

The van Kampen theorem

Omar Antolín Camarena

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1 The van Kampen theorem

The van Kampen theorem allows us to compute the fundamental group of a space from information about the fundamental groups of the subsets in an open cover and their intersections. It is classically stated for just fundamental groups, but there is a much better version for fundamental groupoids:

- The statement and proof of the groupoid version are almost the same as the statement and proof for the group version, except they're a little simpler!
- The groupoid version is more widely applicable than the group version and even when both apply, the groupoid version can be simpler to use.

In fact, I'm convinced the only disadvantage of the groupoid version is psychological: groups are more familiar than groupoids to most people.

1.1 Version for the full fundamental groupoid

By $\pi_{\leq 1}(X)$ we mean the fundamental groupoid of X , which is a category whose objects are the points of X and whose morphisms from p to q are the homotopy classes (relative to endpoints) of paths in X from p to q . It is easy to see these form a category where composition is concatenation of paths, and that every morphism has an inverse (so the category is a groupoid) given by the time-reversal of the path. The classical fundamental group of X at the basepoint $x_0 \in X$ is $\text{hom}_{\pi_{\leq 1}(X)}(x_0, x_0)$.

Theorem 1. *Let X be a space and U and V two open subsets of X such that $X = U \cup V$. Then the following diagram, in which all morphisms are induced by inclusions of spaces, is a pushout square of groupoids:*

$$\begin{array}{ccc} \pi_{\leq 1}(U \cap V) & \longrightarrow & \pi_{\leq 1}(U) \\ \downarrow & & \downarrow \\ \pi_{\leq 1}(V) & \longrightarrow & \pi_{\leq 1}(X) \end{array}$$

Notice that X itself is the push out in the category of topological spaces of the diagram of inclusions $U \leftarrow U \cap V \rightarrow V$; van Kampen's theorem says that this particular type of pushout is preserved by the functor $\pi_{\leq 1}$.

Proof. We'll directly show that $\pi_{\leq 1}(X)$ satisfies the universal property of the pushout. Consider a commutative square of groupoids

$$\begin{array}{ccc} \pi_{\leq 1}(U \cap V) & \longrightarrow & \pi_{\leq 1}(U) \\ \downarrow & & \downarrow \Gamma \\ \pi_{\leq 1}(V) & \xrightarrow{\Lambda} & G \end{array}$$

where G is some arbitrary groupoid. We need to show that this data induces a unique morphism of groupoids $\Phi : \pi_{\leq 1}(X) \rightarrow G$. Let's see the definition is forced so there is at most one such morphism:

- An object of $\pi_{\leq 1}(X)$ is a point $x \in X$ and so lies in either U or V (or both). If $x \in U$, we are forced to set $\Phi(x) = \Gamma(x)$; if $x \in V$, we are forced to set $\Phi(x) = \Lambda(x)$. If $x \in U \cap V$, these definitions agree by the commutative square above.
- A morphism in $\pi_{\leq 1}(X)$ is (the homotopy class of) a path α in X . If α lay solely in U , we would be forced to set $\Phi(\alpha) = \Gamma(\alpha)$. Similarly if α were in V , the definition would be forced. In general, we can always split up α into a composition $\alpha_1 \circ \cdots \circ \alpha_n$ of a bunch of paths, each of which lies completely in U or completely in V , and we are forced to set $\Phi(\alpha) = F_1(\alpha_1) \circ \cdots \circ F_n(\alpha_n)$ where each F_i is either Γ or Λ as necessary.

So there is at most one choice of Φ ; it only remains to show that this choice works, i.e., that the above description is independent of the choice of decomposition and that it really defines a functor. It is clear that it is functorial if well-defined, so let's just show it is well defined. Say that α and β are two homotopic paths between the same pair of points in X and let $H : [0, 1] \times [0, 1] \rightarrow X$ be a homotopy between them. By the Lebesgue covering lemma, we can subdivide the square into tiny little squares so that each one is sent by H to either U or V . Furthermore, we can arrange that the subdivision of $[0, 1] \times \{0\}$ refines the subdivision we chose to define $\Phi(\alpha)$ and similarly the subdivision of $[0, 1] \times \{1\}$ refines the subdivision used for $\Phi(\beta)$.

For each tiny square, we get an equality in the fundamental groupoid of either U or V between composites of the paths obtained by restricting H to the sides: $H|_{\text{right}} \circ H|_{\text{top}} = H|_{\text{bottom}} \circ H|_{\text{left}}$. Applying either Γ or Λ as the case maybe, we get an equality in the groupoid G . Adding them all together proves that $\Phi(\alpha) = \Phi(\beta)$. \square

There is also a more general version of the theorem for open covers consisting of an arbitrary number of open sets, but (1) the basic two-set version covers most of the applications, (2) the proof of the general version is pretty much the same as the version for two sets, only more difficult notationally. Here's a statement:

Theorem 2. (*(May 1999), section 2.7¹*) *Let \mathcal{U} be an open cover of a space X such that the intersection of finitely many members of \mathcal{U} again belongs to \mathcal{U} . We can regard \mathcal{U} as the objects of a category whose morphisms are simply the inclusions. The fundamental groupoid of X is the colimit of the diagram*

¹Note that May states this with an unnecessary hypothesis: that each open be path connected. His proof, however, never uses it and establishes the version stated here.

formed by restricting the fundamental groupoid functor $\pi_{\leq 1}$ to the category \mathcal{U} ; in symbols: $\pi_{\leq 1}(X) = \text{colim}_{U \in \mathcal{U}} \pi_{\leq 1}(U)$.

1.2 Version for a subset of the base points

There is also a version for the fundamental groupoid on a subset of the basepoints. For a set $A \subseteq X$, let $\pi_{\leq 1}(X, A)$ denote the full subcategory of $\pi_{\leq 1}(X)$ on the objects in A . For the van Kampen theorem to hold for these $\pi_{\leq 1}(\cdot, A)$, A needs to satisfy the following condition: A contains at least one point in each component of each of $U \cap V$, U and V . It is actually this version that we are really after, since the whole fundamental groupoid is impractical in that it contains a lot of redundant information.

Our strategy for proving this version will be to deduce it from the version for the full fundamental groupoid. The hypothesis on A guarantees that for $W = U \cap V, U, V, X$ the groupoid $\pi_{\leq 1}(W, A)$ is equivalent to $\pi_{\leq 1}(W)$ and one might hope that replacing each groupoid in a pushout square by an equivalent groupoid yields a new pushout square. This is not quite right: replacing each groupoid by an *isomorphic* groupoid would of course give a new pushout square, but isomorphisms are precisely the relation between objects that pushouts are meant to preserve, they won't preserve mere equivalence. There is a more flexible notion of pushout for groupoids (variously called, weak pushout, homotopy pushout or 2-categorical pushout) that is invariant under equivalence, but we won't talk about that here.

Instead what we'll do to deduce the version for $\pi_{\leq 1}(\cdot, A)$ from the version for the full groupoid is (1) show that the commutative square

$$\begin{array}{ccc} \pi_{\leq 1}(U \cap V, A) & \longrightarrow & \pi_{\leq 1}(U, A) \\ \downarrow & & \downarrow \\ \pi_{\leq 1}(V, A) & \longrightarrow & \pi_{\leq 1}(X, A) \end{array}$$

is a *retract* of the square for the full fundamental groupoids and (2) using the general category theoretical fact that a retract of a pushout square is also a pushout square.

A *retract* of an object X is another object Y with a pair of morphisms $i : Y \rightarrow X$ and $r : X \rightarrow Y$ such that $r \circ i = \text{id}$. We think of i as including Y inside X and r as being a retraction of X onto Y . When we say a commutative square ρ is a retract of another square σ we mean that each corner of ρ is a retract of the corresponding corner of σ , but more than just this: all of the inclusions and retractions have to be compatible with one another in the sense that the cubical diagram formed by the two squares and

the four inclusions commutes, as does the cube formed by the two squares and the four retractions. It only takes, as one says, a straight-forward diagram chase to prove that a retract of a pushout square must also be a pushout square.

Exercise 1. Prove that a retract of a pushout square is also a pushout square.

Now, in our case it is easy to show the second square is a retract of the first. The inclusions are just that: inclusions $\pi_{\leq 1}(X, A) \rightarrow \pi_{\leq 1}(X)$. The retractions are built as follows. To retract $\pi_{\leq 1}(X)$ onto $\pi_{\leq 1}(X, A)$ just pick, for every point $x \in X$ a path α_x from x to some point $a \in A$, but do this in such a way that if x is already in A , α_x is the identity morphism at x . (We can always pick these paths because the hypothesis include that A has at least one point in each component of each of the spaces we use.) Then the retraction we're defining is the morphism that sends each x to the other endpoint of α_x , and each morphism $\beta : x \rightarrow y$ to the morphism $\alpha_y \circ \beta \circ \alpha_x^{-1}$. To ensure that the cube formed by the two van Kampen squares and the four retractions commutes, simply *always pick the same α_x for x in all of the groupoids it appears in.*

2 Applications

2.1 First examples

We can use the van Kampen theorem to compute the fundamental groupoids of most basic spaces.

2.1.1 The circle

The classical van Kampen theorem, the one for fundamental *groups*, cannot be used to prove that $\pi_1(S^1) \cong \mathbb{Z}$! The reason is that in a non-trivial decomposition of S^1 into two connected open sets, the intersection is not connected. That is not an issue for the groupoid version. Take U and V to be semicircles, intersecting at two points $A = \{p, q\}$.

Remark 1. Technically, we need U and V to be open, so we should take them to be open arcs slightly bigger than semicircles, and then their intersection will be a pair of small arcs, one containing p , one containing q . This makes no essential difference and only complicates the language, so we will silently use closed sets whenever we want, with the understanding that it should be checked that fattening them slightly will produce open sets that van Kampen applies to.

Since each of U and V is contractible, both $\pi_{\leq 1}(U, A)$ and $\pi_{\leq 1}(V, A)$ are the groupoid with two objects, p and q and a single isomorphism $p \rightarrow q$. Also, $\pi_{\leq 1}(U \cap V, A)$ is just the *discrete* groupoid on two objects; it has no non-identity morphisms. The pushout $\pi_{\leq 1}(S^1, A)$ is therefore a groupoid on two objects p and q , with *two* isomorphisms $u, v : p \rightarrow q$ and beyond that is as free as possible. So, for example, all the composites $(v^{-1} \circ u)^n$ are distinct (because there is no reason for them not to be). We get that $\pi_1(X, p) = \{(v^{-1} \circ u)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$.

2.1.2 Spheres

We can easily show that all S^n for $n > 1$ are simply-connected. Decompose S^n as two hemispheres H_1 and H_2 , intersecting along the equator, which is an S^{n-1} . Since both hemispheres and their intersection are connected, the group version of van Kampen applies, and therefore $\pi_1(S^n) = \pi_1(H_1) *_{\pi_1(S^{n-1})} \pi_1(H_2)$ is the trivial group: both H_1 and H_2 are contractible (notice that it doesn't matter what $\pi_1(S^{n-1})$ is).

More generally, this shows that the suspension ΣX of any connected space X has zero fundamental group.

2.1.3 Glueing along simply-connected intersections

If $X = U \cup V$ where U and V are connected and $U \cap V$ is simply-connected, then we get that $\pi_1(X) = \pi_1(U) * \pi_1(V)$, where $*$ denotes the *free product* of spaces, which is the coproduct in the category of groups, or the pushout in groupoids over the discrete groupoid with one object. This simple idea has several applications:

1. **Wedges of spaces** If each X_i is a connected space with a reasonable base point (i.e., the base point has a contractible neighborhood), we get that $\pi_1(\bigvee_{i=1}^n X_i) = \pi_1(X_1) * \cdots * \pi_1(X_n)$. For example, the fundamental group of a bouquet of n circles is the free group on n generators.
2. **Removing a point from a manifold.** Given an n -dimensional manifold M , with $n \geq 3$, how are the fundamental groups of M and $M \setminus \{p\}$ related? We can use van Kampen “in reverse” to answer this. Let U be a small neighborhood of p , homeomorphic to a ball in \mathbb{R}^n and let $V = M \setminus \{p\}$. Then $U \cap V \simeq S^{n-1}$ which is simply-connected for $n - 1 > 1$, so we obtain $\pi_1(M) = \pi_1(M \setminus \{p\})$ in that case.²

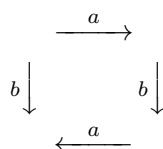
²If $n = 2$, we get $U \cap V = S^1$ with $\pi_1 = \mathbb{Z}$, so the pushout $\pi_1(M) = \pi_1(M \setminus \{p\}) *_{\mathbb{Z}} 1$

3. Attaching cells. The argument in the previous paragraph shows more generally that attaching a cell of dimension 3 or higher to a CW complex does not change its fundamental groupoid. The argument in the footnote shows that attaching a 2-dimensional cell kills the loop it is attached to (and does nothing else).
4. Connected sums. We can also use the observation in the first paragraph to compute fundamental groups of connected sums of manifolds. Recall that given two smooth n -dimensional manifolds M and N , the connected sum $M\#N$ is constructed by removing a ball from each of M and N and glueing these along their boundary, $(M \setminus \text{int}(D^n)) \cup_{\partial D^n} (N \setminus \text{int}(D^n))$. If $n \geq 3$, we simply get a free product

$$\pi_1(M\#N) = \pi_1(M \setminus D^n) * \pi_1(N \setminus D^n) = \pi_1(M) * \pi_1(N)$$

2.1.4 Compact surfaces

Computing the fundamental groups of compact surfaces is easily done by starting with a construction of the surface as the result of identifying some sides of a polygon. For example, the Klein bottle X is obtained from a rectangle



by glueing opposite sides as indicated by the arrows. To compute its fundamental group, draw a smaller square inside this one, let U be the filled in smaller square, and let V be “frame” around it. Then $U \cap V$ is the smaller square and is homotopy equivalent to S^1 , U is contractible and V is homotopy equivalent to a wedge of two circles: the paths a and b indicated in the picture (**exercise**: why are these loops?). We get that $\pi_1(V)$ is free on a and b , and the inclusion $U \cap V \rightarrow V$, sends a generator of $\pi_1(U \cap V)$ to $b^{-1}aba$ as we see by going around the square in the picture clockwise. This means that the fundamental group of the Klein bottle is $\langle a, b \rangle *_{\mathbb{Z}} 1 = \langle a, b : aba = b \rangle$.

Exercise 2. Using the standard construction of the compact orientable surface of genus g as the result of identifying sides of a $2g$ -gon, prove that its fundamental group is $\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g : [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle$.

just kills the loop around $U \cap V$; i.e. $\pi_1(M) = \pi_1(M \setminus \{p\})/N$ where N is the normal subgroup generated by a loop around $U \cap V$ (and all its conjugates).

Exercise 3. Prove that the fundamental group of the real projective plane is cyclic of order 2.

2.2 Four dimensional manifolds can have arbitrary finitely generated fundamental group

What we said about attaching cells easily implies that given any finitely presented group $\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$ can be obtained as the fundamental group of a finite CW-complex with a single 0-cell, a loop for each a_i and a 2-cell attached to each r_i .³ A geometric version of this construction shows that any finitely presented group is the fundamental group of some smooth 4-manifold:

By our calculation of the fundamental group of a connected sum, the connected sum of m copies of $S^1 \times S^3$ has fundamental group isomorphic to the free group on a_1, \dots, a_m . Now we can impose each relation r_j as follows: realize r_j as a simple loop. A tubular neighbourhood of this looks like $S^1 \times D^3$. Do surgery and replace this tubular neighbourhood with $S^2 \times D^2$, which has the same boundary: $\partial(S^1 \times D^3) = S^1 \times S^2 = S^2 \times S^1 = \partial(S^2 \times D^2)$. This kills r_j .

2.3 The Jordan Curve theorem

Now we are going to prove the Jordan Curve Theorem. This is a very well known result, famous for being totally believable, almost obvious even, but surprisingly hard to prove. There are a number of versions of the theorem so I should say precisely which one we'll be proving.

Theorem 3. (The Jordan Curve Theorem.) *Let C be a simple closed curve in the sphere, that is C is a subset of S^2 which is homeomorphic to a circle. Then, the complement of C has exactly two connected components.*

We will present Ronnie Brown's proof (Brown 2006), Section 9.2, with a correction made in (Brown and Antolín Camarena 2015). A further refinement of the theorem, which we will not prove here, is that each of the two components of the complement of C has boundary equal to C . This is also proved in Brown's book. The plan of the proof is to show that, on S^2 :

1. The complement of an arc is connected.

³And if you don't care about finiteness, any group can be similarly obtained from a CW-complex with possibly infinitely many 1- and 2-cells.

2. The complement of a simple closed curve has exactly two components, by showing a) it is disconnected, and b) it cannot have 3 or more components.

Steps 1 and 2b will both use a nice lemma on free groups inside pushouts of groupoids that I think should be motivated before it's stated and proved, so we won't do the proof in logical sequence; rather, we'll do step 2a (assuming 1) first, then the lemma, and then steps 1 and 2b.

2.3.1 The complement of a simple closed curve has exactly two components

In this part we will assume that the complement of any arc (i.e., a subset homeomorphic to a closed interval) on S^2 is connected. This will be proved a little later.

Throughout this section we'll let C be a subset of S^2 homeomorphic to S^1 , and let $C = D \cup E$ where D and E are arcs that meet in exactly two points a and b . We'll also let $U = S^2 \setminus D$, $V = S^2 \setminus E$, so that $U \cap V = S^2 \setminus C$ and $U \cup V = S^2 \setminus \{a, b\} =: X$.

Note that, since C, D and E are all compact, they are closed subsets of S^2 . Moreover, since U, V and X are open subsets of S^2 (and S^2 is locally path-connected), we won't need to distinguish between connectedness and path-connectedness for these subspaces.

Proposition 1. *The complement of a simple closed curve is disconnected.*

Proof. Assume it were connected. Then we can use van Kampen's theorem (even the fundamental *group* version!) to get that

$$\begin{array}{ccc} \pi_1(U \cap V) & \longrightarrow & \pi_1(U) \\ \downarrow & & \downarrow \\ \pi_1(V) & \longrightarrow & \pi_1(X) \end{array}$$

is a pushout square of groups. The lower right corner we know: it is just \mathbb{Z} , since X is (homeomorphic to) an (open) annulus, and thus equivalent to a circle. We'll get a contradiction by showing that both $\pi_1(U) \rightarrow \pi_1(X)$ and $\pi_1(V) \rightarrow \pi_1(X)$ are trivial morphisms sending everything to zero.

It should be intuitive that these morphisms are indeed zero: we're saying that if you have a loop on a sphere that avoids some arc, then the loop can be contracted to a point without going through the endpoints of the arc. To prove it, let's just put one endpoint off limits to begin with: $S^2 \setminus \{b\}$

is homeomorphic to \mathbb{R}^2 (by stereographic projection, for example), and we can even pick a homeomorphism for which a maps to the origin in the plane. Then the arc D corresponds to some curve starting at the origin and going off to infinity, and if we pick a parametrization $\alpha : [0, \infty) \rightarrow \mathbb{R}^2$ for this curve, what we're trying to show is that any loop γ in the plane avoiding the image of α can be contracted to a point while avoiding the origin. Our strategy for that is to translate γ by the vector $-\alpha(t)$: when $t = 0$ we just get γ , but as $t \rightarrow \infty$, γ gets pushed away from the origin until it "looks tiny when viewed from the origin" and then be contracted.

More formally, since the image of γ is compact, it lies inside some big ball of radius R around the origin and there is a t_0 such that $|\alpha(t_0)| > R$. Consider the homotopy $H_t(s) = H(t, s) = \gamma(s) - \alpha(t)$. Because the loop γ avoid the image of α , H never passes through the origin. Also, we have $H_0 = \gamma$ and H_{t_0} is a loop that lies inside the ball B of radius R with center $\alpha(t_0)$. Since $|\alpha(t_0)| > R$, B does not contain the origin and the loop H_{t_0} can be safely contracted to a point inside of B . This shows that the inclusion $U \rightarrow X$, induces the zero map on π_1 , as required. \square

Proposition 2. *The complement of a simple closed curve has exactly two components.*

Proof. We've seen it's disconnected, so it has at least two components; we need only show it can't have three or more components. To do this we'll apply van Kampen to U and V again (but this time we do need the groupoid version). Take a set A of base points that consists of exactly one point from each component of $S^2 \setminus C = U \cup V$. Notice that since U , V and X are connected we don't have to worry about A failing to meet some component of them. Now, van Kampen gives us a pushout of groupoids:

$$\begin{array}{ccc} \pi_{\leq 1}(U \cap V, A) & \longrightarrow & \pi_{\leq 1}(U, A) \\ \downarrow & & \downarrow \\ \pi_{\leq 1}(V, A) & \longrightarrow & \pi_{\leq 1}(X, A) \end{array}$$

Again, we know all about the lower right corner. In particular, if we take a point $p \in A$, we know that the fundamental group of X based at p is just \mathbb{Z} . But here is a heuristic argument indicating that this is *not* consistent with the pushout square: we get $|A| - 1$ independent loops at p coming from following some path from p to $q \in A$ in U and coming back along some path in V . The fact that in $U \cap V$ there are no paths from p to q tells us these loops in X are non-trivial and have no relations between them. This (correctly)

suggests we actually get the free group on $|A| - 1$ generators sitting inside $\pi_1(X, p)$, which shows $|A| = 2$. \square

All that remains is make the heuristic precise, which is the next section.

2.3.2 A lemma about pushouts of groupoids

Lemma 1. *Consider a pushout square of groupoids,*

$$\begin{array}{ccc} C & \xrightarrow{j} & B \\ i \downarrow & & \downarrow \\ A & \longrightarrow & G \end{array}$$

where i, j are bijective on objects, C is totally disconnected, and G is connected. Then G contains as a retract a free groupoid whose vertex groups are of rank $k = n_C - n_A - n_B + 1$, where n_P is the number of components of the groupoid P for $P = A, B, C$ (assuming these numbers are finite).

Further, if C contains distinct objects a, b such that $A(ia, ib), B(ja, jb)$ are nonempty, then F has rank at least 1.

Proof. Let $\text{Fr} : \text{DirectedGraphs} \rightarrow \text{Groupoids}$ be the free groupoid functor. Any groupoid G has a retraction $G \rightarrow \text{Fr}W$ where W is a spanning forest for the underlying graph of G .

Let Z be the set of objects of C (and of A, B and G) regarded as a directed graph with no edges. Pick spanning forests X and Y of the underlying directed graphs of A and B . Then there are retractions

$$C \rightarrow \text{Fr}Z, A \rightarrow \text{Fr}X, B \rightarrow \text{Fr}Y.$$

By “span” in a category we mean a pair of arrows $U \leftarrow W \rightarrow V$; this is the shape of a diagram whose pushout you can take. Then the following diagram in Groupoids commutes and its rows are spans:

$$\begin{array}{ccccc} \text{Fr}X & \longleftarrow & \text{Fr}Z & \longrightarrow & \text{Fr}Y \\ \downarrow & & \downarrow & & \downarrow \\ A & \longleftarrow & C & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \text{Fr}X & \longleftarrow & \text{Fr}Z & \longrightarrow & \text{Fr}Y \end{array}$$

So the span $\text{Fr}X \leftarrow \text{Fr}Z \rightarrow \text{Fr}Y$ is a retract of the span $A \leftarrow C \rightarrow B$. This implies that the pushout, say F , of $\text{Fr}X \leftarrow \text{Fr}Z \rightarrow \text{Fr}Y$ is a retract of G

(which is the pushout of $A \leftarrow C \rightarrow B$). Since the span of free groupoids is actually the image under Fr of the obvious span $X \leftarrow Z \rightarrow Y$ of graphs, and since Fr is a left adjoint, this pushout F is actually just $\text{Fr}W$ where W is the pushout in the category of directed graphs of $X \leftarrow Z \rightarrow Y$.

This graph W is connected because G is connected, so, denoting by $e(Q)$ and $v(Q)$ the number of edges and vertices of a graph, the vertex groups in $\text{Fr}W$ are free of rank $k = e(W) - v(W) + 1$. We have $v(W) = v(X) = v(Y) = v(Z) = n_C$; and, since Z has no edges, $e(W) = e(X) + e(Y)$. Also, since X is a spanning forrest we have $e(X) = v(X) - n_A = n_C - n_A$, and similarly, $e(Y) = n_C - n_A$. Putting this all together, the vertex groups in F have rank $(n_C - n_A) + (n_C - n_B) - n_C + 1 = n_C - n_A - n_B + 1$, as claimed.

For the last part of the theorem, we choose X, Y so that the elements α, β of $A(ia, ib), B(ja, jb)$ respectively are parts of $\text{Fr}X, \text{Fr}Y$ respectively. These map to elements α', β' in F and the element $\beta'^{-1}\alpha'$ will be nontrivial in F ; so F has rank at least 1. This completes the proof. \square

2.3.3 The complement of an arc on the 2-sphere is connected

By an arc $P \subset S^2$ we just mean a subset of S^2 which is homeomorphic to S^1 . We will say P separates two points a and b if they lie in different components of $S^2 \setminus P$.

Lemma 2. *If an arc P separates a and b and $P = P_1 \cup P_2$ with the arcs P_1 and P_2 meeting at a single point p , then at least one of P_1 and P_2 separates a and b .*

Proof. Assume neither P_1 nor P_2 has separates a and b . Take a set of base-points A that contains exactly one point from each component of $S^2 \setminus P$ and choose it so $\{a, b\} \subset A$. We can apply van Kampen's theorem to $U = S^2 \setminus P_1$ and $V = S^2 \setminus P_2$ and $X = U \cup V = S^2 \setminus \{p\}$, and use the last part of the lemma about pushouts of groupoids to get that $\pi_1(X)$ contains \mathbb{Z} as a retract. This is clearly false as X is homemorphic to \mathbb{R}^2 . \square

Now we can prove the complement of an arc on S^2 is connected. Assume not and let the arc P separate a and b . Then by using the above lemma over and over again, we can find a nested sequence of smaller and smaller arcs P_1, P_2, \dots each of which separates a and b . The intersection of all of these arcs is a single point $p \in P$ that can't exist: take any path from a to b that avoids p , by compactness it must avoid P_n for sufficiently large n , which is a contradiction.

3 References

- Brown, Ronald. 2006. *Topology and Groupoids*. Createspace. <http://groupoids.org.uk/pdf/tpg.pdf>.
- Brown, Ronald, and Omar Antolín Camarena. 2015. “Erratum to: Groupoids, the Phragmen–Brouwer Property, and the Jordan Curve Theorem.” *Journal of Homotopy and Related Structures* 10 (3). Springer:669–72. <https://arxiv.org/pdf/1404.0556v3>
- May, J Peter. 1999. *A Concise Course in Algebraic Topology*. University of Chicago press. <https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf>.