MATH 340 Example of pivoting to optimal solution(s). Richard Anstee

We considered the following LP in standard inequality form

$$\begin{array}{rrrrrcl} \max & 4x_1 & +3x_2 & +x_3 & +x_4 & & \\ & x_1 & +2x_2 & & -x_4 & \leq & 3 \\ & 2x_1 & +x_2 & -x_3 & +x_4 & \leq & 2 \\ & & x_2 & +x_3 & & \leq & 2 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

We add slack variables x_5, x_6, x_7 corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$. We form our first dictionary

$$\begin{array}{rcrrrrr} x_5 & = & 3 & -x_1 & -2x_2 & & +x_4 \\ x_6 & = & 2 & -2x_1 & -x_2 & +x_3 & -x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & & 4x_1 & +3x_2 & +x_3 & +x_4 \end{array}$$

It is traditional to use z for the objective function. There is an *obvious solution* to these 4 equations, namely $x_5 = 3$, $x_6 = 2$, $x_7 = 2$ and $x_1 = x_2 = x_3 = x_4 = 0$ with $z = 0$. (this is called a *basic feasible solution*)

We now use Anstee's rule trying to increase a variable from 0 in the current *obvious solution* so we greedily choose x_1 to increase and hence *enter.* We leave $x_2 = x_3 = x_4 = 0$. The choice of x_1 as the variable with the largest coefficient in dictionary expression for z (and in the case of ties choosing the variable of smallest subscript) is called **Anstee's Rule** in this course.

$$\begin{array}{rcrr} x_5 & = & 3 & -x_1 \\ x_6 & = & 2 & -2x_1 \\ x_7 & = & 2 & \\ z & = & & 4x_1 \end{array}$$

we deduce that x_1 can be increased to 1 while decreasing x_6 to 0. We obtain a new dictionary by having x_1 only appear on the left and x_6 is now on the right of the equation signs.

$$\begin{array}{rcrrrrr} x_5 & = & 2 & +\frac{1}{2}x_6 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & +\frac{3}{2}x_4 \\ x_1 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_2 & +\frac{1}{2}x_3 & -\frac{1}{2}x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & 4 & -2x_6 & +x_2 & +3x_3 & -x_4 \end{array}$$

There is an *obvious solution* to these 4 equations, namely $x_5 = 2$, $x_1 = 1$, $x_7 = 2$ and $x_6 = x_2 = x_3 = x_4 = 0$ with $z = 4$. Note how I keep all the entries of each variable in neat columns. It makes adding and subtracting equations much more reliable.

By Anstee's rule we would wish to increase x_3 leaving $x_6 = x_2 = x_4 = 0$

$$\begin{array}{rcrr} x_5 & = & 2 & -\frac{1}{2}x_3 \\ x_1 & = & 1 & +\frac{1}{2}x_3 \\ x_7 & = & 2 & -x_3 \\ z & = & 4 & +3x_3 \end{array}$$

and we deduce that we could increase x_3 to 2 while driving x_7 to 0 and so we say x_3 enters and x_7 leaves.

$$\begin{array}{rcrrrrr} x_5 & = & 1 & +\frac{1}{2}x_6 & -x_2 & +\frac{1}{2}x_7 & +\frac{3}{2}x_4 \\ x_1 & = & 2 & -\frac{1}{2}x_6 & -x_2 & -\frac{1}{2}x_7 & -\frac{1}{2}x_4 \\ x_3 & = & 2 & & -x_2 & -x_7 & \\ z & = & 10 & -2x_6 & -2x_2 & -3x_7 & -x_4 \end{array}$$

There is an *obvious solution* to these 4 equations, namely $x_5 = 1$, $x_1 = 2$, $x_3 = 2$ and $x_6 = x_2 = x_7 = x_4 = 0$ with $z = 10$. But now the equation $z = 10 - 2x_6 - 2x_2 - 3x_7 - x_4$ combined with the four inequalities $x_6 \geq 0$, $x_2 \geq 0$, $x_7 \geq 0$, $x_4 \geq 0$ yields $z \leq 10$. Thus we have found an optimal solution to the LP. In fact $z \leq 10$ with equality if and only if $x_6 = x_2 = x_7 = x_4 = 0$. Now setting $x_6 = x_2 = x_7 = x_4 = 0$ in our third dictionary yields $x_5 = 1$, $x_1 = 2$, $x_3 = 2$. Thus we have found the unique optimal solution in this case.

In general optimal solutions are not unique (although of course the optimal value of the objective function z would be unique!). Consider the following minor variant of our problem where we have increased the coefficient of x_2 in the objective function from 3 to 5.

$$\begin{array}{rrrrrcl} \max & 4x_1 & +5x_2 & +x_3 & +x_4 & \leq & \\ & x_1 & +2x_2 & & -x_4 & \leq & 3 \\ & 2x_1 & +x_2 & -x_3 & +x_4 & \leq & 2 \\ & & x_2 & +x_3 & & \leq & 2 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

We add slack variables x_5, x_6, x_7 corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$. We form our first dictionary

$$\begin{array}{rcrrrrr} x_5 & = & 3 & -2x_1 & -2x_2 & & +x_4 \\ x_6 & = & 2 & -2x_1 & -x_2 & +x_3 & -x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & & 4x_1 & +5x_2 & +x_3 & +x_4 \end{array}$$

Now ignore Anstee's rule and have x_1 enter and x_6 leave which yields the dictionary

$$\begin{array}{rcrrrrr} x_5 & = & 2 & +\frac{1}{2}x_6 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & +\frac{3}{2}x_4 \\ x_1 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_2 & +\frac{1}{2}x_3 & -\frac{1}{2}x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & 4 & -2x_6 & +3x_2 & +3x_3 & -x_4 \end{array}$$

Again ignore Anstee's rule and have x_3 enter and x_7 leave to yield the following dictionary.

$$\begin{array}{rcrrrrr} x_5 & = & 1 & +\frac{1}{2}x_6 & -x_2 & +\frac{1}{2}x_7 & +\frac{3}{2}x_4 \\ x_1 & = & 2 & -\frac{1}{2}x_6 & -x_2 & -\frac{1}{2}x_7 & -\frac{1}{2}x_4 \\ x_3 & = & 2 & & -x_2 & -x_7 & \\ z & = & 10 & -2x_6 & & -3x_7 & -x_4 \end{array}$$

There is an *obvious solution* to these 4 equations, namely $x_5 = 1$, $x_1 = 2$, $x_3 = 2$ and $x_6 = x_2 = x_7 = x_4 = 0$ with $z = 10$. But now the equation $z = 10 - 2x_6 - 3x_7 - x_4$ combined with the three inequalities $x_6 \geq 0$, $x_7 \geq 0$, $x_4 \geq 0$ yields $z \leq 10$. Thus we have found an optimal solution to the new LP. In fact $z \leq 10$ with equality if and only if $x_6 = x_7 = x_4 = 0$ (the coefficient of x_2 is 0 in the expression for z). Now setting $x_6 = x_7 = x_4 = 0$ in our third dictionary yields

$$\begin{array}{rcrr} x_5 & = & 1 & -x_2 \\ x_1 & = & 2 & -x_2 \\ x_3 & = & 2 & -x_2 \\ z & = & 10 & \end{array}$$

We deduce that $0 \leq x_2 \leq 1$ and setting $t = x_2$ we can write all possible optimal solutions to the LP as $x_1 = 2 - t$, $x_2 = t$, $x_3 = 2 - t$, $x_4 = 0$ with $x_5 = 1 - t$, $x_6 = 0$ and $x_7 = 0$ and $z = 10$. Remember that you can check this.