheta_{i \times j}\big|_{T_{M,Q'}} = z^{-e_{i \times j}} \text{ if } i \times j \text{ is a sink in } Q'$$

Example: for $\widetilde{Gr_3(\mathbb{C}^5)}$ we have

$$Q = \begin{tikzcd} \emptyset & & & \\ & 1\times1 \arrow[ul] \arrow[dr] & 1\times2 \arrow[l] \arrow[dr] & 1\times3 \arrow[l] \\ & 2\times1 \arrow[u] & 2\times2 \arrow[u] & 2\times3 \end{tikzcd}$$

$\emptyset$ & 2×3 are sinks

$$\Rightarrow \quad \vartheta_{\emptyset}\big|_{T_{M,Q}} = z^{-e_{\emptyset}} \quad \text{and} \quad \vartheta_{2\times3}\big|_{T_{M,Q}} = z^{-e_{2\times3}}$$

↳ 2×1 and 2×2 are sinks after 1 mutation

compute using pull-back along mutation:

$$\vartheta_{2\times1}\big|_{T_{M,Q}} = z^{-e_{2\times1}}\left(1+z^{-e_{1\times1}}\right)$$

$$\vartheta_{2\times2}\big|_{T_{M,Q}} = z^{-e_{2\times2}}\left(1+z^{-e_{1\times2}}\right)$$

1×3 is sink after 2 mutations

$$\vartheta_{1\times3}\big|_{T_{M,Q}} = z^{-e_{1\times3}}\left(1+z^{-e_{1\times2}}+z^{-e_{1\times2}-e_{1\times1}}\right)$$

Q: How do we know that W is a suitable superpotential?

§2.4 Tropical points

Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield

$\oplus$ does not have inverses

for example, $(\mathbb{Z}, \min, \cdot) =: \mathbb{Z}^t$

Note: Mutation does not involve minus signs
$\Rightarrow$ can consider $\mathbb{P}$-points of $\mathcal{A}$ & $\mathcal{X}$

$$T_N(\mathbb{P}) = N \otimes_{\mathbb{Z}} \mathbb{P} \quad \& \quad T_M(\mathbb{P}) = M \otimes_{\mathbb{Z}} \mathbb{P}$$

this given non-canonical identifications

$$\mathcal{A}(\mathbb{P}) = N \otimes_{\mathbb{Z}} \mathbb{P} \quad \& \quad \mathcal{X}(\mathbb{P}) = M \otimes_{\mathbb{Z}} \mathbb{P}$$

Toric geometry: T_N has dual torus T_M and

$T_M(\mathbb{Z}^t) = M \otimes \mathbb{Z} = M = N^*$ parametrizes

a basis for $\Gamma(T_N, \mathcal{O}_{T_N})$ characters of T_N

Fock-Goncharov conjecture "cluster variety version"

$\boxed{\text{The tropical points } \mathcal{X}(\mathbb{Z}^t) \text{ resp. } \mathcal{A}(\mathbb{Z}^t) \text{ parametrize a basis for } \Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \text{ resp. } \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})}$.

→ false in general, but true for $\mathcal{A}_Q \subset \widetilde{Gr_k \mathbb{C}^n}$

counterexp. due to Gross-Hacking-Keel

Q: What does that mean for our super-potential?

Need: $\Xi := \{ x \in \mathcal{X}_Q(\mathbb{Z}^t) : W^{trop}(x) \geq 0 \}$

cuts out those tropical points that correspond to regular functions on $\overline{\mathcal{A}_Q}$ (not only on $\mathcal{A}_Q$)

[GHKK] This follows from the FG-conj., i.e.

$$\boxed{\operatorname{Spec}\Big(\bigoplus_{x \in \Xi} \mathbb{C} \cdot \vartheta_x\Big) = \overline{\mathcal{A}_Q}} = \widetilde{Gr_k(\mathbb{C}^n)}$$

this is an algebra, in fact its $\mathbb{C}[Gr_k(\mathbb{C}^n)]$

Remark: This is a mirror symmetry statement for affine log Calabi-Yau varieties [Gross-Siebert]

have nowhere vanishing top form
e.g. algebraic tori, cluster varieties

§3 Relating two superpotentials

Have seen Marsh-Rietsch's potential

$$W_q : \check{X}^\circ \longrightarrow \mathbb{C} \qquad W_q = q\,\omega_{n-k} + \sum_{i \neq n-k} w_i$$

$\check{X}^\circ \subset Gr_{n-k}(\mathbb{C}^n)$

and GHKK's cluster superpotential

$$W : \mathcal{X}_Q \longrightarrow \mathbb{C} \;, \quad W = \sum_{i \text{ frozen}} \vartheta_i$$

Q: How are they related? → by a p^*-map!

For a pair $(\mathcal{A}, \mathcal{X})$ there exists a global map

$$p : \mathcal{A} \longrightarrow \mathcal{X}$$

On a chart we have $p|_{T_N} : T_N \longrightarrow T_M$

defined by $p^*: N \longrightarrow M$

$$n \longmapsto \{n, -\}$$

only well-defined for n unfrozen

↳ p^*-maps are not unique!

<u>The trick</u>: any p^*-map commutes with mutation $\Rightarrow$ get a global map $p: \mathcal{A} \to \mathcal{X}$.

<u>Theorem</u> [B - Cheung - Magee - Nájera Chávez]

$\exists!$ p-map $p: \mathcal{A}_Q \longrightarrow \mathcal{X}_Q$ s.t.

$$p^*(\vartheta_i) = W_i \quad \text{and so} \quad p^*(W) = W_{q=1}$$

GHKK

Marsh-Rietsch

Proof: 1) Show that $p^*(v_i) = W_i$ determines a unique map $p^*: N \longrightarrow M$

2) show it's a p^*-map.

a bit tricky as neither v_i nor W_i are given in the initial chart

Corollary 1: $p^*: N \to M$ has 1-dim kernel K and $\bar{p}^*: N/K \longrightarrow K^\perp$ induces an isom.

$$\bar{p}: \mathcal{A}_{Q,k} \xrightarrow{\sim} X_{Q,k}$$

with $\mathcal{A}_{Q,k} \subset Gr_k(\mathbb{C}^n)$ and $K \cong Pic(Gr_k \mathbb{C}^n)$

canonically

Moreover $\bar{p}$ extends to $\check{\mathbb{X}}^\circ \longrightarrow \mathbb{X}^\circ$ and to $Gr_k(\mathbb{C}^n) \longrightarrow Gr_{n-k}(\mathbb{C}^n)$

About toric degenerations:

(1) Rietsch-Williams define toric degenerations of Grassmannians using valuations assoc. to each chart.

They show

$$\underbrace{\Delta_s(D)}_{\text{Newton-Okounkov body for a chart}} = \Big\{ \underbrace{W_{q=1}|_{T_s}^{trop}}_{\text{tropicalized potential in the same chart}}(v) \geq 0 \Big\} \quad \star$$

↳ $\star$ is a consequence of our result, we show $\Delta_s(D) = \Xi$ and $\Xi = \{ W_{q=1}|_{T_s}^{trop}(v) \geq 0 \}$

2

Gross-Hacking-Keel-Kontsevich construction also gives toric degenerations of $\overline{\mathcal{A}}_Q$ for every chart

↳ a corollary of our Thm is that they coincide with Rietsch-Williams degenerations.

the toric degen. by GHKK is defined by Ξ, the RW one by $\Delta_G(D)$

but we have already seen: $\Xi = \Delta_G(D)$