Toric degenerations from Newton-Okounkov bodies

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Why toric degenerations?

Toric varieties are nice:

$$\left\{\begin{array}{c}\text{algebraic and}\\ \text{geometric}\\ \text{properties}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{combinatorial}\\ \text{properties}\\ \text{of polytope, fan}\end{array}\right\}$$

$\rightsquigarrow$ get close by toric degenerations:

$$X \text{ and } Y \text{ fibres in flat family } \pi : \mathcal{X} \to \mathbb{A}^n.$$

many applications:
mirror symmetry, algebraic statistics, symplectic geometry ...
$\rightsquigarrow$ each needs certain properties

Motivation: Example

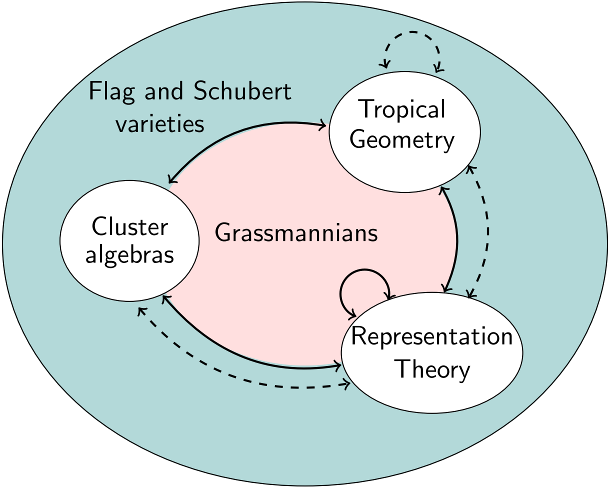
Example

$$\mathrm{Gr}(2, \mathbb{C}^4) = \{\text{2-dimensional subspaces of } \mathbb{C}^4\},$$

$$I = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle \subset \mathbb{C}[p_{12}, \ldots, p_{34}],$$

$$I_t = \langle p_{12}p_{34} - p_{13}p_{24} + t p_{14}p_{23} \rangle \subset \mathbb{C}[t][p_{12}, \ldots, p_{34}],$$

Various constructions of toric degenerations



Valuation and Newton-Okounkov body

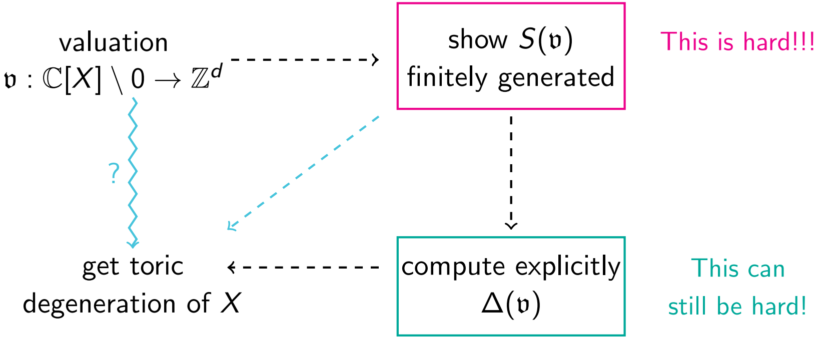
$$X^d = V(I) \hookrightarrow \mathbb{P}^{n-1} \qquad \rightsquigarrow \qquad \mathbb{C}[X] = \mathbb{C}[x_1,\dots,x_n]/I$$

- $\mathfrak{v} : \mathbb{C}[X] \setminus \{0\} \to (\mathbb{Z}^d, \prec)$:

$$\begin{aligned} \mathfrak{v}(fg) &= \mathfrak{v}(f) + \mathfrak{v}(g), \\ \mathfrak{v}(f+g) &\succeq \min\{\mathfrak{v}(f), \mathfrak{v}(g)\}, \\ \mathfrak{v}(cf) &= \mathfrak{v}(f). \end{aligned}$$

- image $S(\mathfrak{v})$
- $\Delta(\mathfrak{v}) := \overline{\mathrm{Conv} \bigcup_{k\geq 0} \{\mathfrak{v}(f)/k \mid f \in \mathbb{C}[X]_k\}}$.

Strategy


valuation
$\mathfrak{v} : \mathbb{C}[X] \setminus 0 \to \mathbb{Z}^d$
show $S(\mathfrak{v})$ finitely generated
This is hard!!!
?
get toric degeneration of X
compute explicitly $\Delta(\mathfrak{v})$
This can still be hard!

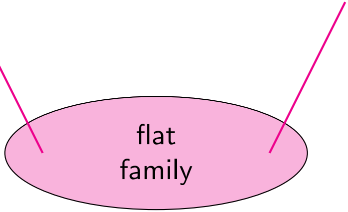
(Higher) Gröbner degenerations

$$X = V(I) \qquad \overset{M \in \mathbb{Z}^{d\times n}}{\rightsquigarrow} \qquad V(\mathrm{in}_M(I))$$

$$I \subseteq \mathbb{C}[x_1, \dots, x_n] \qquad \mathrm{in}_M(I) \subseteq \mathbb{C}[x_1, \dots, x_n]$$

$$f = a_1\mathbf{x}^{n_1} + \cdots + a_s\mathbf{x}^{n_s} \qquad \mathrm{in}_M(f) = a_{i_1}\mathbf{x}^{n_{i_1}} + \cdots + a_{i_r}\mathbf{x}^{n_{i_r}}$$

$$1 \le i_1 < \cdots < i_r \le s$$



But: in general not toric!

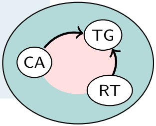
Machinery

valuation $\mathfrak{v}$ on $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I$ $\rightsquigarrow$ $M_{\mathfrak{v}} \in \mathbb{Z}^{d\times n}$ columns $\mathfrak{v}(\bar{x}_i)$ $\rightsquigarrow$ $\text{in}_{M_{\mathfrak{v}}}(I) \subset \mathbb{C}[x_1, \ldots, x_n]$

Theorem 1 (B.)

If $\text{in}_{M_{\mathfrak{v}}}(I)$ is prime:

- $S(\mathfrak{v})$ is finitely generated,
- $\Delta(\mathfrak{v}) = \text{conv}(\mathfrak{v}(\bar{x}_i)_i)$.



Representation Theory

X variety with group action

$\rightsquigarrow$ use action to define valuation

$\rightsquigarrow$ birational sequences

Problems:

1. $S(\mathfrak{v})$ finitely generated?
2. Compute $\Delta(\mathfrak{v})$?

Valuation from fixing coordinates

suppose $\psi : \mathbb{C}(\mathrm{Gr}(k, \mathbb{C}^n)) \cong \mathbb{C}(y_1, \dots, y_d)$, want

$$\mathfrak{v} : \mathbb{C}[\mathrm{Gr}(k, \mathbb{C}^n)] \setminus \{0\} \to (\mathbb{Z}^d, \prec)$$

1. $f = \sum_{\mathbf{m} \in \mathbb{N}^\mathbf{d}} a_\mathbf{m} \mathbf{y}^\mathbf{m} \neq 0$

$$\mathfrak{v}(f) := \min_{\prec} \{\mathbf{m} \mid a_\mathbf{m} \neq 0\}$$

2. $f/g \in \mathbb{C}(y_1, \dots, y_d) \setminus 0$

$$\mathfrak{v}(f/g) := \mathfrak{v}(f) - \mathfrak{v}(g)$$

3. $p \in \mathbb{C}[\mathrm{Gr}(k, \mathbb{C}^n)] \setminus 0$

$$\mathfrak{v}(p) := \mathfrak{v}(\psi(p))$$

$\rightsquigarrow S(\mathfrak{v})$ finitely generated? $\Delta(\mathfrak{v})$?

Birational sequences for $\mathrm{Gr}(k, \mathbb{C}^n)$

$\{\epsilon_i\}_i$ basis $\mathbb{R}^n$, $\epsilon_i - \epsilon_j$ for $i < j$,

$$U^- := \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix}, U_k^- := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}, U_{(i,j)}^- := \mathrm{id} + yE_{(j,i)}$$

Definition (Fang-Fourier-Littelmann)

$S = (\epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_d} - \epsilon_{j_d})$ is *birational*, if image of

$$\psi_S := \mathrm{mult} : U_{(i_1,j_1)}^- \times \cdots \times U_{(i_d,j_d)}^- \to U^-$$

is birational to U_k^-.

Problem: very few S known

(Non-)examples for $\mathrm{Gr}(2, \mathbb{C}^4)$

1. Non-Example $S = (\epsilon_1 - \epsilon_4, \epsilon_1 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y_4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_3 & y_4 & 1 & 0 \\ y_1+y_2 & 0 & 0 & 1 \end{pmatrix} \lightning$$

2. Example: $S = (\epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y_4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_3 & y_4 & 1 & 0 \\ y_1+y_2y_3 & y_2y_4 & y_2 & 1 \end{pmatrix}$$

Compute $\mathfrak{v}_S$ for $\mathrm{Gr}(2, \mathbb{C}^n)$

$$V(\omega_2) = \bigwedge^2 \mathbb{C}^n = U(\mathfrak{n}^-).(e_1 \wedge e_2), \; p_{ij} = (e_i \wedge e_j)^*$$

Theorem[FFL]: $\mathfrak{v}_S(p_{ij}) = \min_{\prec}\{\mathbf{m} \in \mathbb{N}^d \mid \mathbf{f}^{\mathbf{m}}.(e_1 \wedge e_2) = e_i \wedge e_j\}$

Example

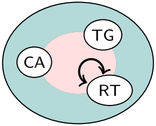
$S = (\epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3)$

$\mathfrak{v}_S$	$\epsilon_1 - \epsilon_4$	$\epsilon_3 - \epsilon_4$	$\epsilon_1 - \epsilon_3$	$\epsilon_2 - \epsilon_3$
p_{12}	0	0	0	0
p_{13}	0	0	0	1
p_{23}	0	0	1	0
p_{14}	0	1	0	1
p_{24}	1	0	0	0
p_{34}	1	0	0	1

$M_S = (\mathfrak{v}_S(p_{ij})) \in \mathbb{Z}^{4\times 6}$

Results for $\mathrm{Gr}(2, \mathbb{C}^n)$

New class: iterated (birational) sequences



Proposition (B.)

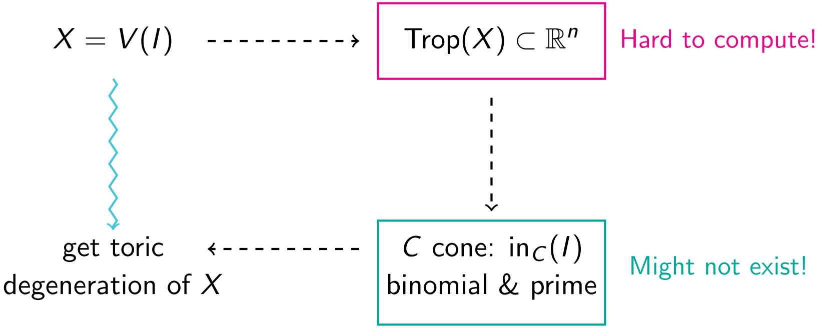
S iterated sequence for $\mathrm{Gr}(2, \mathbb{C}^n)$, then

$$\mathrm{in}_{M_{\mathfrak{v}_S}}(I) \text{ is prime.}$$

In particular,

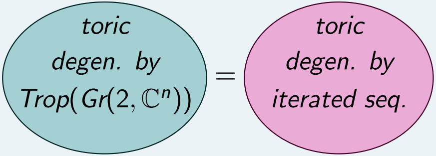
1. $S(\mathfrak{v}_S)$ finitely generated!
2. $\Delta(\mathfrak{v}_S) = \mathrm{conv}(\mathfrak{v}_S(p_{ij})_{i,j})$!

Tropical Geometry

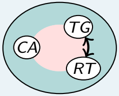


Theorem (B.)

For $Gr(2,\mathbb{C}^n)$ we have



up to isomorphism.



Algorithm:

iterated sequences $S \rightsquigarrow$ maximal cone $C_S \subset \mathrm{Trop}(\mathrm{Gr}(2,\mathbb{C}^n))$

Cluster Algebras

Commutative ring with

1. seeds
2. mutation

Example

$\mathbb{C}[\mathrm{Gr}(2,\mathbb{C}^4)]$ is cluster algebra:

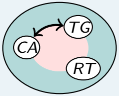
$$\left\{\begin{matrix} p_{12}, p_{23}, p_{34} \\ p_{14}, p_{13} \end{matrix}\right\} \xleftrightarrow{\frac{p_{12}p_{34}+p_{14}p_{23}}{p_{13}} = p_{24}} \left\{\begin{matrix} p_{12}, p_{23}, p_{34} \\ p_{14}, p_{24} \end{matrix}\right\}$$

Cluster Algebra $\mathbb{C}[Gr(k, \mathbb{C}^n)]$

fix seed $\mathcal{P}$ $\rightsquigarrow$ valuation $\mathfrak{v}_{\mathcal{P}}$ $\rightsquigarrow$ $S(\mathfrak{v}_{\mathcal{P}})$ finitely generated? Compute $\Delta(\mathfrak{v}_{\mathcal{P}})$?

Theorem (B. and B.-Fang-Fourier-Hering-Lanini)

For $Gr(2, \mathbb{C}^n)$ we have $in_{M_{\mathfrak{v}_{\mathcal{P}}}}(I)$ *is prime* $\forall$ $\mathcal{P}$.



In particular,

1. $S(\mathfrak{v}_{\mathcal{P}})$ *finitely generated,*
2. $\Delta(\mathfrak{v}_{\mathcal{P}}) = conv(\mathfrak{v}_{\mathcal{P}}(p_{ij})_{i,j})$.

Thank you!

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