Toric degenerations with projections and standard monomials

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Toric degenerations

Definition

Let X be a projective variety. A *toric degeneration* of X is a flat morphism $\xi : \mathfrak{X} \to \mathbb{A}^1$ with generic fibre isomorphic to X and special fibre $\xi^{-1}(0)$ a toric variety.

Examples:

1. an *abstract* degeneration, *e.g.* a toric scheme over $\mathbb{A}^1 = \mathrm{Spec}(k[t])$;
2. an *embedded* degeneration, *e.g.* $\mathfrak{X} = V(xy - x^2 + ty^3) \subset \mathbb{P}^1_{x,y} \times \mathbb{A}^1_t$;
3. a toric degeneration *admits a projection* if it is an embedded toric degeneration with a projection $\xi^{-1}(1) \twoheadrightarrow \xi^{-1}(0)$.

From abstract to embedded toric degenerations

<u>Question:</u> If the toric fibre $\xi^{-1}(0)$ has a very ample line bundle, can we extend this embedding to all of the family?

Conjecture (Takuya Murata)

If a toric degenerations $\xi : \mathfrak{X} \to \mathbb{A}^1$ is proper and $\mathcal{L}$ is an invertible flat $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathcal{L}|_{\xi^{-1}(0)}$ is very ample, then ξ it is an embedded degeneration; i.e. there exists an embedding $\mathfrak{X} \hookrightarrow \mathbb{P}^N \times \mathbb{A}^1$ such that

$$\begin{array}{ccc} \mathfrak{X} & \hookrightarrow & \mathbb{P}^N \times \mathbb{A}^1 \\ & {\scriptstyle \xi} \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

is a commutative diagram.

Embedded toric degenerations

Given an embedded toric degeneration

$$\begin{array}{ccc} \mathfrak{X} & \hookrightarrow & \mathbb{P}^N \times \mathbb{A}^1 \\ & \underset{\xi}{\searrow} & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

we have $\mathfrak{X} = \text{Proj}(\mathfrak{R})$ for some flat Noetherian $k[t]$-algebra $\mathfrak{R}$. Then the generic fibre is

$$\xi^{-1}(1) = X = \text{Proj}(R)$$

where $R := \mathfrak{R}/(t-1)\mathfrak{R}$. Similarly, the special fibre is $\xi^{-1}(0) = X_0 = \text{Proj}(R_0)$ where $R_0 := \mathfrak{R}/t\mathfrak{R}$.

Assumption: X and X_0 are irreducible, so R is a positively graded domains and R_0 is a finitely generated algebra of a graded semigroup with identity.

[KM19]/[Mur20]: May assume $\mathfrak{R}$ is the *Rees algebra of a valuation* on R.

Toric degenerations from valuations

Let $R = \bigoplus_{i \geq 0} R_i$ be a graded k-algebra and domain. A *valuation* on R is a map $\nu : R \setminus \{0\} \to (\mathbb{Z}^d, <)$ such that for all $f, g \in R \setminus \{0\}$ and $c \in k$

$$\nu(fg) = \nu(f) + \nu(g), \quad \nu(cf) = \nu(f), \quad \nu(f + g) \geq \min{}_<\{\nu(f), \nu(g)\}$$

Notice: $S := \mathrm{im}(\nu)$ is a semigroup.

Moreover, ν induces a filtration of R: for every $m \in \mathbb{Z}^d$

$$F_m := \{f \in R : \nu(f) \leq m\} \quad \text{and} \quad F_{<m} := \{f \in R : \nu(f) < m\}.$$

Theorem (Anderson)

Let $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank valuation with finitely generated value semigroup S. Then there exists a toric degeneration of $X = Proj(R)$ with special fibre $X_0 = Proj(k[S])$ defined by the Rees algebra of ν.

A vector space basis $\mathbb{B}$ of R is *adapted to ν* if $\mathbb{B} \cap F_m$ is a vector space basis for all m.

Equations for embedded toric degenerations

The polytope defining the (normalization of the) toric variety Proj($k[S]$) is the *Newton–Okounkov polytope*

$$\Delta(R,\nu) := \overline{\text{conv}\left(\bigcup_{i>0}\{\nu(f)/i : f \in R_i\}\right)}.$$

Hence, we can compute equations for $X_{\Delta(R,\nu)} = \bar{X}_0$ from $\Delta(R,\nu)$.

(proper) abstract toric degeneration	$\rightsquigarrow$	(embedded) toric degeneration by valuation	$\rightsquigarrow$	equations for normalization of X_0

Question: How about equations for X and the family $\mathfrak{X}$?
$\rightsquigarrow$ can be obtained using *Gröbner theory*.

Gröbner degenerations

Let $k = \bar{k}$ with $\mathrm{char}(k) = 0$ and $R = k[x_1, \dots, x_n]/I$ for I homogeneous.

For every $w \in \mathbb{R}^n$ we have the *initial ideal* $\mathrm{in}_w(I) := (\mathrm{in}_w(f) : f \in I)$, for example $\mathrm{in}_{(1,1)}(xy - x^2 + y) = xy - x^2$, and a flat family

$$\xi_w : \mathfrak{X} \to \mathbb{A}^1$$

with generic fibre $\mathrm{Proj}(R)$ and special fibre $\mathrm{Proj}(R_w)$, where $R_w := k[x_1, \dots, x_n]/\mathrm{in}_w(I)$.

Definition

The *Gröbner fan* $\mathrm{GF}(I)$ of I is $\mathbb{R}^n$ with fan structure

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(I) = \mathrm{in}_w(I)$$

The *tropicalization* $\mathcal{T}(I)$ of I is the closed subfan of $\mathrm{GF}(I)$ consisting of those w for which $\mathrm{in}_w(I)$ contains no monomials.

Correspondence Theorem and Corollary

Theorem (L.B.'20, K.Kaveh–C.Manon '19)

Let R be a positively graded algebra and domain, $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ full-rank valuation with finitely generated value semigroup. Then there exists an isomorphism of graded algebras

$$k[x_1, \dots, x_n]/I \cong R$$

such that Anderson's toric variety $Proj(k[S])$ is isomorphic to the toric variety of a Gröbner toric degeneration for some $w \in \mathcal{T}(I)$:

$$Proj(k[S]) \cong Proj(R_w).$$

Corollary

The value semigroup $S \subset \mathbb{Z}^d$ is isomorphic to a semigroup $S' \subset \mathbb{Z}^d_{\geq 0}$.

Example: projected toric degenerations

Example

Consider the toric degeneration

$$\mathfrak{X} = V(y^2z - x^3 - txz^2) \subset \mathbb{P}^2_{x:y:z} \times \mathbb{A}^1_t$$

of the elliptic curve $X = V(y^2z - x^3 - xz^2)$ to the toric variety $X_0 = V(y^2z - x^3)$.
The projection $X \to \mathbb{P}^1$ given by $[x : y : z] \mapsto [y : z]$ composed with the normalization map $\mathbb{P}^1 \to X_0$ defines a projection

$$X \twoheadrightarrow X_0.$$

Algebraically, this correspond to an embedding of the semigroup algebra $R_0 = k[x, y, z]/(y^2z - x^3)$ into $R = k[x, y, z]/(y^2z - x^3 - xz^2)$.

Question: Which (embedded) toric degenerations admit such a projection?

Toric subalgebras

Let $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank valuation with finitely generated semigroup S.

Algebraically, we are looking for an embedding of $k[S]$ as a *toric subalgebra* into R.

Idea: Map the generators of $k[S]$ onto basis elements of R.

Example

In the above example, R has a k-basis $\mathbb{B} = \{x^a y^b z^c : a < 3\}$. The semigroup S defining R_0 is generated by $(1,0),(1,1),(1,3) \subset \mathbb{N} \times \mathbb{Z}$. So we may embed

$$k[S] \hookrightarrow R, \quad \chi^{(m,n)} \mapsto y^m z^n \in \mathbb{B}.$$

This map is neither graded nor finite, so it defines a dominant map

$$\mathrm{Spec}(R) \to \mathrm{Spec}(k[S])$$

$\rightsquigarrow$ these maps are not too hard to find, e.g. in cluster algebras.

Example: cluster algebras

A *cluster algebra* A is a commutative algebra generated recursively by

- *seeds*: maximal algebraically independent sets whose elements are called *cluster variables*, that are related to each other via
- *mutation*: an operation that creates a new seed from a given one by replacing one cluster variable by a binomial with positive coefficients in the other cluster variables.

The monomials in cluster variables of one seed are called *cluster monomials* and they are linearly independent in A.

Cluster algebras and valuations

Proposition (L.B.–M.Cheung–T.Magee–A.Nájera Chávez, H.Oya–N.Fujita)

Let A be a (graded) cluster algebra that satisfies the *full Fock–Goncharov conjecture*. For every seed s there exists a full-rank valuation

$$g_s : A \setminus \{0\} \to \mathbb{Z}^d$$

with finitely generated semigroup. The associated Newton–Okounkov polytope $\Delta(A, g_s)$ is the tropicalization of *Gross–Hacking–Keel–Kontsevich's superpotential* for the associated cluster variety.

The Proposition applies to, for example, Grassmannians, flag varieties, configuration spaces, the del Pezzo surface of degree 5 ...

Cluster algebras and toric degenerations

Corollary (L.B.–T.Murata)

The toric degeneration of $Spec(A)$ to the affine toric variety $\tilde{X}_{\Delta(A,g_s)}$ induced by $g_s : A \setminus \{0\} \to \mathbb{Z}^d$ admits a dominant map

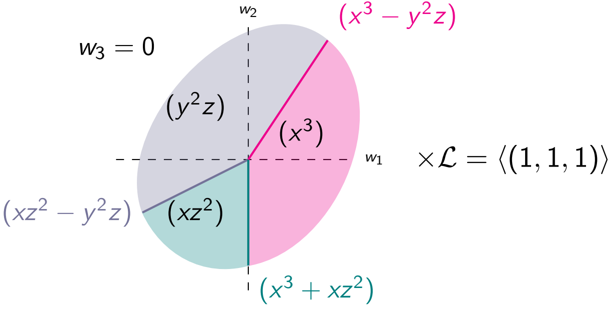
$$Spec(A) \to \tilde{X}_{\Delta(A,g_s)}.$$

Strategy of Proof:

1. By the *Correspondence Corollary* S is isomorphic to a monoid $S' \subset \mathbb{Z}^d_{\geq 0}$.
2. Consider the cluster variables $x_1, \dots, x_d$ of the seed s, so all monomials in $x_1, \dots, x_d$ are *cluster monomials*.
3. Define $k[S] \cong k[S'] \hookrightarrow A$ by $\chi^m \mapsto \chi^{m'} \mapsto x_1^{m'_1} \cdots x_d^{m'_d}$.

Standard monomial bases from the Gröbner fan

Take $I = (x^3 + xz^2 - y^2z) \subset \mathbb{C}[x,y,z]$. Then $GF(I)$ is $\mathbb{R}^3$ with the fan structure below and $\mathcal{T}(I)$ is its 1-skeleton.



It is not hard to see in general that maximal cones $C \in GF(I)$ have monomial initial ideals $\mathrm{in}_C(I)$ and define *standard monomial basis* for R

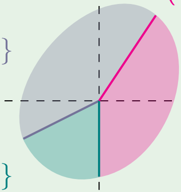
$$\mathbb{B}_C := \{\bar{\mathrm{x}}^m \in R : \mathrm{x}^m \notin \mathrm{in}_C(I)\}.$$

Standard monomial bases for the elliptic curve

Example ($I = (x^3 + xz^2 - y^2z)$)

$(x^3 - y^2z)$

$\mathbb{B}_{(y^2z)} = \{x^a y^b z^c : b < 2 \text{ or } c < 1\}$



$\mathbb{B}_{(x^3)} = \{x^a y^b z^c : a < 3\}$

$\mathbb{B}_{(xz^2)} = \{x^a y^b z^c : a < 1 \text{ or } c < 2\}$

For $R_0 = k[x,y,z]/(x^3 - yz^2)$ we have two *adapted bases* $\mathbb{B}_{(x^3)}$ and $\mathbb{B}_{(yz^2)}$. For every choice of maximal algebraically independent set of generators in R_0 its monomials are standard in $\mathbb{B}_{(x^3)}$ *or* $\mathbb{B}_{(yz^2)}$:

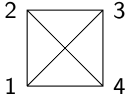
$$\begin{aligned} \{x,y\} &\rightsquigarrow x^a y^b \in \mathbb{B}_{(y^2z)} \quad \text{as } c < 1, \\ \{x,z\} &\rightsquigarrow x^a z^c \in \mathbb{B}_{(y^2z)} \quad \text{as } b < 2, \\ \{y,z\} &\rightsquigarrow y^b z^c \in \mathbb{B}_{(x^3)} \quad \text{as } a < 3. \end{aligned}$$

Standard monomial bases for $\mathrm{Gr}_2(\mathbb{C}^n)$

For the Plücker ideal of $\mathrm{Gr}_2(\mathbb{C}^n)$ we have a correspondence:

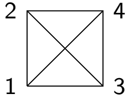
labelling $(i_1, \dots, i_n)$ of the vertices of the n-gon $\rightsquigarrow$ standard monomials have "non-crossing" support

Example: For $(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$ defining $\mathrm{Gr}_2(\mathbb{C}^4)$ we have three standard monomial bases:



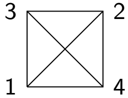
gives $\mathbb{B} = \{\mathrm{p}^m : p_{13}p_{24} \nmid \mathrm{p}^m\}$

$\rightsquigarrow$ *cluster monomials*



gives $\mathbb{B} = \{\mathrm{p}^m : p_{14}p_{23} \nmid \mathrm{p}^m\}$

$\rightsquigarrow$ *Young's standard monomials*



gives $\mathbb{B} = \{\mathrm{p}^m : p_{12}p_{34} \nmid \mathrm{p}^m\}$

Standard monomials and cluster monomials

Theorem (L.B.–F.Mohammadi–A.Nájera Chávez)

1. *The basis of cluster monomials for $Gr_2(\mathbb{C}^n)$ is the standard monomial basis arising from the cyclic labelling of the vertices of the n-gon.*
2. *The basis of cluster monomials for $Gr_3(\mathbb{C}^6)$ is a standard monomial basis of the ideal defining $Gr_3(\mathbb{C}^6)$ in its cluster embedding:*

$$J \subset k[p_{123}, \dots, p_{456}, X, Y]$$

where X, Y are cluster variables of degree 2 in Plücker coordinates and $J \cap k[p_{123}, \dots, p_{456}]$ is the Plücker ideal.

$\rightsquigarrow$ All cluster monomials are *non-standard* in the sense of Young's standard monomials.

From embedded toric degenerations to projections

Conjecture (L.B.–T.Murata)

Given an embedded toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ determined by a full-rank valuation $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ with finitely generated semigroup S there exists an embedding $k[S] \hookrightarrow R$ inducing a projection

$$Proj(R) \twoheadrightarrow Proj(k[S]).$$

Strategy of proof:

1. Use the *Correspondence Theorem*, so that $R = k[x_1, \ldots, x_n]/I$ and there is a cone $\tau \in \mathcal{T}(I)$ with $k[S] = k[x_1, \ldots, x_n]/\mathrm{in}_\tau(I)$;
2. fix a maximal algebraically independent set $s = \{x_{i_1}, \ldots, x_{i_d}\}$ in $k[S]$;
3. refine τ by a term order such that monomials in s are standard;
4. adjust the grading of S and map S onto standard monomials;
5. check that a suitable localization $(k[S]_f)_0 \hookrightarrow R_f$ is finite.

Adjusting the grading of S

Example

Recall $S = \langle (1,0),(1,1),(1,3)\rangle_{\geq 0}$ is graded by $\deg(a,b) = a$, but in R monomials $y^a z^b$ have degree total $a+b$. Se we embed S into a semigroup S' graded by total degree:

Adjusting the grading of S

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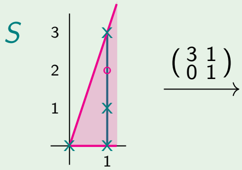

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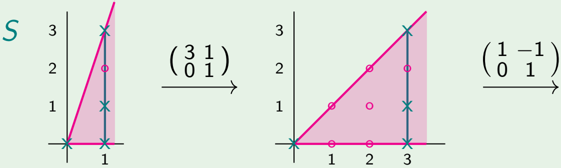

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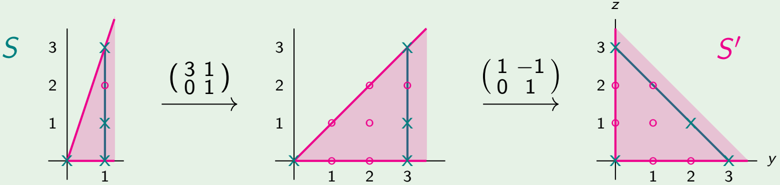

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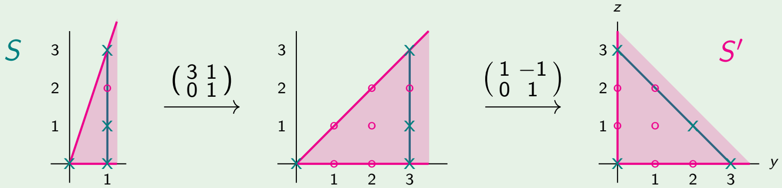

Recall that $y^a z^b \in R$ are standard monomials. So we may embed

$$k[S] \hookrightarrow R \quad \text{by} \quad (1,0) \mapsto y^3, \quad (1,1) \mapsto y^2 z \quad \text{and} \quad (1,3) \mapsto z^3.$$

References

BM Lara Bossinger, Takuya Murata. Embeddings of toric degenerations (working title). *in preparation.* (2021)

BCMN Lara Bossinger, Mandy Cheung, Timothy Magee, Alfredo Nájera Chávez. On cluster duality for Grassmannians. *in preparation.* (2021)

BMN Lara Bossinger, Fatemeh Mohammadi, Alfredo Nájera Chávez. Gröbner degenerations of Grassmannians and universal cluster algebras. *arxiv preprint arXiv:2007.14972 [math.AG]* (2020)

B Lara Bossinger. Full-Rank Valuations and Toric Initial Ideals. *Int. Math. Res. Not.* rnaa071 (2020)

FZ Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

GHKK Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608 (2018)

KM Kiumars Kaveh and Christopher Manon. Khovanskii bases, higher rank valuations, and tropical geometry. *SIAM J.Appl. Algebra Geom.*, 3(2):292–336 (2019)

Murata Takuya Murata. Toric degenerations of projective varieties with an application to equivariant Hilbert functions. *PhD Thesis University of Pittsburgh* (2020)

Scott Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc.* (3) 92 (2006), no. 2, 345–380.

Proof of the Correspondence Theorem

We call a set of algebra generators $b_1, \ldots, b_n$ of R a *Khovanskii basis* for ν if $\nu(b_1), \ldots, \nu(b_n)$ generate S.

Idea of Proof: Choose a finite Khovanskii basis $b_1, \ldots, b_n \in R$. Take

$$\pi : k[x_1, \ldots, x_n] \to R, \quad x_i \mapsto b_i$$

and $I := \ker(\pi)$. Then by [B, Main Theorem] exits $w \in \mathcal{T}(I)$ such that

$$in_w(I) \quad \text{is toric} \quad \Leftrightarrow \quad S \quad \text{is finitely generated.}$$

Moreover, $k[S] \cong k[x_1, \ldots, x_n]/in_w(I)$. ■