Some problems concerning curves in projective three space

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Abstract. These are the lecture notes of the seminar talk delivered by Robin Hartshorne at the "Seminario nacional de geometría algebraica" in México, October 13th 2021. In the talk, he discussed some open problems concerning curves, together with examples, and ideas about how to study the problems.

1. Perrin's problem

Problem 1.1. [**Pe87**] Given a family of curves in $\mathbb{P}^3$, how many points in general position can we assign to a curve in the family?

Let us fix a degree d and a genus g. Given an irreducible component $\mathscr{C}$ of the Hilbert scheme of degree d and genus g curves $\mathscr{H}_{d,g}$, we can rephrase the problem above as follows: find the maximum number $m = m(\mathscr{C})$ such that m general points in $\mathbb{P}^3$ are contained in an element of $\mathscr{C}$.

The analogous problem in $\mathbb{P}^2$ has a simple answer. Plane curves $C \subseteq \mathbb{P}^2$ of degree d form a linear system $|C|$ of dimension

$$\dim\, H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \binom{d+2}{2}$$

from which we conclude that

$$m(|C|) = \dim\, |C| = \frac{d(d+3)}{2}.$$

The situation in $\mathbb{P}^3$ is more delicate. Let us work out an example.

Example 1.2. If $\mathscr{C} \subseteq \mathscr{H}_{3,0}$ is the component parametrizing twisted cubic curves, then $m(\mathscr{C}) = 6$.

Proof. The dimension of the componenent $\mathscr{C}$ is 12; and each point imposes 2 conditions in the curves of $\mathscr{C}$. Therefore, $m(\mathscr{C}) \leq 6$.

Now consider 6 general points $p_1, \ldots, p_6 \in \mathbb{P}^3$. Take a general point p_7 in the line L passing through p_1 and p_2. The linear system $\mathcal{Q}$ of quadric surfaces containing $p_1, \ldots, p_7$ has dimension 2. If we take $Q_1, Q_2 \in \mathcal{Q}$ then the intersection $Q_1 \cap Q_2$ is the union of L and a twisted cubic curve C which contains p_3, p_4, p_5, p_6.

Claim: We can choose Q_1 and Q_2 so that $p_1, \ldots, p_6 \in C$. □

EXAMPLE 1.3. Consider the Hilbert scheme $\mathscr{H}_{9,10}$. This scheme has two irreducible components:

1) The family $\mathscr{H}'$ consisting of complete intersections of 2 cubic surfaces, of dimension 36.
2) The family $\mathscr{H}''$ consisting of curves of bi-degree $(3,6)$ on a nonsingular quadric surface, of dimension 36.

It turns out that for the first family, $m(\mathscr{H}') = 18$ is the expected number. However, 9 general points are contained in a unique quadric surface, which implies that $m(\mathscr{H}'') = 9$. This exhibits that the number m can vary depending on the component of $\mathscr{H}_{d,g}$.

In general Problem 1.1 is still open, even for families of ACM curves. Daniel Perrin studied this problem in [**Pe87**] and solved it in some cases. One of such cases, somewhat unintuituve, considers a cohomological property of the normal bundle.

PROPOSITION 1.4. [**Pe87**, Prop. 5.6, Cor. 5.7] *Let $C \in \mathscr{H}_{d,g}$ be a smooth and irreducible curve. If the normal bundle of the curve $N = N_{C/\mathbb{P}^3}$ satisfies*

$$H^0(C, N(-2)) = 0,$$

then:

a) $H^1(N) = 0$,
b) the component $\mathscr{H}_C \subset \mathscr{H}$ containing C has the expected dimension $4d$,
c) $m(\mathscr{H}_C) = 2d$.

SKETCH OF THE PROOF: If $Z \subset C$ is the divisor cut out by a general quadric surface, we get an exact sequence

$$0 \to N(-2) \to N \to N_Z \to 0.$$

If $H^0(C, N(-2)) = 0$ then the deformations of C surject onto the deformations of Z, which means that one can move the $2d$ points Z in general directions and there will be a deformation of C following them. □

In proving the previous result, Perrin cites a result by Ellia which needs the following result.

THEOREM 1.5 (Kleppe). *There is an isomorphism*

$$H^1(C, N_{C/\mathbb{P}^3}) = Ext^2(\mathcal{I}_C, \mathcal{I}_C).$$

The previous result by Kleppe is not obvious. However, in the case of ACM curves of degree 6 and genus 3 and elementary proof of this result can be worked out. These curves are included in the following.

PROPOSITION 1.6. [**Pe87**, Prop. 5.11.bis] *Let s be a positive integer. If C is a smooth curve admitting a resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-s-1)^s \to \mathcal{O}_{\mathbb{P}^3}(-s)^{s+1} \to \mathcal{I}_C \to 0$$

then

$$H^0(C, N_{C/\mathbb{P}^3}(-2)) = 0.$$

QUESTION 1.7. The number $m(\mathscr{H}_C)$ is subject to two restrictions:

(a) $m(\mathscr{H}_C)$ is less than half the dimension of $\mathscr{H}_C$; and
(b) $m(\mathscr{H}_C)$ is restricted by the maximum number of general points that a surface (containing C) of low degree can pass through.

Are these the only restrictions?

It would be interesting to answer Question 1.7 in the case of ACM curves.

Definition 1.8. (Joe Harris) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible curve of degree d and genus g. We denote the component of the Hilbert scheme $\mathscr{H}_{d,g}$ that contains C by $\mathscr{H}_C$ and we say that $\mathscr{H}_C$ parametrizes *flexible* curves if

1) $\dim \mathscr{H}_C = 4d$, and
2) $m(\mathscr{H}_C) = 2d$.

Example 1.9. Let C be a curve of degree $d = 12$ and genus $g = 17$ with the following pure quadratic minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-6)^2 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{I}_C \to 0.$$

It follows that $h^0(N) = 48$ and also that $\mathscr{H}_C$ is generically smooth. Notice that $h^0(N(-2)) = 6$. Are the curves of $\mathscr{H}_C$ flexible?

2. Normal bundle of space curves

Problem 2.1. Let $C \subset \mathbb{P}^3$ be a smooth and irreducible curve and $N = N_{C/\mathbb{P}^3}$ be its a normal bundle. When is N (semi-)stable? When is $H^1(C, N(-2)) = 0$?

The normal bundle N of a (locally complete intersection) curve C is important for many reasons. For example, $H^0(C, N)$ can be identified with the Zariski tangent space at $[C]$ of the Hilbert scheme $\mathscr{H}_C$; whereas $H^1(C, N)$ is the space of obstructions. Furthermore, if $H^1(C, N) = 0$ then $\mathscr{H}_C$ will be smooth at $[C]$; and by the Riemann-Roch theorem, one can compute that $h^0(C, N) = 4 \cdot \deg C$.

Proposition 1.4 concludes the flexibility of the family in which a curve C sits as long as its normal bundle satisfies $h^0(N(-2)) = 0$. Since such a proposition depends on a non trivial result by Kleppe, let us work out an example for which we can show the vanishing of h^0 directly.

Example 2.2. Consider an ACM curve $C \in \mathscr{H}_{6,3}$, whose ideal sheaf has minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{O}_{\mathbb{P}^3}(-3)^4 \to \mathcal{I}_C \to 0.$$

Taking $Hom(-, \mathcal{O}_C(-2))$ we get an exact sequence

$$0 \to N(-2) \to \mathcal{O}_C(1)^4 \to \mathcal{O}_C(2)^3. \tag{2.1}$$

By the Riemann-Roch theorem we get

$$h^0(C, \mathcal{O}_C(1)) = 4$$
$$h^0(C, \mathcal{O}_C(2)) = 10.$$

Since C is ACM, the following diagram is commutative and has isomorphisms in the vertical arrows:

$$\begin{tikzcd} H^0(C,\mathcal{O}_C(1))^4 \arrow[r] & H^0(C,\mathcal{O}_C(2))^3 \\ H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(1))^4 \arrow[r] \arrow[u] & H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))^3 \arrow[u] \end{tikzcd}$$

The bottom arrow is injective (this can easily be verified). Therefore, by the exact sequence (2.1) we get that

$$H^0(C, N(-2)) = 0.$$

PROPOSITION 2.3. *Let C be a smooth, irreducible curve. If*

$$H^0(C, N(-2)) = 0$$

then N is semi-stable.

PROOF. Suppose there is a line bundle $L \subset N$ with

$$\deg(L) > \frac{1}{2}\deg(N) = 2d + g - 1.$$

Applying the Riemann-Roch formula to $L(-2)$ we get

$$\begin{aligned} h^0(C, L(-2)) &\geq \deg(L(-2)) + 1 - g \\ &> 2d + g - 1 - 2d + 1 - g \\ &= 0. \end{aligned}$$

Thus

$$0 \neq H^0(C, L(-2)) \subset H^0(C, N(-2)).$$

□

The previous proposition provides a sufficient condition for the stability of N. Note that a necessary condition for N to be stable is the inequality

$$g < d(s-2) + 1,$$

where s is the minimal degree of a surface containing C. Indeed, if Y is a degree s surface containing C, we have an injective morphism C

$$\mathcal{O}_{\mathbb{P}^3}(-s) = \mathcal{I}_Y \to \mathcal{I}_C/\mathcal{I}_C^2.$$

Dualizing this inclusion yields a surjection

$$N \to \mathcal{O}_{\mathbb{P}^3}(s)$$

and the inequality guarantees that this surjection does not destabilize N. One could ask when is this condition sufficient for N to be stable.

PROBLEM 2.4. If C is an ACM curve and $g < d(s-2) + 1$, is N stable?

For a low degree ACM curve, one may compute its h-vector and try to figure it out component by component.

3. Peskine's problem

PROBLEM 3.1. Consider a family of irreducible, smooth curves C_t in $\mathbb{P}^3_k$. If for all $t \neq 0$, the curve C_t is a complete intersection, is C_0 a complete intersection too? Assume $char(k) = 0$.

Notice that, a priori, we do not assume that the degrees of the defining equations are constant.

Smoothness of the curves C_t (even for $t = 0$), the characteristic of the field as well as the dimension of the projective space are all essential hypothesis. There are counterexamples otherwise. For example, there are families of curves where C_t is a smooth complete intersection of 2 quadrics for $t \neq 0$, and the central fiber C_0 is a plane cubic with an incident line; which not a complete intersection. Furthermore, Kumar has given counterexamples to Problem 3.1 when the characteristic of the field $char(k) = p > 0$ [**Ku91**].

Ellia and Hartshorne proved that for complete intersections of degrees $a \leq 4$ and $b \leq 5$ the answer to Question 3.1 is affirmative [**EllHa99**]. In this paper, the authors conjectured the following.

CONJECTURE 3.2. For any curve $C \subseteq \mathbb{P}^3$ let $s(C)$ be the least degree of a surface containing C. Assume that C_t is a flat family of smooth irreducible curves such that $s(C_t) = s$ for $t \neq 0$. If $deg(C_t) \geq s^2$, then $s(C_0) = s$.

EXAMPLE 3.3. The inequality $deg(C_t)$ is necessary: the curves in Example 2.2 can be specialized to smooth curves of bi-degree $(2, 4)$ on a smooth quadric surface [**Ha10**, Ex. 8.8].

REMARK 3.4. Conjecture 3.2 implies Peskine's problem. In fact, if a curve C is a complete intersection of surfaces of degrees $a \leq b$ then $s(C) = a$. If C_t is a family as in Peskine's problem, then $deg(C_t) = ab \geq a^2$; therefore, Conjecture 3.2 implies that $a = s(C_0)$. Let S be a degree a surface containing C_0. By the semicontinuity of $h^0(\mathbb{P}^3, \mathcal{I}_{C_t}(b))$, there is a degree b surface containing C_0 with no common components with S as C_0 was assumed to be smooth. Given that $deg(C_0) = ab$, it follows that C_0 is the complete intersection of these surfaces.

EXAMPLE 3.5. Consider smooth curves of degree $d = 9$ and genus $g = 10$. The Hilbert scheme of such curves has two components parametrizing:

a) complete intersections of degrees $a = b = 3$, and
b) curves of bi-degree $(3, 6)$ in smooth quadric surfaces.

In this context, Peskine's problem asks whether or not there can be a family inside the first component specializing to a point in the second one. This setting does not provide a counterexample to Peskine's problem as no such family exists.

One possible strategy to produce counterexamples to Conjecture 3.2 is to consider rational curves. The locus of degree d rational curves inside its Hilbert scheme is smooth and irreducible of dimension $4d$. If $d \geq 9$ then the general such curve is not contained in a cubic surface. We could consider the family of curves contained in cubics and try to produce specializations to a smooth curve contained in a quadric. This would be a counterexample.

4. Behaviour of Rao modules

Definition 4.1. The Rao module, or deficiency module, of a curve $C \subset \mathbb{P}^3$ is the module

$$M_C := \bigoplus_{n \in Z} H^1(\mathbb{P}^3, \mathcal{I}_C(n)).$$

Question 4.2. How does the Rao module behave in a family?

It follows from semi-continuity that the dimensions in each degree can only increase. But what about the module structure? For example, is there any limitation on the Rao module of a curve that is a limit of ACM curves? (see [**HaDesPe97**]).

Theorem 4.3 (Rao). [**Ra79**, Proposition 2.3, Theorem 2.6] *Two locally Cohen-Macaulay curves in $\mathbb{P}^3$ are (evenly) linked if and only if their Rao modules are isomorphic, up to a shift. Furthermore, given any finite-length module M there is a smooth curve with Rao module isomorphic to (a shift of) M.*

One way to study Question 4.2 is to find a relative version of the previous theorem. However, the definition of Rao modules for families of curves is not straightforward. For example, one can consider the functor that associates to T the function $\mathcal{F} : t \mapsto M_{C_t}$ and try to invetigate if two families of curves C_t and C'_t parametrized by T

$$\begin{tikzcd} C'_t, C_t \arrow[r] & \mathbb{P}^3 \times T \arrow[d, "\pi"'] \\ & T, \end{tikzcd}$$

are evenly linked (by a family of liasons) if and only if the $\mathcal{F}(\pi(C'_t)) = \mathcal{F}(\pi(C_t))$. However, this association behaves badly and does not provide enough information.

Another possibility is to consider the cohomology groups $H^1(\mathbb{P}^3, \mathcal{I}_{C_t})$, which naturally leads to $R^1 f_* \mathcal{I}_C(d)$. However this is not sufficient either as problems with base extensions occur. For example, tensoring with $\mathcal{O}_{T,t}$ does not always recover the Rao module of C_t.

The correct notion can be stated in terms of *triads*. These objects have desired properties and I refer the reader to [**HaDesPe97**], [**HaDesPe98**] and [**HaDesPe00**] for the technical details.

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