Gröbner theory of Grassmannian cluster algebras

Lara Bossinger (jt. Fatemeh Mohammadi and Alfredo Nájera Chávez)

Universidad Nacional Autónoma de México, IM-Oaxaca

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Question: How are different toric degenerations of X related?

Toric degenerations from valuations

$A = \bigoplus_{i\geq 0} A_i$ graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ a valuation with image $S(A, \mathfrak{v})$ a finitely generated semigroup of rank $d := \dim_{\mathrm{Krull}} A$.

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[Anderson] Exists a toric degeneration of $Proj(A)$ with special fibre a projective toric variety whose normalization is $TV(\Delta(A, \mathfrak{v}))$, where

$$\Delta(A, \mathfrak{v}) := \overline{\mathrm{conv}\left(\bigcup_{i\geq 1}\left\{\frac{\mathfrak{v}(f)}{i} : f \in A_i\right\}\right)}$$

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A set $\{b_1, \ldots, b_n\} \subset A$ of algebra generators is a *Khovanskii basis* for $\mathfrak{v}$ if $\mathfrak{v}(b_1), \ldots, \mathfrak{v}(b_n)$ generate *image*$(\mathfrak{v})$.

Gröbner toric degenerations

Reminder: $f = x^2 + y \in \mathbb{C}[x, y]$ and $w = (1, 1)$, then $in_w(f) = y$ and for $J \subset \mathbb{C}[x_1, \ldots, x_n]$ ideal $in_w(J) := (in_w(f) : f \in J)$

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Let $A := \mathbb{C}[x_1, \dots, x_n]/J$ with J homogeneous prime ideal and $w \in Trop(J)$ such that $in_w(J)$ is binomial and prime (i.e. *toric*).

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Then exists a flat family with generic fibre $Proj(A)$ and special fibre the toric variety $Proj(\mathbb{C}[x_1,\dots,x_n]/in_w(J))$, called a *Gröbner toric degeneration*.

Motivating result

Theorem (Kaveh–Manon, B.)

Let A be a positively graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ full rank valuation with finitely generated value semigroup.

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Let A be a positively graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ full rank valuation with finitely generated value semigroup. Then there exists an isomorphism of graded algebras

$$\mathbb{C}[x_1, \dots, x_n]/J \cong A$$

such that the toric variety of the Newton–Okounkov polytope is isomorphic to the toric variety of a Gröbner toric degeneration for some $w \in Trop(J)$:

$$TV(\Delta(A, \mathfrak{v})) \cong Proj(\mathbb{C}[x_1, \dots, x_n]/in_w(J))^{nor}.$$

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Idea of Proof: Choose a finite Khovanskii basis $b_1, \ldots, b_n \in A$. Take

$$\pi : \mathbb{C}[x_1, \ldots, x_n] \to A, \quad x_i \mapsto b_i$$

and $J := \ker(\pi)$.

[1] $w_{\mathfrak{v}}$ is obtained from $M_{\mathfrak{v}} := (\mathfrak{v}(b_i))_{i \in [n]} \in \mathbb{Z}^{d \times n}$ by an *order preserving projection* $e : \mathbb{Z}^d \to \mathbb{Z}$, i.e. $w_{\mathfrak{v}} := e(M_{\mathfrak{v}})$ and $\mathrm{in}_{w_{\mathfrak{v}}}(J) = \mathrm{in}_{M_{\mathfrak{v}}}(J)$.

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$$in_{w_{\mathfrak{v}}}(J) \quad \text{is toric} \quad \Leftrightarrow \quad S(A, \mathfrak{v}) \quad \text{is finitely generated.}$$

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Recall: $\mathfrak{v}$ defines a filtration on A: $F_{m;\mathfrak{v}} := \{f \in A : \mathfrak{v}(f) \leq m\}$ for all $m \in \mathbb{Z}^d$ and $\leq$ a fixed total order. A vector space basis $\mathbb{B}$ of A is *adapted* to $\mathfrak{v}$ if $\mathbb{B} \cap F_{m;\mathfrak{v}}$ is a vector space basis for each $F_{m;\mathfrak{v}}$.

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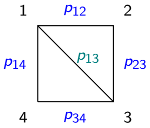

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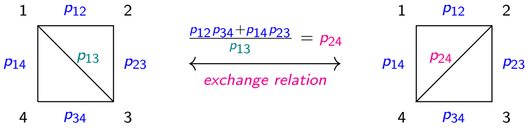

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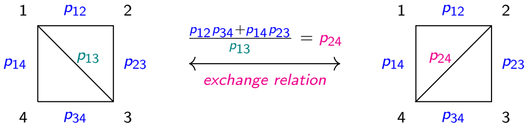

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Application: Toric degenerations via cluster algebras

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$\rightsquigarrow$ All these degenerations *share the ϑ-basis*, i.e. $A_s^{\mathrm{prin}} = \bigoplus_{\vartheta \in \Theta} \vartheta$ for all s.

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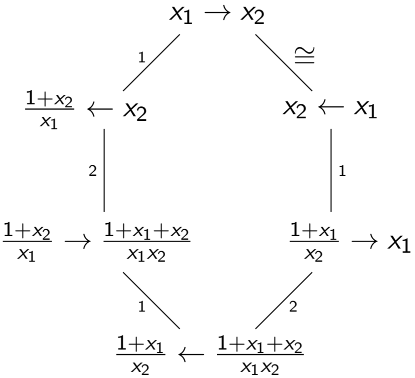
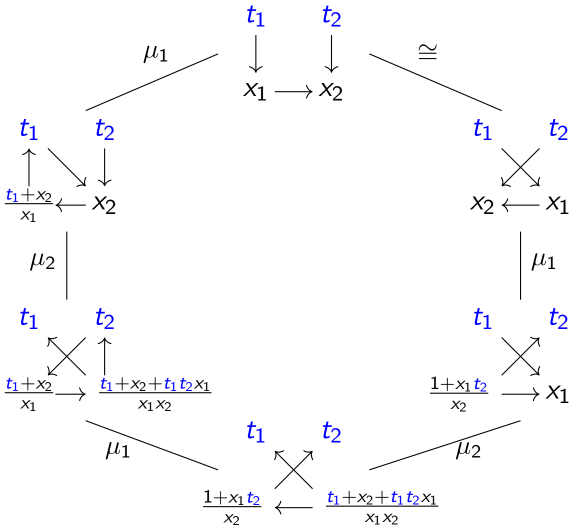


$$A_{x_1 \to x_2} = \left\langle x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\rangle \subset \mathbb{C}(x_1, x_2).$$

Example: principal coefficients



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- ⊕ A^{univ} knows all toric degenerations $X_{s,0}$,
- ⊖ A^{univ} is defined only *recursively*.

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For all seeds s of $A_{k,n}$ exists a full rank valuation $g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$ and a basis called ϑ-basis adapted to all of them simultaneously. The *cluster algebra with principal coefficients at s* $A_{k,n}^{\mathrm{prin},s}$ is a flat $\mathbb{C}[t_x : x \in s_{\mathrm{mut}}]$-algebra defining the toric degeneration.

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[Fomin–Zelevinsky]/[Reading]: $\exists$ flat $\mathbb{C}[t_x : x \text{ m.c.v.}]$-algebra $A_{k,n}^{\text{univ}}$ and projections $pr_s : \mathbb{C}[t_x : x \text{ m.c.v}] \to \mathbb{C}[t_x : x \in s_{\text{mut}}]$ for all seeds s that extend to

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For each seed s we can apply the Motivating Theorem and get an ideal J_s and a Gröbner toric degeneration of J_s corresponding to $A_{k,n}^{\text{prin},s}$.

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For each seed s we can apply the Motivating Theorem and get an ideal J_s and a Gröbner toric degeneration of J_s corresponding to $A_{k,n}^{\text{prin},s}$.

Question: How are different J_s related and what is $A_{k,n}^{\text{univ}}$ in this context?

Gröbner fan and standard monomial bases

Definition/Proposition (Mora–Robbiano)

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(J) = \mathrm{in}_w(J).$$

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$$\mathbb{B}_{C,\tau} := \{\bar{\mathbf{x}}^\alpha \in A_\tau \mid \mathbf{x}^\alpha \notin \text{in}_C(J)\}.$$

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Then $\mathbb{B}_{C,\tau}$ is a vector space basis for A_τ called *standard monomial basis*.

Example

Take $J = (x_1^2 x_2^2 + x_1^4 - x_2 x_3^3) \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(J)$ is $\mathbb{R}^3$ with the fan structure:

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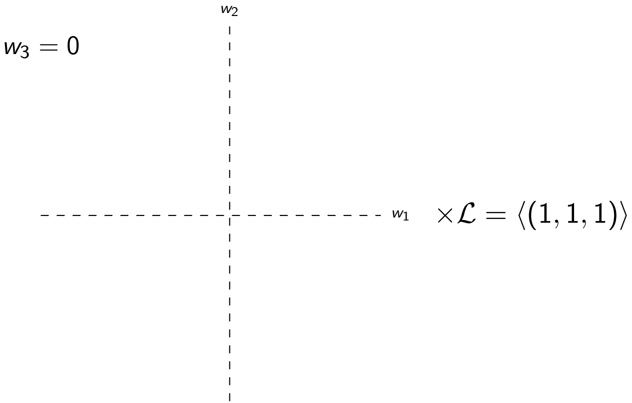

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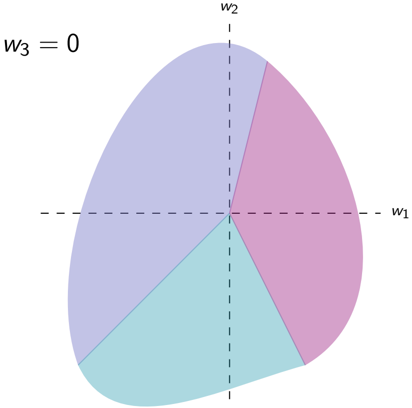

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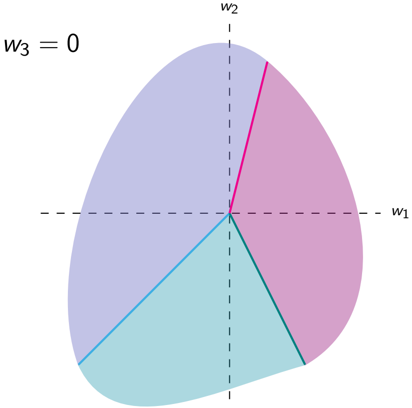

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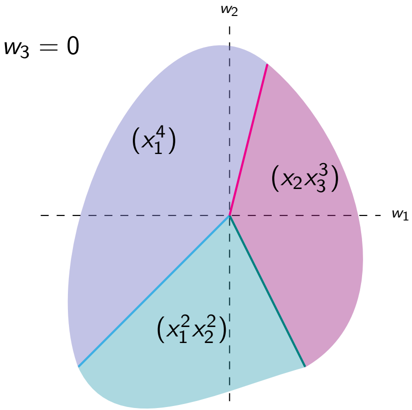

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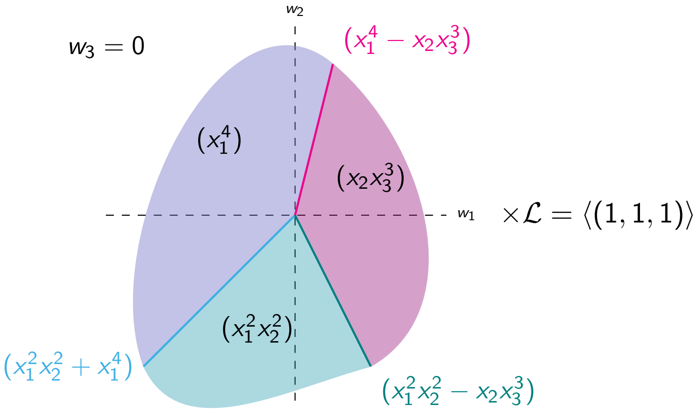

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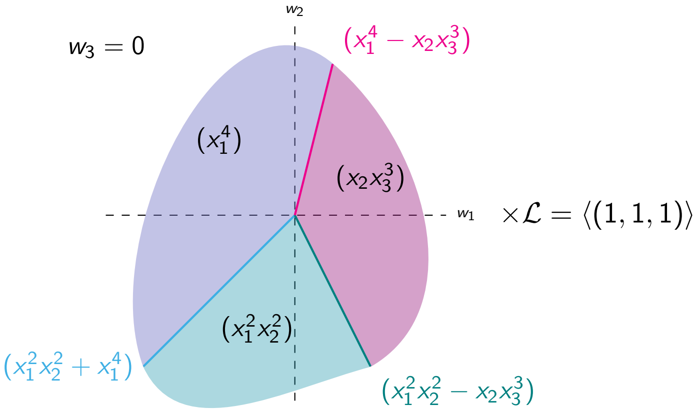

E.g. $\mathbb{B}_{\langle r_1, r_2\rangle} = \{\bar{\mathbf{x}}^a : x_2x_3^3 \nmid \mathbf{x}^a\}$ gives a basis for A, A_{r_1}, A_{r_2} and $A_{\langle r_1, r_2\rangle}$.

Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \dots, r_m$ representatives of primitive ray generators of $\overline{C} \in \mathrm{GF}(J)/\mathcal{L}$. Let $\mathbf{r}$ be the matrix with rows $r_1, \dots, r_m$.

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$$\mu(f) := (\min_{c_\alpha \neq 0} \{r_1 \cdot \alpha\}, \ldots, \min_{c_\alpha \neq 0} \{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

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In $\mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ the *lift* of f is

$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \dots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n)\mathbf{t}^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{\mathbf{r}\cdot\alpha - \mu(f)}.$$

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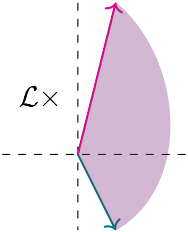
Definition/Proposition

The *lifted ideal* $\tilde{J} := (\tilde{f} : f \in J) \subset \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ is generated by $\{\tilde{g} : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a *Gröbner basis* for J and C.

Example

Take $f = x_1^2x_2^2 + x_1^4 + x_2x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF((f))$ the maximal cone C spanned by the rows of $r := \begin{pmatrix} 1 & 4 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ and $\mathcal{L}$.



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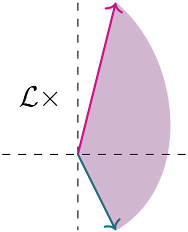

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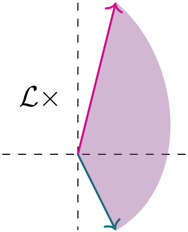

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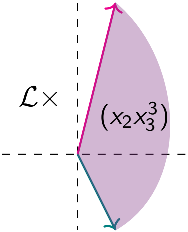

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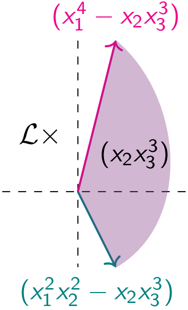

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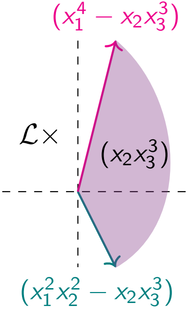

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Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$ and recall $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\mathrm{in}_\tau(J)$.

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$\tilde{A}$ is a free $\mathbb{C}[t_1, \dots, t_m]$-module with basis $\mathbb{B}_C$ and so the morphism

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The cluster variables of $A_{3,6}$ are Plücker coordinates and two more, so

$$A_{3,6} \cong \mathbb{C}[X, Y, p_{123}, \dots, p_{456}]/J_{3,6}$$

and $J_{3,6} \cap \mathbb{C}[p_{123}, \dots, p_{456}]$ is the Plücker ideal $I_{3,6}$.

A minimal generating set of $J_{3,6} \subset \mathbb{C}[p_{123}, \dots, p_{456}, X, Y]$:

$$p_{145}p_{236} - p_{123}p_{456} - X, \qquad p_{124}p_{356} - p_{123}p_{456} - Y,$$
$$p_{136}p_{245} - p_{126}p_{345} - X, \qquad p_{125}p_{346} - p_{126}p_{345} - Y,$$
$$p_{146}p_{235} - p_{156}p_{234} - X, \qquad p_{134}p_{256} - p_{156}p_{234} - Y,$$
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Totally positive ideals

Reminder: $J \subset \mathbb{R}[x_1, \ldots, x_n]$ is *totally positive* if $J \cap \mathbb{R}_{\geq 0}[x_1, \ldots, x_n] = \varnothing$.
The *totally positive part of Trop*(J) is

$$Trop^+(J) := \{w \in Trop(J) : in_w(J) \text{ totally positive}\}.$$

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Totally positive ideals

Reminder: $J \subset \mathbb{R}[x_1, \ldots, x_n]$ is *totally positive* if $J \cap \mathbb{R}_{\geq 0}[x_1, \ldots, x_n] = \varnothing$.
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Hence, $Trop^+(J) \subset Trop(J) \subset GF(J)$ are *closed subfans*.

Example: The initial ideal $(x_1^2x_2^2 + x_1^4)$ is not totally positive, but $(x_1^2x_2^2 - x_2x_3^3)$ and $(x_1^4 - x_2x_3^3)$ are.

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$$\text{rays of } Trop^+(J_{k,n}) \quad \overset{1:1}{\longleftrightarrow} \quad \text{cluster variables}$$

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[Ilten–Nájera Chávez–Treffinger]: generalized (1) for graded cluster algebras of finte type and (2)/(3) for ADE types.

The reduced Gröbner basis of $J_{3,6}$ for C consists contains the above minimal generating set and additionally the following elements:

$$
\begin{array}{ll}
p_{235}Y - p_{125}p_{234}p_{356} - p_{123}p_{256}p_{345}, & p_{134}X - p_{136}p_{145}p_{234} - p_{123}p_{146}p_{345}, \\
p_{146}Y - p_{124}p_{156}p_{346} - p_{126}p_{134}p_{456}, & p_{256}X - p_{156}p_{236}p_{245} - p_{126}p_{235}p_{456}, \\
p_{136}Y - p_{123}p_{156}p_{346} - p_{126}p_{134}p_{356}, & p_{346}X - p_{136}p_{234}p_{456} - p_{146}p_{236}p_{345}, \\
p_{245}Y - p_{125}p_{234}p_{456} - p_{124}p_{256}p_{345}, & p_{125}X - p_{123}p_{156}p_{245} - p_{126}p_{145}p_{235}, \\
p_{145}Y - p_{125}p_{134}p_{456} - p_{124}p_{156}p_{345}, & p_{124}X - p_{126}p_{145}p_{234} - p_{123}p_{146}p_{245}, \\
p_{236}Y - p_{126}p_{234}p_{356} - p_{123}p_{256}p_{346}, & p_{356}X - p_{136}p_{235}p_{456} - p_{156}p_{236}p_{345}, \\
p_{135}Y - p_{125}p_{134}p_{356} - p_{123}p_{156}p_{345}, & p_{135}X - p_{136}p_{145}p_{235} - p_{123}p_{156}p_{345}, \\
p_{246}Y - p_{124}p_{256}p_{346} - p_{126}p_{234}p_{456}, & p_{246}X - p_{146}p_{236}p_{245} - p_{126}p_{234}p_{456}.
\end{array}
$$

$$g = XY - p_{123}p_{156}p_{246}p_{345} - p_{126}p_{135}p_{234}p_{456} - p_{126}p_{156}p_{234}p_{345} - p_{123}p_{156}p_{234}p_{456} - p_{123}p_{126}p_{345}p_{456}.$$

The first monomial of each relation lies in $in_C(J_{3,6})$.

Arbitrary Grassmannians (in progress)

[GHKK 18]/[Fujita–Oya 20]/[B.–Cheung–Magee–Nájera Chávez]:

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toric degenerations of $\mathrm{Gr}(k, n)$ induced by valuations on $A_{k,n}$	$\longleftrightarrow$	maximal cones in $\mathrm{Trop}(J)$ for some J $A_{k,n} \cong k[x_1, \ldots, x_m]/J$

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Conjecture (B.)

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Conjecture (B.)

For every seed s of $A_{k,n}$ exists a maximal prime cone τ_s in $\mathrm{Trop}^+(J)$ for an appropriate ideal J with $A_{k,n} \cong k[x_1, \dots, x_m]/J$, s.t. if J is appropriate for two adjacent seeds s, s' then τ_s and $\tau_{s'}$ share a facet.

Remark: J is *appropriate* for s if $S(A_{k,n}, g_s) = \langle g_s(\bar{x}_1), \dots, g_s(\bar{x}_m) \rangle$.

Examples

1. For every seed s of $A_{2,n}$

$$S(A_{2,n}, g_s) = \langle g_s(p_{ij}) : 1 \leq i < j \leq n \rangle,$$

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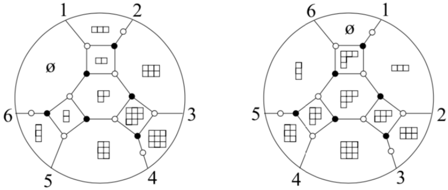

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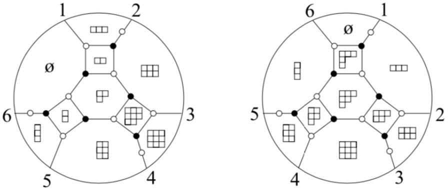

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3. Conjecturally, the ideal J presenting a finite type cluster algebra A w.r.t all cluster variables is appropriate for all seeds.

Thank you!

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Combinatorics of $C \in GF(J_{3,6})$

Let $\{e_{123}, \dots, e_{456}, e_x, e_y\}$ be the standard basis of $\mathbb{R}^{22}$ and

$$E_i := \sum_{k,j \neq i} e_{ijk} + e_x + e_y.$$

The *lineality space* of $GF(J_{3,6})$ is $\mathcal{L} = \langle E_1, \dots, E_6 \rangle$. Let $f_{i,j} := \sum_{k \notin \{i,j\}} e_{ijk}$ and $\pi : \mathbb{R}^{22} \to \mathbb{R}^{20}$ away from e_x, e_y.

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#	rays of $C/\mathcal{L}$	projections
6	$a_i := e_{i,i+1,i+2}$	$e_{i,i+1,i+2}$
6	$b_i := f_{i,i+1} + \delta_{i \text{ odd}} e_y + \delta_{i \text{ even}} e_x$	$f_{i,i+1}$
2	$c_i^- := b_i + e_{i-2,i-1,i} + e_{i-2,i-1,i+1}$	$g_{i,i+1,i+2,i-3,i-2,i-1}$
2	$c_i^+ := b_i + e_{i,i+2,i+3} + e_{i+1,i+2,i+3}$	$g_{i+2,i+1,i,i-3,i-2,i-1}$

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Notice: $c_i^\pm = c_j^\pm mod \mathcal{L}$ if $i = j \mod 2$ and
$g_{i,i+1,i+2,i-3,i-2,i-1} + g_{i+2,i+1,i,i-3,i-2,i-1} = f_{i+1,i+2} + f_{i-1,i} + f_{i-2,i-3}$

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The ideal $J_{3,6}$ is invariant under the action of $G := \langle (123456), w_0 \rangle \subset \mathfrak{S}_6$, so $GF(J_{3,6})$ has an induced *G-action*.

[3] FFFGG is a bipyramid in $\text{Trop}(I_{3,6})$ and each G-orbit maps onto one of the pyramids.

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#	G-orbit of max cone in $C \cap Trop(J_{3,6})$	type of projection	#
18	$\{a_i, a_{i-2}, b_{i-2}, b_{i+3}\}$	EEFF	180
12	$\{a_i, c_i^{\pm}, b_{i-2}, b_{i+4}\}$	EFFG	180
12	$\{a_i, a_{i+2}, b_i, c_i^{+}\}$ or $\{a_i, a_{i-2}, b_{i+1}, c_{i+1}^{-}\}$	EEFG	360
4	$\{a_{i-2}, a_i, a_{i+2}, c_i^{\pm}\}$	EEEG	240
4	$\{b_{i-2}, b_i, b_{i+2}, c_i^{\pm}\}$	FFFGG[3]	15

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The type of projected cone refers to the $\mathfrak{S}_6$-orbits in Trop$(I_{3,6})$, respectively Trop$^{+}(I_{3,6})$, as used [SS04]&[BCL17], the number is the number of maximal cones in Trop$(I_{3,6})$ of this type.

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