Tropical Geometry of Grassmannians & their cluster structure

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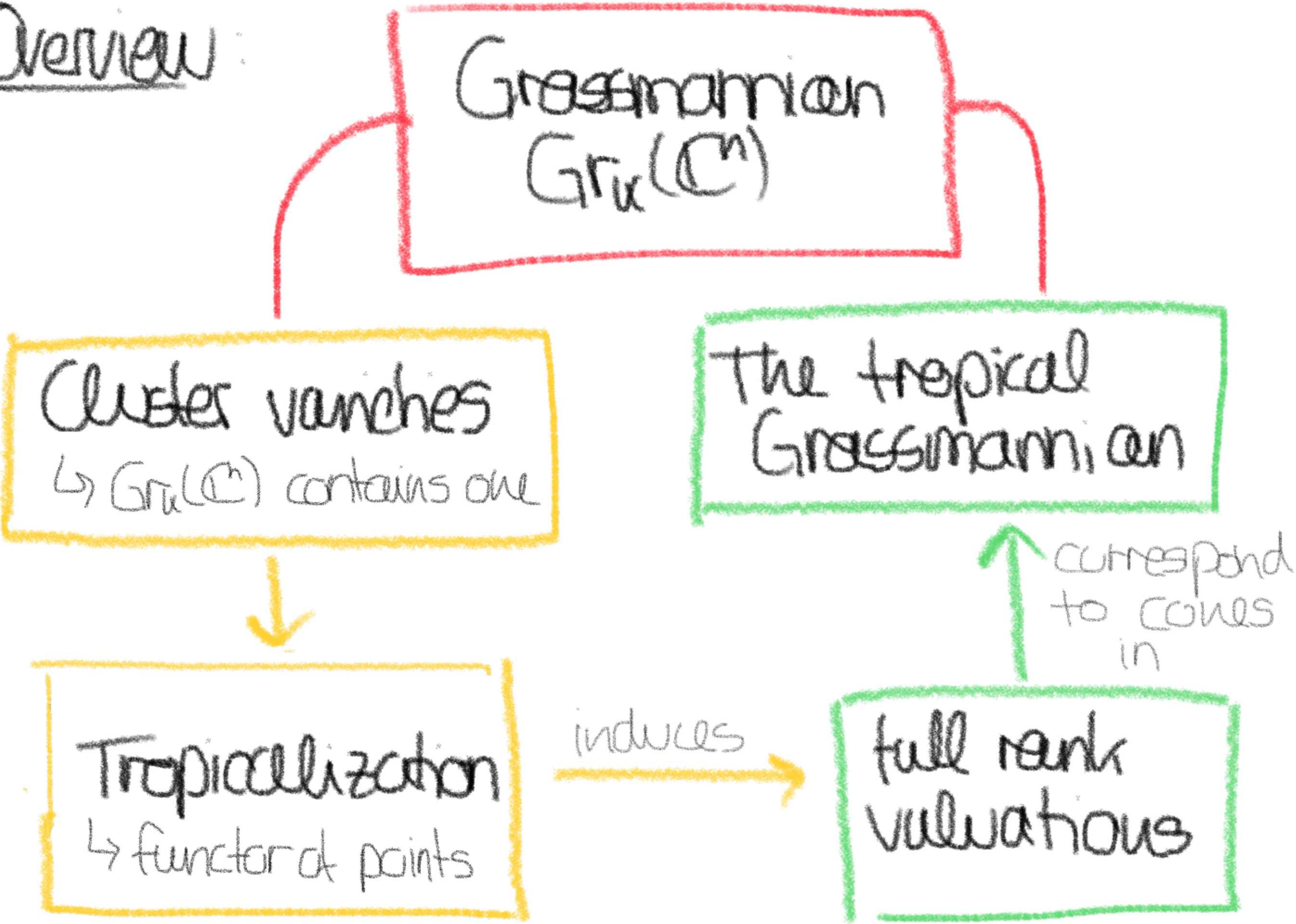
December 4 2020

Overview


Grassmannian $Gr_k(\mathbb{C}^n)$
Cluster varieties
↳ $Gr_k(\mathbb{C}^n)$ contains one
The tropical Grassmannian
Tropicalization
↳ functor of points
induces
full rank valuations
correspond to cones in

§1 Cluster varieties

Notation: $N = \mathbb{Z}^r$ a lattice & $M = N^*$ the dual

$T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ a torus & $T_M := M \otimes_{\mathbb{Z}} \mathbb{C}^*$ the dual

Def: Given $n \in N$ & $m \in M$ define mutation

$$\mu_{(n,m)} : T_N \dashrightarrow T_N$$

$$\mu^*_{(n,m)}(z^{m'}) = z^{m'}(1+z^m)^{m'(n)}$$

dually, $\mu_{(m,n)} : T_M \dashrightarrow T_M$

Fix $\{\cdot,\cdot\} : N \times N \to \mathbb{Z}$ skew-sym. bilinear

& $s = (e_1, \dots, e_r)$ basis of N, $v_k := \{e_k, -\} \in M$

call s a seed

Exercise: tropicalizing $\mu_{(-e_k, v_k)}$ to $\mu_k^T : N \to N =: T_N(\mathbb{Z})$

gives a new seed $\mu_k(s) = (e_1', \dots, e_r')$ with

$$e_i' := e_i + \max\{0, [e_i, e_k]\} e_k \qquad i \neq k$$

set $e_k' := -e_k$ → this is a pseudo reflection

Definition: The $\mathcal{A}$-cluster variety ass. to $\{,\}$ & s is

$$\boxed{\mathcal{A}_s := \bigcup_{s' \sim s} T_{N,s} \Big/ \text{glued along mutations of form } \mu_{(-e_k, v_k)}}$$

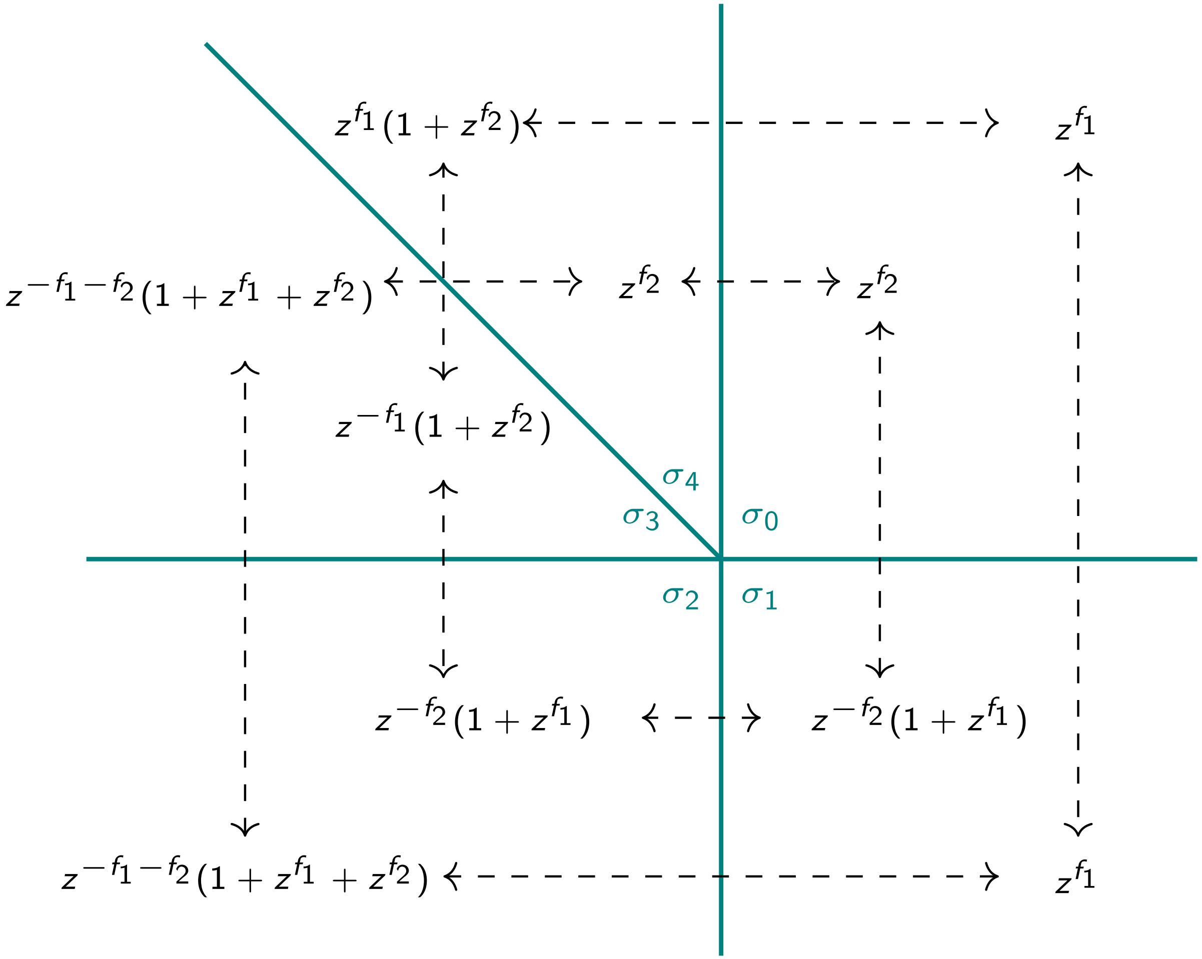
The $\mathcal{X}$-cluster variety ass. to $\{,\}$ & s is

$$\boxed{\mathcal{X}_s := \bigcup_{s' \sim s} T_{M,s} \Big/ \text{glued along mut. of form } \mu_{(v_k, e_k)}}$$

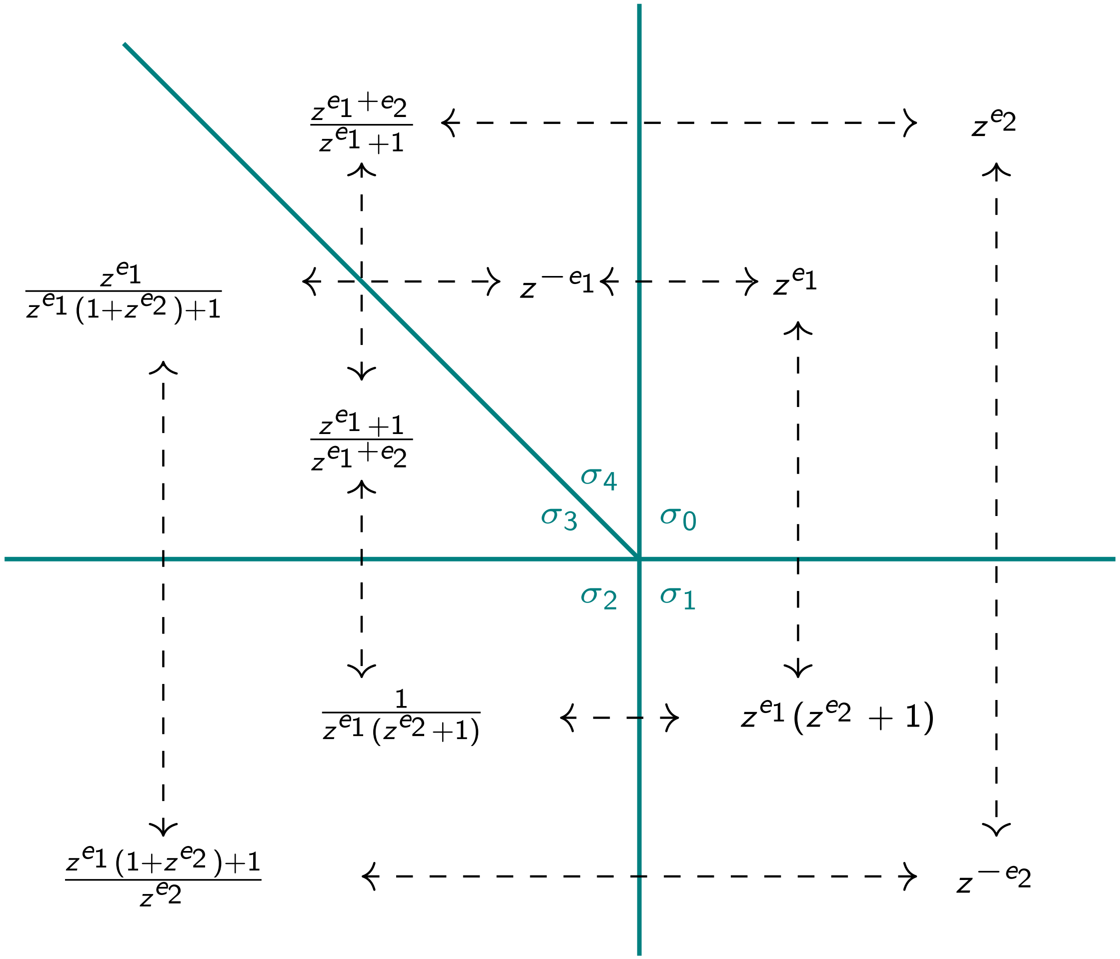
→ note the switch!

Example: $\mathcal{A}$ in case $\mathtt{A}_2$

Text



Example: $\mathcal{X}$ in case $\mathtt{A}_2$



§1.2 The cluster variety inside $Gr_k \mathbb{C}^n$

$\{\cdot,\cdot\}$ skew-sym. $\longleftrightarrow$ Q quiver w/ vertices $1,\dots,r$
$s = (e_i)$ & $\{e_i, e_j\} = \#\{i \to j\} - \#\{j \to i\}$

Exp: $1 \to 2$ corresps to (e_1, e_2) with $\{\cdot,\cdot\}$ given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Def:

$$Q = \begin{tikzcd}
\emptyset & & & \\
1\times1 \arrow[u] \arrow[dr] & 1\times2 \arrow[l] \arrow[dr] & \cdots \arrow[l] \arrow[dr] & 1\times k \arrow[l] \\
2\times1 \arrow[u] \arrow[dr] & 2\times2 \arrow[l] \arrow[u] \arrow[dr] & \cdots \arrow[l] \arrow[dr] & 2\times k \arrow[l] \\
\vdots \arrow[u] \arrow[dr] & \vdots \arrow[u] \arrow[dr] & \cdots \arrow[dr] & \vdots \\
(n-k)\times1 \arrow[u] & (n-k)\times2 \arrow[u] & \cdots & (n-k)\times k
\end{tikzcd}$$

blue means frozen, i.e. never mutate in this direction

Thm (Scott) Q defines an $(n-k)k+1$ dimensional $\mathcal{A}$-cluster variety inside $\widetilde{Gr_k(\mathbb{C}^n)}$

$s_0 = (e_{ixj})$ basis of N & (f_{ixj}) dual basis of M:

$$T_{N, s_0} = N \otimes \mathbb{C}^* = \operatorname{spec}(\mathbb{C}[z^{\pm f_{ixj}}])$$

Laurent poly. in $z_1, \dots, z_{k(n-k)+1}$

and $\boxed{z^{f_{ixj}} = P_{[1,k-j] \cup [k-j+i+1, k+i]}}$

Plücker coord!

Reminder: the Plücker embedding of $Gr_k(\mathbb{C}^n)$ is

$$\begin{array}{ccc} Gr_k(\mathbb{C}^n) & \hookrightarrow & \mathbb{P}(\Lambda^k \mathbb{C}^n) \\ \rotatebox{90}{$\in$} & & \\ V = \langle v_1, \dots, v_k \rangle & \longmapsto & [v_1 \wedge \dots \wedge v_k] \end{array}$$

with coordinates $(e_{i_1} \wedge \dots \wedge e_{i_k})^* =: \boxed{P_{i_1 \dots i_k}}$

$\{e_1, \dots, e_n\}$ basis for $\mathbb{C}^n$

Tropicalizing cluster varieties

Recall: $\mu^*_{(n,m)}(z^{m'}) = z^{m'}(1+z^{m})^{m'(n)}$ → substraction-free

↳ can consider cluster varieties over semifields "fields without substraction"

Let $\mathbb{P}$ be a semifield, then $T_N(\mathbb{P}) = N \otimes_{\mathbb{Z}} \mathbb{P}$

⇒ for every seed we have not canonically!

$\mathcal{A}(\mathbb{P})\big|_{T_{N,s}} \cong N \otimes_{\mathbb{Z}} \mathbb{P}$, $\mathcal{X}(\mathbb{P})\big|_{T_{M,s}} \cong M \otimes_{\mathbb{Z}} \mathbb{P}$

Definition: The tropicalization of a cluster variety is $\mathcal{A}(\mathbb{Z}^T)$ resp. $\mathcal{X}(\mathbb{Z}^T)$, for $\mathbb{Z}^T = (\mathbb{Z}, \max, +)$

→ What is this good for?

Recall from toric geometry: T_N has dual torus T_M and $T_M(\mathbb{Z}^T) = M \otimes \mathbb{Z} = M = N^* \longrightarrow$ characters of T_N

parametrizes a basis for $\Gamma(T_N, \mathcal{O}_{T_N})$

Cluster variety version: "Fock-Goncharov conjecture"

$\mathcal{X}(\mathbb{Z}^T)$ resp. $\mathcal{A}(\mathbb{Z}^T)$ parametrizes a basis for $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ resp. $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

→ false in general (see Gross-Hacking-Keel)

true for $\mathcal{A} \subset \widetilde{Gr_k(\mathbb{C}^n)}$

For $\mathcal{A} \subset \widetilde{Gr_k(\mathbb{C}^n)}$, $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ has a basis indexed by $\mathcal{X}(\mathbb{Z}^T)$ called the ϑ-basis

Notation: $\Theta = \{ \vartheta_p : p \in \mathcal{X}(\mathbb{Z}^T) \}$

recall: for every seed s: $\mathcal{X}(\mathbb{Z}^T) \equiv M$

denote this by $g_s : \mathcal{X}(\mathbb{Z}^T) \longrightarrow M$

Def: For a seed s, and $\vartheta_p \in \Theta$

$g_s(p)$ is called *g-vector* of ϑ_p w.r.t s

§1.4 g-vectors for Grassmannians

Let $s \sim s_0$, $s = (e'_{i \times j})$ basis of N

consider $(f'_{i \times j})$ the dual basis of M

Then $f'_{i \times j} \in \mathcal{X}(\mathbb{Z}^T)$ and $z^{f'_{i \times j}} \in \Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$

[GHKK] $z^{f'_{i \times j}} \in \Theta$ called a <u>cluster variable</u>

many other brilliant mathematicians

Theorem [Fomin–Zelevinsky 2007, ..., Gross–Hacking–Keel–Kontsevich 2018]

The g-vectors of all $z^{f_1' x_j}$ form a simplicial fan called the g-fan.

Proposition [B–Cheung–Magee–Nájera Chávez]

For every seeds the map $\Theta \to M$, $\vartheta_p \mapsto g_s(p)$ extends to a valuation

$$g_s : \Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \setminus \{0\} \longrightarrow M$$

Reminder: Let $<$ be a total order on $\mathbb{Z}^r$ & A a k-algebra, then $v: A \setminus \{0\} \to \mathbb{Z}^r$ is a valuation if $v(fg) = v(f) + v(g)$

$v(f+g) \leq \min_< \{v(f), v(g)\}$, $v(cf) = v(f)$

$\forall f, g \in A$ & $c \in k$.

↳ this is our chance to pass to the tropical Grassmannian!

The tropical Grassmannian

Let $A_{k,n} := \mathbb{C}[Gr_k(\mathbb{C}^n)]$ homog. coord. ring of $Gr_k(\mathbb{C}^n)$ w.r.t. Plücker embedding.

Then $A_{k,n} \cong \mathbb{C}[p_I : I \subset [n], |I| = k] / I_{k,n}$ ← Plücker ideal

Def: The tropical Grassmannian [Speyer - Sturmfels] is

$$\mathrm{Trop}(I_{k,n}) := \left\{ w \in \mathbb{R}^{\binom{n}{k}} : \underbrace{in_w(I_{k,n})}_{\text{initial ideal}} \not\ni \text{monomials} \right\}$$

⚠ We work over $\mathbb{C}$ with trivial valuation

⇝ $\mathrm{Trop}(I_{k,n})$ is the support of a fan, we fix the fan structure given by the Gröbner fan, i.e.

$$v, w \in C^{\circ} \overset{\text{def}}{\iff} in_v(I_{k,n}) = in_w(I_{k,n})$$

recall, e.g. $in_{(1,0)}(x_1 + x_1 x_2^2 + x_2) = x_2$
& $in_w(I_{k,n}) := \langle in_w(f) : f \in I_{k,n} \rangle$

Fact: Given $v: A\setminus\{0\} \longrightarrow (\mathbb{Z}^r, <)$ a valuation of full rank (i.e. the value semigroup = image of v has rank = Krull-dimension of A)

and $A \cong k[x_1,\dots,x_n]/I$ can construct $w_v \in \mathrm{Trop}(I)$:

Step 1: $M_v := \left[v(\bar{x}_1), \dots, v(\bar{x}_n) \right] \in \mathbb{Z}^{r\times n}$

Step 2: choose an order preserving projection $e: (\mathbb{Z}^r, <) \longrightarrow \mathbb{Z}$ linear s.t. for certain $a, b \in \mathbb{Z}^r$ with $a < b$ also $e(a) < e(b)$

corresponding to elements of a Gröbner basis of I

Step 3: $w_v := e(v(\bar{x}_1)), \dots, e(v(\bar{x}_n)) \in \mathrm{Trop}(I)$ [B]

Theorem [B '20]

Let $v : A \setminus \{0\} \longrightarrow \mathbb{Z}^d$ be a full-rank valuation then $\mathrm{im}(v)$ is a fin. gen. semigroup $\iff \exists$ presentation $A \cong k[x_1, \dots, x_n]/I$ s.t. $\mathrm{in}_{w_v}(I)$ is binomial & prime.

$w_v \in \mathrm{Trop}(I)$ $\Rightarrow w_v \in C^\circ \subset \mathrm{Trop}(I)$ maximal cone

Prop. [GHKK, BCMN] For every seed s of $\mathcal{A} \subset \widetilde{\mathrm{Gr}_k \mathbb{C}^n}$ the value semigroup of $g_s|_{\mathbb{C}[\mathrm{Gr}_k(\mathbb{C}^n)]} : \mathbb{C}[\mathrm{Gr}_k(\mathbb{C}^n)] \to M$ is finitely generated.

$\hookrightarrow$ by [Anderson] got a toric degen. for $\mathrm{Gr}_k(\mathbb{C}^n)$ & it agrees with GHKK's degeneration

$\underline{Q}^*$: Given a seed s, what's the presentation of $\mathbb{C}[Gr_k(\mathbb{C}^n)]$ in the Theorem?

↳ open in general, known for $Gr_2(\mathbb{C}^n)$ & $Gr_3(\mathbb{C}^6)$

$Gr_2(\mathbb{C}^n)$ and $Gr_3(\mathbb{C}^n)$

Why these Grassmannians?

↳ they only have finitely many seeds!

Answer to Q^* for $Gr_2(\mathbb{C}^n)$: the Plücker ideal.

<u>Thm</u> [Speyer-Sturmfels] Let C be a maximal cone in $\mathrm{Trop}(I_{2,n})$ then $\forall w \in C^\circ$ $in_w(I_{2,n})$ is <u>binomial & prime</u>. i.e. gives a Gröbner toric degen.

$\rightsquigarrow$ this is not true for $Gr_3(\mathbb{C}^6)$

Fact: $\exists$ seeds s for $A \subset \widetilde{Gr_3(\mathbb{C}^6)}$ such that

$w_{g_s} \in \mathrm{Trop}(I_{3,6})$ from above satisfies

$\mathrm{in}_{w_{g_s}}(I_{3,6})$ not prime

Theorem [B-Mohammadi-Nájera Chavez]

weighted proj. space

$\exists \quad Gr_3(\mathbb{C}^6) \longrightarrow \mathbb{P}(\underbrace{1,\cdots,1}_{20},2,2)$ s.t.

the (weighted) homog. coord. ring is

$\mathbb{C}[p_{ijk}, X, Y]/J \cong A_{3,6}$.

Moreover, for every seed s we have

$\mathrm{in}_{w_{g_s}}(J)$ is binomial & prime.

Some computational fun facts:

	# max. cones	# prime max. cones	# positive max. cones	# prime positive max. cones
$\mathrm{Trop}(I_{3,6})$	1035	990	50	46
$\mathrm{Trop}(J)$	?	?	50	50

combinatorially equivalent to the $\boxed{g\text{-fan}}$

Can play the same game with the flag variety $F\ell_n = V(I_n)$ Plücker ideal

eg. $n=4$

$\mathrm{Trop}(I_4)$	78	72	14	12
$\mathrm{Trop}(J_4)$	105	99	14	14

Thank you!

References

My work:

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[B-Mohammadi-Nájera Chávez] "Families of Gröbner degenerations, Grassmannians & universal cluster algebras, arxiv: 2007.14972

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Background on Cluster varieties:

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- Gross-Hacking-Keel-Kontsevich "Canonical bases for cluster varieties" JAMS 2018
- Gross-Hacking-Keel "Birational geometry of cluster varieties" – Algebraic geometry 2015
- Scott "Grassmannians & cluster algebras" Proc. London Math. Soc. 2006
- Fomin-Zelevinsky "Cluster algebras IV: Coefficients" Compositio Math. 2007

Other background

- Speyer-Sturmfels "The tropical Grassmannian" Adv. Geom. 2004
- Anderson "Okounkov bodies & toric degenerations" Math. Ann. 2013