Two-Floodlight Illumination of Convex Polygons

V. Estivill-Castro

Laboratorio Nacional de Informática Avanzada, Xalapa, Veracruz, México

J. Urrutia

Department of Computer Science, University of Ottawa, Ottawa, Ontario, Canada.

Abstract

A floodlight of size α is a light source that projects light in a cone of size α. In this paper we study the problem of illuminating a convex polygon using floodlights. We give an $O(n^2)$ time algorithm to find an optimal pair of floodlights to illuminate a convex polygon P with n vertices; that is a pair of floodlights to illuminate a convex polygon in such a way that the sum of their sizes is minimized. When our polygon is cocircular (i.e. all of the vertices of P lie on a circle) such a pair of floodlights can be found in linear time.

Introduction

Art Gallery problems have always been of great interest in Computational Geometry. Many articles on guarding or illumination problems dealing with numerous variations of the classical Art Gallery Theorem of Chvátal have been published [CV, FI]. Problems dealing with guarding of simple polygons, orthogonal polygons [O'R, ECU], families of convex sets on the plane [FT, UZ, CRCU1], edge guards [O'R], vertex guards etc. have received much attention. For a survey of recent results on this area, see [SHER]. In this paper we continue the study of illumination problems using floodlights.

A floodlight is a light source that illuminates the area within a cone. Floodlight illumination problems were introduced in [BGLOSU]. One of the motivations for the study of floodlight illumination problems, is that many illumination or guarding devices cannot search or illuminate all around themselves in all directions. Floodlights, for example illuminate only a circular cone. As it turns out, many communication devices used in the broadcasting of satellite and telecommunication signals can be idealized as floodlights. For example, satellites transmit signals using dishes that restrict the area of coverage to a cone.

Perhaps the main open problem in this area is that known as the *Stage Illumination Problem*. In this problem, we are given a line segment L (the stage) and a set of floodlights with fixed sizes and apexes, and we are asked to decide if we can rotate the floodlights around their apexes in such a way that L is completely illuminated.

Floodlight illumination problems can be grouped in two classes, decision type problems and optimization problems. In the decision type problems, we usually have a set of floodlights and an object, or collection of objects to be illuminated [BGLOSU, BBCUZ, SS, ECO'RUX, O'RX]. In the optimization type problems, we again have an object to be illuminated and are asked to find a set of floodlights satisfying certain restrictions that illuminate our objects subject to the condition that the sum of the sizes of our floodlights or the number of floodlights used is minimized [CRCU, ECO'RUX, ECU, O'RX].

In this paper we study the Two-Floodlight Illumination Problem, that is, the problem of finding two floodlights that illuminate a convex polygon with n vertices, in such a way that the sum of their sizes is minimized. We give an $O(n^2)$ algorithm to solve the Two-Floodlight Illumination Problem. Despite the fact that we are restricting ourselves to two floodlights and a convex polygon, our problem turns out to have a rich geometric structure which is used to derive our algorithm. We show that, when the polygon is cocircular,that is, when all its vertices lie on a circle, the Two-Floodlight Illumination Problem can be solved in linear time.We have been unable to obtain a faster algorithm to solve the Two-Floodlight Illumination Problem, but contrary to the natural intuition that $O(n \log(n))$ should be achievable, we suspect that $O(n^2)$ *is* optimal.

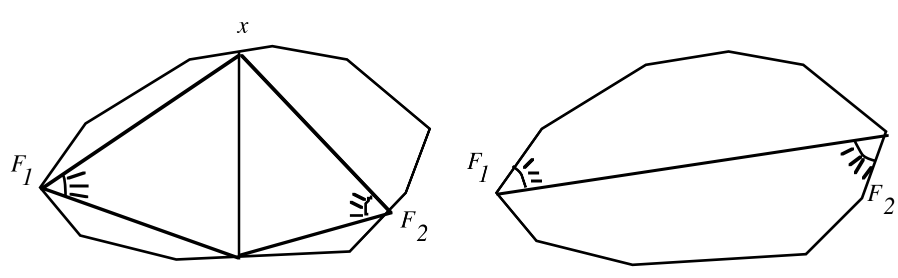
This paper is structured as follows: in Section 2 we develop an $O(n^2)$ algorithm to solve the Two-Floodlight Illumination Problem for convex polygons with the restriction that no four vertices of our polygon lie on a circle. In Section 3, we find a linear time algorithm for the Two-Floodlight Illumination Problem when all the vertices of a polygon lie on a circle. A solution for the general case can easily be obtained by combining the cocircular and the general position case, and will not be treated here. In Section 4, we present some open problems.

2. Optimal 2-Floodlight Illumination of Convex Polygons in General Circular Position

Some terminology and definitions will be given now. A floodlight F_i of size α is a light source that projects light in a cone of size α. F_i will be called an $\alpha-$ *floodlight* If the apex of an $\alpha-$ *floodlight* is located at a point p and p is a vertex of a polygon, we call it a *vertex* $\alpha-$ *floodlight*.

Consider two points a and b on the boundary of a polygon P, not necessarily vertices of P. The vertex interval (a,b) of P is defined to be the set of vertices of P that we meet when we move in the clockwise direction on the boundary of P from a to b. For short, we will refer to the interval (a,b) rather than to the vertex interval (a,b). Notice that $(a,b)\neq(b,a)$. In particular, when a and b are interior points of edges of P, $(a,b)\cup(b,a)$ is the set of vertices of P.

A pair of floodlights F_1 and F_2 that illuminates a polygon P will be called a floodlight illuminating pair and will be called a FLIP of P. If F_1 and F_2 are such that the sum of their apertures is minimized, we call them an optimal FLIP. We call F_1, F_2 an *opposite* FLIP if the intersection of the regions illuminated by F_1 and F_2 is a quadrilateral with all of its vertices on the boundary of P, see Figure 1(a) . If the interior of the regions illuminated by a FLIP F_1 and F_2 are disjoint, we call F_1, F_2 a dividing FLIP, see Figure 1(b). Observe that an optimal FLIP must be either an opposite FLIP or a dividing FLIP.



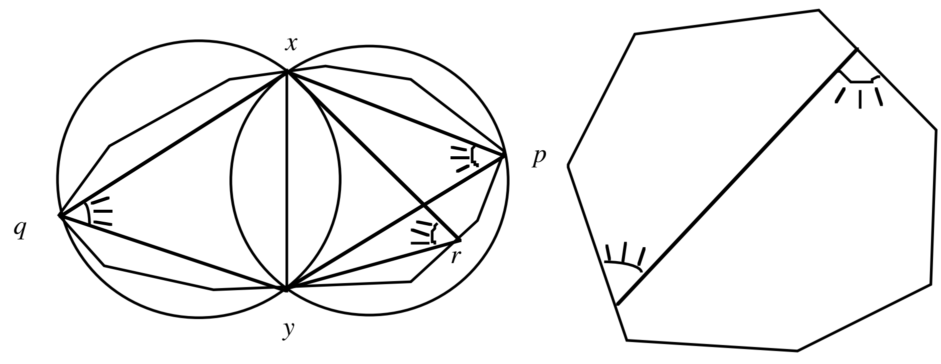
(a) (b)

Figure 1

Lemma 1: Let F_1, F_2 be an optimal FLIP of a polygon P. Then the apexes of F_1 and F_2 are located at vertices of P.

Proof: Suppose that F_1, F_2 is an optimal FLIP of a polygon P. Two cases arise:

a) F_1 and F_2 is an opposite FLIP with apexes at q and r such that r is not a vertex of P. Let x, r, y, q be the vertices of the intersection of the area illuminated by F_1 and F_2. Consider the circle $C(x, y, p)$ that passes through x, y and a vertex p of P such that $C(x, y, p)$ contains all the vertices of the interval (x, y) of P. Notice that the angle spanned by p and the line segment $l(xy)$ is smaller than that spanned by r and $l(xy)$. Therefore if we replace F_2 by a floodlight F_3 with apex in p and illuminating the angular sector determined by x, p and y, then F_1 and F_3 also illuminate P and the sum of their sizes is smaller than that of F_1 and F_2, see Figure 2(a).



(a) (b)

Figure 2

b) F_1 and F_2 form a *partitioning* pair of floodlights, see Figure 2(b). The reader may easily verify that in this case, we can slide the apex of F_1 and F_2 towards two vertices of P such that the sum of the sizes of F_1 and F_2 decrease.

QED.

Suppose now that we want to find a FLIP F_1 and F_2 of a polygon P in such a way that the apexes of F_1 and F_2 *must be located on two fixed vertices* p and q of P and the sum of the sizes of F_1 and F_2 are minimized. Call such a FLIP an optimal $(p,q) - FLIP$. We now show:

Lemma 2: Given two vertices p and q of a polygon P, we can find an opposite optimal $(p,q) - FLIP$ in linear time.

Proof: Our objective here is to find two points, x and y, on the boundary of P such that the sum of the angles $\alpha = ypx$ and $\beta = xqy$ is minimized. Consider the angles γ and δ formed by pxq and qyp as in Figure 3.

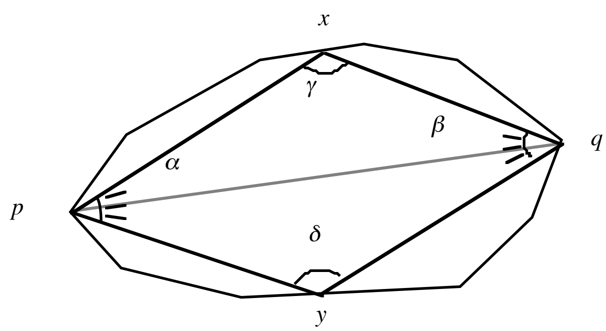
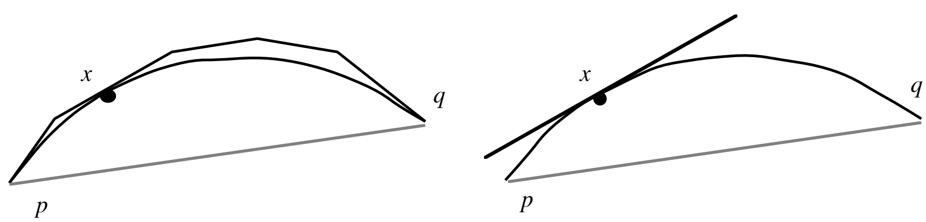


Figure 3

We now observe that minimizing $\alpha + \beta$ is equivalent to maximizing $\gamma + \delta$. However, the maximization of γ and δ can be done independently! Thus to minimize $\alpha + \beta$, we can proceed independently to maximize each of γ and δ. To locate the point x that maximizes γ we observe that this point corresponds to the largest circle $C(p,q)$ through p and q that is tangent to an edge e of P in the chain of edges of P from p to q in the clockwise direction, leaving the other edges of that chain outside of $C(p,q)$, see Figure 4(a).



(a) (b)

Figure 4

Thus all we need to do is to locate for each edge e of the chain of edges of P from p to q the point x on e that maximizes the angle pxq, see Figure 4(b). Clearly this can be done in constant time per edge, so x can be found in linear time. The same procedure is applied to find y. A degenerate case could happen here when $C(p,q)$ is tangent to the edge of P containing p, q or both. In the first and last cases simply take $x = p$, in the second case take $x = q$.

QED.

We now provide a criterion to reduce the number of pairs of vertices of P that can be apexes of an optimal *opposite* FLIP of P. Consider the set D of all diagonals of P joining all pairs of vertices of P with the property that there is a circle through their endpoints that contains P. Under our general circular position on the vertices of P, it is easy to see that the elements of D induce a triangulation $T(P)$ of P. Using standard techniques for calculating Voronoi Diagrams of convex polygons, D and $T(P)$ can be found in linear time [AGSS].

A subset of three vertices $\{u,v,w\}$ of the vertex set of P is called a $c-triple$ if $\{u,v,w\}$ is the set of vertices of a triangle of $T(P)$. It is well known that circle $C(u,v,w)$ determined by $\{u,v,w\}$ contains P. Under our general position assumption, the number of $c-triples$ of P is exactly $n-2$, see Figure 5. We say that two $c-triples$ $\{u,v,w\}$ and $\{x,y,z\}$ are adjacent if they share two common elements, i.e. $u=x$, $v=y$ and $w \neq z$. The vertices w and z will be called the antipodal vertices of the adjacent $c-triples$ $\{u,v,w\}$ and $\{x,y,z\}$.

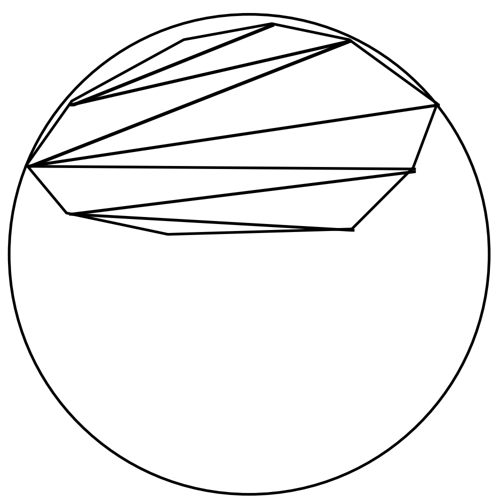


Figure 5

Given two vertices p and q of a polygon P, we say that a $c-triple$ $\{r,s,t\}$ separates p and q if p and q are both different from r, s and t and when we traverse the boundary of P from p to q (and from q to p) in the clockwise order, we meet either one of r, s or t before we meet q (resp. p), see Figure 6.

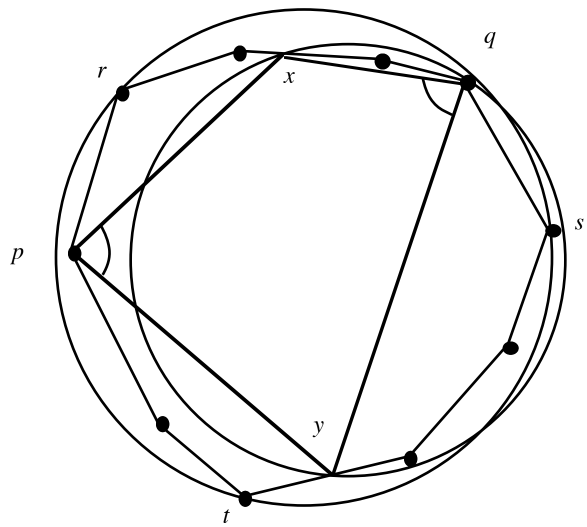


Figure 6

Lemma 3: If two vertices p and q of a polygon P are apexes of an optimal FLIP of a polygon P then there is no c – *triple* that separates p and q.

Proof: Suppose there is a c – *triple*$\{r,s,t\}$ that separates p and q, see Figure 6. We now show that p and q. cannot be apexes of an optimal FLIP F_1, F_2 of P.. Suppose otherwise that there is an optimal opposing FLIP F_1, F_2 of P such that p is the apex of F_1 and q is the apex of F_2. Suppose further that F_1 and F_2 meet at two points x and y in the interior of two edges of P. As in the proof of Lemma 1, the circle through x, y and q contains in its interior all of the vertices of P between x and y in the clockwise direction, except for q itself. Similarly for the circle trough y, p and x. However, the reader may now verify that one of these circles has to intersect the circle through r, s and t at least four times, which is impossible.

QED.

From Lemma 3 the following result follows immediately:

Corollary 4: Let p and q, be the apexes of an optimal FLIP a polygon P. Then there is a circle through p and q containing P or p and q are the antipodal vertices of two adjacent c – *triples* $\{u,v,p\}$ and $\{q,u,v\}$ of P.

We are now ready to give our algorithm O-FLIP to obtain an optimal floodlight illumination pair of a polygon P.

O-FLIP

Input a polygon P with n vertices.

Step 1: Find the triangulation $T(P)$ of P, and all $c-triples$ of P.

Step 2: Find all antipodal pairs of vertices for all the pairs of adjacent $c-triples$ of P.

Step 3: For each pair of antipodal vertices p and q of a polygon P, find an optimal opposite $(p,q)-FLIP$ in linear time.

Step 4: For each pair of vertices p and q of P find in constant time the partitioning pair of floodlights F_1, F_2 with apexes at p and q that illuminates P and minimizes the sum of the sizes of F_1 and F_2.

Output the FLIP of minimum weight identified in Steps 3 and 4.

Theorem 1: Given a convex polygon P with n vertices in general position, O-FLIP finds an optimal FLIP of P in $O(n^2)$.

Proof: Steps 1 and 2 can be carried out in linear time. Step 3 requires $O(n^2)$ time, since there are at most n-2 antipodal pairs, and solving each of them requires $O(n)$ time. In Step 4, we need to test all pairs of vertices p and q of P. However, for every pair of vertices p and q of P, there are exactly two partitioning FLIP's of P, thus the complexity of Step 4 is $O(n^2)$.

QED.

3. Optimal 2-Floodlight Illumination of Cocircular Polygons

A polygon P is called cocircular if all of its vertices lie on a circle. In this section we will prove that finding an optimal FLIP of a cocircular polygon can be done in linear time. We start by proving the following lemma:

Lemma 4: Let P be a cocircular polygon, then any optimal FLIP of P is a dividing FLIP.

Proof: Suppose that P is cocircular and that it has an opposing optimal FLIP F_1, F_2. Let $C(P)$ be the circle containing the vertices of P. Assume that the apexes of F_1, F_2 are at two vertices p and q of P and that the vertices of the quadrilateral defined by F_1, F_2 are p, x, q, y, see Figure 7.

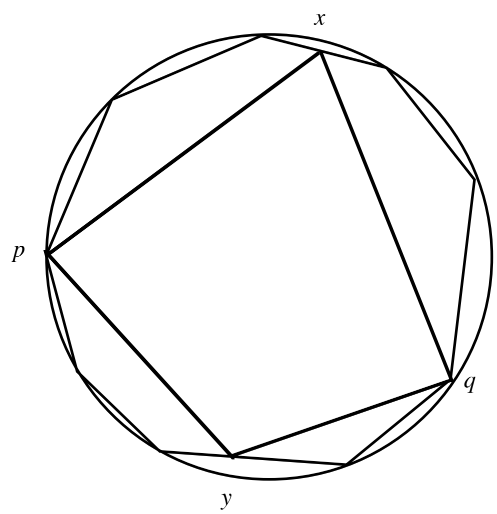


Figure 7

By Lemma 1, the circle $C(x,q,y)$ through x, q and y contains all the vertices of the interval (x,y) of P. This implies that $C(x,q,y)$ intersects $C(P)$ in at least three points, once in the middle of the arc of $C(P)$ joining the endpoints of the edge of P containing x, a second time in the middle of the arc of $C(P)$ joining the endpoints of the edge of P containing y and a third time at q. Since this is not possible our lemma follows.

QED.

By Lemma 4, all we have to do to solve the cocircular optimal FLIP problem, is to find the optimal partitioning FLIP.

Consider a cocircular polygon P and the circle $C(P)$ containing the vertices of P. Assume that $C(P)$ has radius 1 and that we measure the lenght of arcs of $C(P)$ in radians. The vertex set of P partitions the circle $C(P)$ into n arcs of sizes $\alpha_1, \ldots, \alpha_n$. The following two observations will prove useful:

Observation 1: Any FLIP F_1, F_2 of P illuminates all of $C(P)$ except two subarcs α_i and α_j of P. The sum of the sizes of F_1, F_2 is $\frac{2\pi - \alpha_i - \alpha_j}{2}$, see Figure 8.

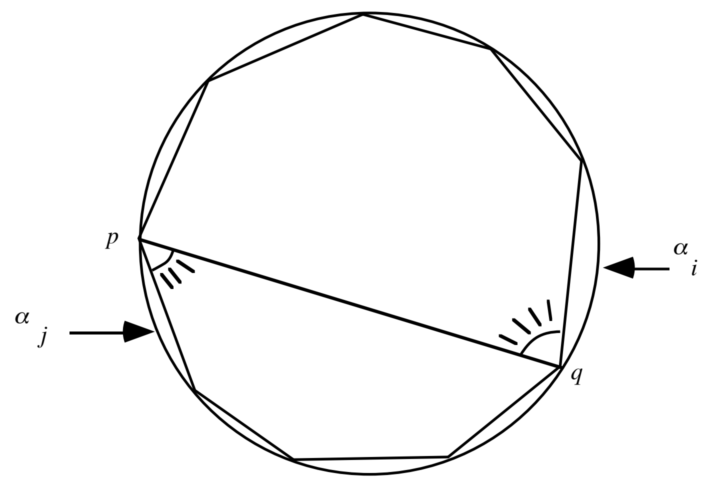


Figure 8

Observation 2: Given any two α_i and α_j of P, there is a FLIP of P that illuminates all of $C(P)$ except the arcs α_i and α_j. Notice here that if we want to avoid illuminating two consecutive arcs α_i and α_j of P, all we need to do is to illuminate P with a single floodlight placed at the vertex that separates α_i from α_j.

Theorem 2: Finding an optimal FLIP of a cocircular polygon can be done in linear time.

Proof: Choose the two largest arcs α_i and α_j of $C(P)$ and illuminate P as in Observation 2. Our result now follows.

QED.

It is now easy to see that the results of Sections 2 and 3 can be used to solve the general Two Floodlight Illumination problem in $O(n^2)$ time. The details are left to the reader. Summarizing, we have:

Theorem 3: Finding an optimal FLIP for an arbitrary convex polygons can be done in $O(n^2)$ time.

4. Conclusions

In this paper we have given an $O(n^2)$ algorithm to solve the two-floodlight illumination problem. We do not know if this algorithm is optimal. In spite of numerous efforts we have been unable to improve on the complexity of our algorithm. The main problem here is that of finding a criterion to lower the number of pairs of vertices of a polygon P which are candidates to be the apexes of an optimal FLIP.

An open question is that of deciding if the complexity of the two-floodlight illumination problem is $O(n^2)$ or $O(n \log(n))$ or better. The k-floodlight illumination problem for convex polygons seems to be hard to solve. For instance, we cannot even prove that an optimal set of k floodlights that illuminate a convex polygon must have their apexes at vertices of the polygon. We believe this to be true. In general, can we find an $O(n^k)$ algorithm to solve the optimal k-floodlight illumination problem?

Another open problem is that of finding optimal floodlight illuminating sets for simple polygons.

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