Green function and Martin kernel for higher-order fractional Laplacians in balls

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Abstract

We give the explicit formulas for the Green function and the Martin kernel for all integer and fractional powers of the Laplacian $s > 1$ in balls. As consequences, we deduce interior and boundary regularity estimates for solutions to linear problems and positivity preserving properties. Our proofs rely on a characterization of suitable s-harmonic functions and on a differential recurrence equation.

Keywords. Boggio's formula · s-harmonic functions · maximum principles
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1 Introduction

In this paper, we show that the Green function $\mathscr{G}_s$ and the Martin kernel M_s for any power $s > 1$ of the Laplacian $(-\Delta)^s$ in the unit ball $B \subset \mathbb{R}^N$, $N \in \mathbb{N}$, are given by

$$\mathscr{G}_s(x,y) := k_{N,s}|x-y|^{2s-N} \int_0^{\rho(x,y)} \frac{v^{s-1}}{(v+1)^{\frac{N}{2}}}\, dv \qquad \text{for } x,y \in \mathbb{R}^N,\ x \neq y, \tag{1.1}$$

where

$$\rho(x,y) := \frac{(1-|x|^2)_+(1-|y|^2)_+}{|x-y|^2}, \qquad k_{N,s} := \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}} 4^s \Gamma(s)^2} \tag{1.2}$$

and

$$M_s(x,\theta) \;=\; \lim_{B \ni y \to \theta} \frac{\mathscr{G}_s(x,y)}{(1-|y|^2)^s} = \frac{k_{N,s}}{s} \frac{(1-|x|^2)_+^s}{|\theta - x|^N} \qquad \text{for } x \in \mathbb{R}^N,\ \theta \in \partial B. \tag{1.3}$$

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Here, $(-\Delta)^s$ is the pseudo-differential operator associated to the symbol $|\cdot|^{2s}$. Further notation is explained in Section 2 below, where in particular a pointwise evaluation in terms of finite differences can be found (see (2.1)). Our main result regarding the Green function is the following.

Theorem 1.1. *Let $s>1$, $N\in\mathbb{N}$, $f\in C^\alpha(\overline{B})$ for some $\alpha\in(0,1)$, $2s+\alpha\notin\mathbb{N}$, and*

$$u:\mathbb{R}^N\to\mathbb{R} \quad \textit{be given by} \quad u(x):=\int_B \mathscr{G}_s(x,y)\,f(y)\,dy, \tag{1.4}$$

then $u\in C^{2s+\alpha}(B)\cap C_0^s(B)$ is the unique pointwise solution (in $\mathscr{H}_0^s(B)$) of

$$(-\Delta)^s u=f \quad \textit{in } B, \qquad u\equiv 0 \quad \textit{on } \mathbb{R}^N\backslash B, \tag{1.5}$$

and there is $C>0$ such that

$$\|\operatorname{dist}(\cdot,\partial B)^{-s}u\|_{L^\infty(B)}<C\|f\|_{L^\infty(B)}. \tag{1.6}$$

For the relevance and applications of the higher-order fractional Laplacian we refer to [4,21]. The function $\mathscr{G}_s$ is known as *Boggio's formula*, see [7,8,10,14]. Since $\mathscr{G}_s$ is a positive function, Theorem 1.1 shows that problems on balls enjoy a positivity preserving property. This is not the case for general domains, see [4]. The proof of Theorem 1.1 is based on a differential recurrence formula for $\mathscr{G}_s$ in terms of $\mathscr{G}_{s-1}$ and an explicit function P_{s-1} which is $(s-1)$-harmonic in the ball, see Lemma 3.1 below. Since the validity of Boggio's formula is known for $s\in(0,1]$, this allows us to implement an induction argument to extend this result to all $s>1$. We remark that our approach also provides an alternative proof for $s\in\mathbb{N}$. Two key elements in the proof are an elementary pointwise calculation of $-\Delta_x\mathscr{G}_s(x,y)$ for $y\neq x$ and $s>1$ (see Lemma 3.1) and the introduction of *higher-order Martin kernels* (1.3), which we use to characterize a large class of s-harmonic functions, see Proposition 1.2 below. Martin kernels were introduced in [19] for $s=1$ to provide an analogue of Poisson kernels in nonsmooth domains and in [6] for $s\in(0,1)$ to give representation formulas for s-harmonic functions which are singular at the boundary of the domain (a purely nonlocal phenomenon). Our construction is similar to the one presented in [1] and we generalize it to $s>1$.

Note that the regularity of solutions —in particular, integrability, which is used to show uniqueness —is more involved for higher-order fractional powers of the Laplacian. For instance, consider the function $u(x)=(1-|x|^2)_+^s$ for $s>0$, which is a pointwise solution of $(-\Delta)^s u=C$ in B for some constant $C>0$ (see *e.g.* [11,21]). Clearly u belongs to $H^{2s}(B)$ if s is an integer, since in this case u is a polynomial. For general s, however, u may have derivatives which blow-up at the boundary, for example terms involving $(1-|x|^2)_+^{s-2}$ are *not* in $L^2(B)$ if $s\in(1,\frac{3}{2})$. To circumvent this difficulty and show that $u\in\mathscr{H}_0^s(B)$, we use standard interpolation theory as in [18,24].

In [10] the authors show independently the validity of Boggio's formula for all $s>0$ considering only smooth functions with compact support as right-hand sides. The proofs in [10] are very different from ours and rely on covariance under Möbius transformations and computations using Hypergeometric functions.

Our main result regarding Martin kernels is the following.

Proposition 1.2. *Let $s > 0$ and μ be a finite Radon measure on ∂B. The function*

$$u(x) = \int_{\partial B} M_s(x,z)\, d\mu(z) \qquad \text{for } x \in \mathbb{R}^N$$

is s-harmonic in B, that is, $(-\Delta)^s u(x) = 0$ for every $x \in B$.

Proposition 1.2 was known for $s \in (0,1)$, see [1, 6]. See also Remark 4.6 for more on s-harmonic functions.

To close this introduction, we remark that the explicit formulas for the Green function and the Martin kernel are the first step towards developing a comprehensive theory for linear and nonlinear problems in general domains involving the higher-order fractional Laplacians. In particular, the transition between the Laplacian and the bilaplacian Δ^2 —whose solutions exhibit very different qualitative properties—can be studied in detail. In this regard, we refer to [3], where Theorem 1.1 and Proposition 1.2 are complemented with other kernels and suitable boundary traces to fully characterize solutions of nonhomogeneous Dirichlet boundary value problems. For example, solutions of (1.5) which do not satisfy (1.6) are constructed in the following corollary. In particular, the failure of (1.6) implies that the Green representation in (1.4) does not hold.

Corollary 1.3. *Let $s > 1$, $j \in (0,s) \cap \mathbb{N}$, and μ be a finite Radon measure on B. Then the function $u_j : \mathbb{R}^N \to \mathbb{R}$, given by $u_j(x) = \int_B \mathscr{G}_{s-j}(x,y) \int_B \mathscr{G}_j(y,z)\, d\mu(z) dy$ is a distributional solution of $(-\Delta)^s u_j = \mu$. In particular, if $d\mu(z) = f(z)\, dz$ for some $f \in C^\alpha(\overline{B})$ then $u_j \in C_0^{s-j}(B)$.*

The organization of the paper is the following. The notation is introduced in Section 2 and the proofs of Theorem 1.1 and Corollary 1.3 are written in Section 3 together with some remarks on s-harmonic functions. In the Appendix, we prove the differential recurrence equation involving Boggio's formula and we present results regarding the interchange of derivatives.

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2 Notation

Let $N \in \mathbb{N}$ and $U, D \subset \mathbb{R}^N$ be nonempty measurable sets. We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function and $|U|$ the Lebesgue measure. The notation $D \subset\subset U$ means that $\overline{D}$ is compact and contained in the interior of U. For $x \in \mathbb{R}^N$ and $r > 0$ let $B_r(x)$ denote the open ball centered at x with radius r, moreover we fix $B := B_1(0)$.

For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \ : \ (1 + |\xi|^2)^{\frac{s}{2}}\, \widehat{u} \in L^2(\mathbb{R}^N) \right\}$, where $\widehat{u}$ denotes the Fourier transform of u. Moreover, if U is open and $s > 0$, we define

$$\mathscr{H}_0^s(U) := \{u \in H^s(\mathbb{R}^N) \ : \ u = 0 \text{ on } \mathbb{R}^N \setminus U\}$$

and, if U is smooth, we put $H^s(U) := \{u 1_U \ : \ u \in H^s(\mathbb{R}^N)\}$.

For $m \in \mathbb{N}_0$, $\sigma \in (0,1]$, $s = m+\sigma$, and U open, we write $C^s(U)$ (resp. $C^s(\overline{U})$) to denote the space of m-times continuously differentiable functions in U (resp. $\overline{U}$) whose derivatives of order m are locally σ-Hölder continuous in U (or Lipschitz continuous if $\sigma = 1$). Moreover, for $s \in [0,\infty]$, $C^s_c(U) := \{u \in C^s(\mathbb{R}^N) : \operatorname{supp} u \subset\subset U\}$ and $C^s_0(U) := \{u \in C^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus U\}$, where $\operatorname{supp} u := \overline{\{x \in U \,:\, u(x) \neq 0\}}$ is the support of u. Observe that, under this definition, $C^1(B)$ is *not* the set $\mathscr{C}^1(U)$ of continuously differentiable functions in U, but the set of locally Lipschitz functions.

Let $u : U \to \mathbb{R}$ be a function. We use $u^+ := u_+ := \max\{u,0\}$ and $u^- := -\min\{u,0\}$ to denote the positive and negative part of u respectively and if $f : U \times D \to \mathbb{R}$ we write $(-\Delta_x)^s f(x,y)$ to denote derivatives with respect to x, whenever they exist in some appropriate sense.

For $m \in \mathbb{N}_0$, $\sigma \in (0,1)$, and $s = m+\sigma$, the operator $(-\Delta)^s$ can be defined via finite differences (see [2, equation (1)]), namely, for $u \in C^{2s+\alpha}(U) \cap L^\infty(\mathbb{R}^N)$

$$\begin{aligned} (-\Delta)^s u(x) &:= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{\delta_{m+1} u(x,y)}{|y|^{N+2s}}\, dy, \qquad x \in \mathbb{R}^N, \\ \text{where} \qquad \delta_{m+1} u(x,y) &:= \sum_{k=-m-1}^{m+1} (-1)^k \binom{2(m+1)}{m+1-k} u(x+ky) \qquad \text{for } x,y \in \mathbb{R}^N \end{aligned} \tag{2.1}$$

is a finite difference of order $2(m+1)$, and $c_{N,s}$ is a *positive* normalization constant (for the precise value, see [2, equation (2)]). The Fourier symbol of $(-\Delta)^s$ as given in (2.1) is $|\xi|^{2s}$ (see [22, Lemma 25.3] or [2, Theorem 1.9]); moreover, if $u \in C^{2s+\alpha}(U) \cap L^\infty(\mathbb{R}^N)$ then $(-\Delta)^s u(x) = (-\Delta)^m (-\Delta)^\sigma u(x)$ for every $x \in U$ (see [2, Corollary 1.3]), but in general the fractional Laplacian $(-\Delta)^\sigma$ cannot be interchanged freely with the Laplacian $(-\Delta)$, this would require extra regularity assumptions on u, particularly across the boundary ∂U (see [2]). For instance, for $u \in C^\infty_c(\mathbb{R}^N)$ we have

$$(-\Delta)^s u = (-\Delta)^\sigma (-\Delta)^m u = (-\Delta)^m (-\Delta)^\sigma u \quad \text{in } \mathbb{R}^N.$$

We use $\mathscr{D}'$ to denote the space of distributions in $\mathbb{R}^N$, *i.e.* $\mathscr{D}' := (C^\infty_c(\mathbb{R}^N))'$, and denote $\langle \cdot\,, \cdot \rangle : \mathscr{D}' \times C^\infty_c(U) \to \mathbb{R}$ as the dual pairing of $\mathscr{D}'$ and $C^\infty_c(U)$ in U. As usual, for suitable $u : U \to \mathbb{R}$ we identify u with its associated distribution $T_u : C^\infty_c(U) \to \mathbb{R}^N$ given by $\langle T_u\,, f \rangle = \int_{\mathbb{R}^N} u(x) f(x)\, dx$ for all $f \in C^\infty_c(U)$.

For $\Omega \subset \mathbb{R}^N$ open and bounded and $f \in \mathscr{D}'$ a function $u \in L^1(\mathbb{R}^N)$ is called a *distributional solution* of $(-\Delta)^s u = f$ if $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$ and

$$\langle (-\Delta)^s u, \varphi \rangle = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{2.2}$$

Moreover, if $f = 0$, then u is called *distributionally s-harmonic in* Ω.

3 Representation of solutions in the ball

Let $m \in \mathbb{N}_0$, $\sigma \in (0,1]$, $s = m+\sigma$, $N \in \mathbb{N}$, and $d(x) := \min_{y \in \mathbb{R}^N \setminus B} |x-y|$ for $x \in \mathbb{R}^N$ be the distance of x to the complement of B. In this section provide a representation formula for solutions in a ball in terms of a kernel $\mathscr{G}_s$ given by *Boggio's formula* (1.1). We show that

$u(x) = \int_B \mathscr{G}_s(x,y) f(y)\, dy$ for $x \in \mathbb{R}^N$ if and only if u is a solution (in a suitable sense) of $(-\Delta)^s u(x) = f$ in B and $u \equiv 0$ on $\mathbb{R}^N \setminus B$.

A key ingredient in our proofs is the following iteration formula.

Lemma 3.1. *If $s > 1$ then $-\Delta_x \mathscr{G}_s(x,y) = \mathscr{G}_{s-1}(x,y) - k_{N,s} 4(s-1) P_{s-1}(x,y)$ for all $x, y \in B$, $x \neq y$, where*

$$P_{s-1}(x,y) := \frac{(1-|x|^2)_+^{s-2}(1-|y|^2)_+^{s-1}(1-|x|^2|y|^2)}{[x,y]^N} \tag{3.1}$$

for $x, y \in \mathbb{R}^N$, $x \neq y$, and $[x,y] := \sqrt{|x|^2|y|^2 - 2x\cdot y + 1}$.

The proof of Lemma 3.1 is done by an elementary—but lengthy—direct computation and for the reader's convenience we give a proof in Appendix A.

Remark 3.2.

1. For $\sigma = \frac{1}{2}$, $N = 1$, the substitution $t = \sqrt{v}$ yields $G_{1,\frac{1}{2}}(x,y) = \frac{1}{\pi} \ln\left(\frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{|x-y|}\right)$, which agrees with [8, Theorem 3.1, formula (3.2)] and for $s \in \mathbb{N}$, the change of variables $\tilde{v} = \sqrt{v+1}$ yields $\mathscr{G}_s(x,y) = 2k_{N,s}|x-y|^{2s-N} \int_1^{p(x,y)} (v^2-1)^{s-1} v^{1-N}\, dv$, with $p(x,y) = [x,y]|x-y|^{-1}$, which is another known expression for Boggio's formula, see [14].
2. By rescaling we have that Theorem 1.1 holds in balls of radius $r > 0$ using $\rho_r(x,y) = (r^2-|x|^2)(r^2-|y|^2)r^{-2}|x-y|^{-2}$ in place of ρ in (1.1).

The following is a useful auxiliary Lemma.

Lemma 3.3. *Let $N \in \mathbb{N}$, $R, s, r > 0$, and $\varepsilon \in (0, \min\{N, s\})$. Then*

$$R^{2s-N} \int_0^{\frac{r}{R^2}} \frac{t^{s-1}}{(t+1)^{\frac{N}{2}}}\, dt \le \frac{2}{s} R^{\varepsilon - N} r^{s-\frac{\varepsilon}{2}}.$$

Proof. Let $\delta \in (0,1)$ such that $\varepsilon := \frac{N\delta}{2} \in (0, \min\{N, s\})$. By a change of variables we have that

$$R^{2s-N} \int_0^{\frac{r}{R^2}} \frac{t^{s-1}}{(t+1)^{\frac{N}{2}}}\, dt = R^{-N} \int_0^{r} \frac{t^{s-1}}{(tR^{-2}+1)^{\frac{N}{2}}} \frac{R^\varepsilon}{R^\varepsilon}\, dt = R^{\varepsilon-N} \int_0^r \frac{t^{s-1}}{(tR^{\delta-2}+R^\delta)^{\frac{N}{2}}}\, dt.$$

Note that the function $R \mapsto tR^{\delta-2} + R^\delta$ has a unique minimum in $(0,\infty)$ at $R_0 = k\sqrt{t}$ with $k = \sqrt{\frac{2-\delta}{\delta}}$. Therefore

$$\begin{aligned} R^{\varepsilon-N} \int_0^r \frac{t^{s-1}}{(tR^{\delta-2}+R^\delta)^{\frac{N}{2}}}\, dt &\le R^{\varepsilon-N} \int_0^r \frac{t^{s-1}}{(tR_0^{\delta-2}+R_0^\delta)^{\frac{N}{2}}}\, dt = R^{\varepsilon-N} \int_0^r \frac{t^{s-1}}{(t^{\frac{\delta}{2}}(k^{\delta-2}+k^\delta))^{\frac{N}{2}}}\, dt \\ &\le R^{\varepsilon-N} \int_0^r \frac{t^{s-1-\frac{\varepsilon}{2}}}{k^\varepsilon}\, dt = \frac{k^{-\varepsilon}}{s-\frac{\varepsilon}{2}} R^{\varepsilon-N} r^{s-\frac{\varepsilon}{2}} \le \frac{2}{s} R^{\varepsilon-N} r^{s-\frac{\varepsilon}{2}}, \end{aligned}$$

since $\varepsilon < s$ and $k^{-\varepsilon} = \frac{\delta^{\frac{\varepsilon}{2}}}{(2-\delta)^{\frac{\varepsilon}{2}}} \le \delta^{\frac{\varepsilon}{2}} \le \delta^{\frac{N\delta}{4}} \le 1$, because $\delta \in (0,1)$. ∎

3.1 Interior and boundary regularity

Lemma 3.4. *Let $s>1$, $1<p\leq\infty$, $f\in L^p(B)$, and $v(x):=\int_B P_{s-1}(x,y)f(y)\,dy$, $x\in B$. If $p>\frac{N}{s}$, then $v\in C^\infty(B)$ and for all $\alpha\in\mathbb{N}_0^N$ there is $C=C(N,s,\alpha)>0$*

$$\|d^{2-s+|\alpha|}\partial^\alpha v\|_{L^\infty(B)}\leq C\|f\|_{L^p(B)}. \tag{3.2}$$

Proof. In the following let $C_i=C_i(N,s,p)>0$, $i=1,2,\dots$ be constants. Let $x,y\in B$, then

$$[x,y]=\sqrt{|x|^2|y|^2-2x\cdot y+1}\geq 1-|x||y|\geq 1-|y|\geq\frac{1}{2}(1-|y|^2), \tag{3.3}$$

and therefore $P_{s-1}(x,y)\leq(1-|x|^2)^{s-2}C_1[x,y]^{s-N}$ for $s>1$. Moreover,

$$[x,y]\geq C_2\left|y-\frac{x}{|x|}\right|\qquad\text{for all } x\in B\setminus B_{\frac{3}{4}}(0). \tag{3.4}$$

Indeed, denote $|x|=r$, $\theta=\frac{x}{|x|}$ and note that $[r\theta,y]=|ry-\theta|$ and, for $r>3/4$,

$$\begin{aligned}|ry-\theta|^2=|(r-1)y+y-\theta|^2&=(1-r)^2|y|^2-2(1-r)\langle y,y-\theta\rangle+|y-\theta|^2\\&\geq-2(1-r)\langle\theta,y-\theta\rangle-2(1-r)|y-\theta|^2+|y-\theta|^2\\&\geq-2(1-r)|y|+2(1-r)-2(1-r)|y-\theta|^2+|y-\theta|^2\\&\geq-2(1-r)|y-\theta|^2+|y-\theta|^2=\frac{|y-\theta|^2}{2},\end{aligned}$$

which implies (3.4). Note that (3.4) gives that there is $C_3>0$ such that

$$\sup_{x\in B}\int_B[x,y]^{s-N}\,dy\leq C_3. \tag{3.5}$$

Next, let $f\in L^p(B)$, $p\in(1,\infty]$, $s>\frac{N}{p}$, and define $v(x)=\int_B P_{s-1}(x,y)f(y)\,dy$ for $x\in B$. Note that for every $\alpha\in\mathbb{N}_0^N$ there is $C=C(\alpha)>0$ such that $|\partial^\alpha v(x)|\leq C(\alpha)\|f\|_{L^p(B)}$ for all $x\in B_{\frac{3}{4}}(0)$. Moreover, for $|x|>\frac{3}{4}$ we have with $q=\frac{p}{p-1}$ for $p<\infty$ and $q=1$ for $p=\infty$

$$\begin{aligned}|v(x)|&\leq(1-|x|^2)^{s-2}\|f\|_{L^p(B)}\left(\int_B(1-|y|^2)^{(s-1)q}(1-|x|^2|y|^2)^q[x,y]^{-Nq}\,dy\right)^{\frac{1}{q}}\\&\leq 2^s(1-|x|^2)^{s-2}\|f\|_{L^p(B)}\left(\int_B[x,y]^{(s-N)q}\,dy\right)^{\frac{1}{q}}\leq C_4(1-|x|^2)^{s-2}\|f\|_{L^p(B)},\end{aligned}$$

since $(s-N)+\frac{N}{q}=s-\frac{N}{p}>0$ and using (3.3) and (3.5). Arguing similarly one can obtain (3.2) for derivatives of order k, since terms of the form $(1-|x|^2)^{s-2}[x,y]^{-N-k}$ can be bounded by $(1-|x|^2)^{s-2-k}[x,y]^{-N}$. Thus, proceeding as above, $|\partial^\alpha v(x)|\leq C_5\|f\|_{L^p(B)}(1-|x|^2)^{s-2-|\alpha|}$ for all $i\in\{1,\dots,N\}$, and the Lemma follows. ∎

Proposition 3.5. *Let $1 \le p \le \infty$, $k \in \mathbb{R}$, $s > 0$, $f : B \to \mathbb{R}$ such that $d^k f \in L^p(B)$, and u as in (1.4). If $s > k$, then there is $C = C(N,s,k,p) > 0$ such that $\|d^{-s}u\|_{L^p(B)} \le C\|d^k f\|_{L^p(B)}$.*

Proof. First, note that given $\varepsilon > 0$ there is $C = C(\varepsilon) > 0$ such that $\int_B |x-y|^{\varepsilon-N} d(x)^{-p\frac{\varepsilon}{2}}\, dx \le C$ for all $y \in B$ and $p < \frac{2}{\varepsilon}$. In the following let $C_i = C_i(N,s,p,k) > 0$, $i = 1,2,\dots$ be constants. First let $1 \le p < \infty$ and fix $0 < \varepsilon < \min\{1, s-k, \frac{1}{p}\}$. Then, by Lemma 3.3 and Hölder's inequality,

$$\begin{aligned}
\|d^{-s}u\|_{L^p(B)}^p &\le C_1 \int\limits_B \left(\int\limits_B |x-y|^{\varepsilon-N} d(x)^{-\frac{\varepsilon}{2}} d^{s-k-\frac{\varepsilon}{2}}(y) d^k(y) |f(y)|\, dy \right)^p dx \\
&\le C_2 \int\limits_B \left(\int\limits_B |x-y|^{\varepsilon-N} d(x)^{-\frac{\varepsilon}{2}} d^k(y) |f(y)|\, dy \right)^p dx \\
&\le C_3 \int\limits_B \left(\int\limits_B |x-y|^{\varepsilon-N}\, dy \right)^{p-1} \left(\int\limits_B d(x)^{-\frac{p\varepsilon}{2}} |x-y|^{\varepsilon-N} d^{kp}(y) |f(y)|^p\, dy \right) dx \\
&\le C_4 \int\limits_B \int\limits_B d(x)^{-p\frac{\varepsilon}{2}} |x-y|^{\varepsilon-N} d^{kp}(y) |f(y)|^p\, dy\, dx \\
&= C_5 \int\limits_B d^{kp}(y) |f(y)|^p \int\limits_B d(x)^{-p\frac{\varepsilon}{2}} |x-y|^{\varepsilon-N}\, dx\, dy \le C_6 \|d^k f\|_{L^p(B)}.
\end{aligned}$$

Next let $p = \infty$, $x \in \mathbb{R}^N \setminus \{0\}$. Then

$$\begin{aligned}
|d^{-s}(x)u(x)| &\le k_{N,s} \|d^k f\|_{L^\infty(B)} d^{-s}(x) \int\limits_B |x-y|^{2s-N} d^{-k}(y) \int\limits_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}} \frac{t^{s-1}}{(t+1)^{\frac{N}{2}}}\, dtdy \\
&\le 2^s k_{N,s} \|d^k f\|_{L^\infty(B)} \int\limits_B |x-y|^{2s-N} d^{s-k}(y) \int\limits_0^{|x-y|^{-2}} \frac{t^{s-1}}{((1-|y|^2)(1-|x|^2)t+1)^{\frac{N}{2}}}\, dtdy \\
&\le 2^s k_{N,s} \|d^k f\|_{L^\infty(B)} \int\limits_B |x-y|^{2s-N} \int\limits_0^{|x-y|^{-2}} d^{s-k}(y) \frac{t^{s-1}}{((1-|y|^2)t+1)^{\frac{N}{2}}}\, dtdy.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int\limits_B |x-y|^{2s-N} \int\limits_0^{|x-y|^{-2}} d^{s-k}(y) \frac{t^{s-1}}{((1-|y|^2)t+1)^{\frac{N}{2}}}\, dtdy \\
&\qquad \le \int\limits_B |x-y|^{2s-N}\, dy + \int\limits_B |x-y|^{2s-N} d^{s-k}(y) \int\limits_1^{\max\{|x-y|^{-2},1\}} \frac{t^{s-1}}{((1-|y|^2)t+1)^{\frac{N}{2}}}\, dt\, dy
\end{aligned}$$

$$\leq C_7 + \int_B \int_{\min\{1,|x-y|^2\}}^{1} d^{s-k}(y) \frac{t^{s-1}}{((1-|y|^2)t+|x-y|^2)^{\frac{N}{2}}}\, dt\, dy$$

$$\leq C_7 + \int_B d^{s-k}(y) \int_0^1 \frac{1}{((1-|y|^2)t+|x-y|^2)^{\frac{N}{2}}}\, dt\, dy$$

$$\leq C_8 + C_8 \int_B d^{s-k-1}(y) \left| ((1-|y|^2)t+|x-y|^2)^{1-\frac{N}{2}} \Big|_0^1 \right| dy$$

$$\leq C_9 + C_9 \int_B d^{s-k-1}(y)|x-y|^{2-N}\, dy < \infty.$$

Hence the statement also holds for $p=\infty$. □

The following remarks are used in the proof of Theorem 3.7 below.

Remark 3.6. For $s \in \mathbb{R}$ let $H^s(B)$ and $\mathscr{H}_0^s(B)$ as in Section 2.

1. For every $s \geq 0$ and $u : \mathbb{R}^N \to \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \backslash B$, there is $k > 0$ such that

$$k\|u\|^2_{\mathscr{H}_0^s(B)} \leq \|u\|^2_{H^s(B)} + \|d^{-s}u\|^2_{L^2(B)} \leq \frac{1}{k}\|u\|^2_{\mathscr{H}_0^s(B)}, \tag{3.6}$$

   see [24, Section 4.3.2, eq. (7)].

2. By [24, Section 5.7.1 page 402], the Laplacian with Dirichlet boundary conditions gives an isomorphic mapping from $H^{2+s}(B)$ onto $H^s(B)$ for all $-1 < s < \infty$, $s \neq -\frac{1}{2}$, and therefore,

$$(-\Delta)^{-1} : H^s(B) \to H^{s+2}(B) \qquad \text{for all } s > -1,\ s \neq -\frac{1}{2}. \tag{3.7}$$

   The inverse Laplacian $(-\Delta)^{-1}$ is represented by the Green function $\mathscr{G}_1$ whenever the corresponding integral is finite.

3. Let $(\mathscr{H}_0^s(B))'$ denote the dual space of $\mathscr{H}_0^s(B)$. Then, by [24, Theorem 2.10.5/1] (see also [20]),

$$(\mathscr{H}_0^s(B))' = H^{-s}(B) \qquad \text{for } s \in \mathbb{R}. \tag{3.8}$$

Theorem 3.7. *Let $s > 0$, $f \in C^\alpha(\overline{B})$ for some $\alpha \in (0,1)$, $2s+\alpha \notin \mathbb{N}$, and u as in (1.4). Then*

$$u \in C^{2s+\alpha}(B) \cap C_0^s(B) \cap \mathscr{H}_0^s(B).$$

Proof. For $s \in \mathbb{N} \cup (0,1)$ the result is known, see [14, Section 4.2.1] and [1, 8, 15, 17, 23]. We argue by induction on s. Let $s > 1$, $s \notin \mathbb{N}$, and consider the case $2\sigma + \alpha \in (0,1)$ (the other cases can be proved similarly). By the induction hypothesis, we have that $\mathscr{G}_{s-1}(\cdot,y), P_{s-1}(\cdot,y) \in L^1(B)$ and, by Lemma 3.1,

$$\mathscr{G}_s(x,y) = \int_B \mathscr{G}_1(x,z)\mathscr{G}_{s-1}(z,y)\, dz - C\int_B \mathscr{G}_1(x,z)P_{s-1}(z,y)\, dz \qquad \text{for } x,y \in B \tag{3.9}$$

with $C = 4k_{N,s}(s-1)$. If u is given by (1.4), then (3.9) implies that $u = u_1 - Cu_2$, where

$$u_1(x) := \int_B \mathscr{G}_1(x,z)v_1(z)\,dz, \quad v_1(z) := \int_B \mathscr{G}_{s-1}(z,y)f(y)\,dy,$$
$$u_2(x) := \int_B \mathscr{G}_1(x,z)v_2(z)\,dz, \quad v_2(z) := \int_B P_{s-1}(z,y)f(y)\,dy.$$

Then $v_1 \in C^{2s-2+\alpha}(B)$, by the induction hypothesis, and then $u_1 \in C^{2s+\alpha}(B)$, by classical elliptic regularity. Furthermore, $v_2 \in C^\infty(B)$, by Lemma 3.4, and thus $u_2 \in C^\infty(B)$. Therefore $u \in C^{2s+\alpha}(B)$ and $u \in C_0^s(B)$, by Proposition 3.5.

It remains to show that $u \in \mathscr{H}_0^s(B)$. By (3.6) and Proposition 3.5, it suffices to show that $u \in H^s(B)$. Since $v_1 \in \mathscr{H}_0^{s-1}(B) \subset H^{s-1}(B)$, by the induction hypothesis, we obtain that $u_1 \in H^{s+1}(B) \subset H^s(B)$.

We now show that $u_2 \in H^s(B)$ arguing differently according to the value of s.

Assume first that $1 < s < \frac{3}{2}$. Then there is $C > 0$ such that

$$\int_B v_2(x)\varphi(x)\,dx \le C\int_B (1-|x|^2)^{s-2}\varphi(x)\,dx \le C\|d^{-(2-s)}\varphi\|_{L^2(B)} \le C\|\varphi\|_{\mathscr{H}_0^{2-s}(B)} \tag{3.10}$$

for $\varphi \in \mathscr{H}_0^{2-s}(B)$, by (3.6). Then the functional $\mathscr{H}_0^{2-s}(B) \ni \varphi \mapsto \int_B v_2\varphi\,dx$ is linear and bounded. Therefore, $v_2 \in (\mathscr{H}_0^{2-s}(B))' = H^{s-2}(B)$, by (3.8), and thus $u_2 \in H^s(B)$, by (3.7) (note that $\int_B \mathscr{G}_1(x,y)\delta(y)^{s-2}\,dy$ is finite for every $x \in B$ because $s > 1$).

Now, let $s = \frac{3}{2}$ and fix $p \in (\frac{2N}{N+1}, 2)$. Then $v_2 \in L^p(B)$ and thus $u_2 \in W^{2,p}(B) \subset H^s(B)$, by Sobolev embeddings (see *e.g.* [24, Section 4.6.1]) and (3.7).

Furthermore, if $2 > s > \frac{3}{2}$, then Lemma 3.4 implies that $v_2 \in L^2(\mathbb{R}^N)$ and then $u_2 \in H^2(B) \subset H^s(B)$, by (3.7) and Sobolev embeddings.

For $s = m + \sigma > 2$ with $\sigma \le \frac{1}{2}$, fix

$$q := (1 - \frac{\sigma}{2})^{-1} \qquad \text{and} \qquad p := \frac{2-2\sigma}{1-\sigma(2-\sigma)}. \tag{3.11}$$

Then, by Lemma 3.4 and complex interpolation (see [18, Proposition 2.4]),

$$v_2 \in W^{m-2,p}(B) \cap W^{m-1,q}(B) \subset [W^{m-2,p}(B), W^{m-1,q}(B)]_\sigma = H^{s-2}(B).$$

Therefore $v_2 \in H^{s-2}(B)$, which yields $u_2 \in H^s(B)$, by (3.7).

Finally, if $s = m + \sigma > 2$ and $\sigma > \frac{1}{2}$, then $v_2 \in H^{m-1}(B) \subset H^{s-2}(B)$, by Lemma 3.4. But then $u_2 \in H^s(B)$, by (3.7), also in this case and the proof is finished. □

Remark 3.8.

1. If $u_s := \int_B \mathscr{G}_s(\cdot,y)f(y)\,dy \in H^s(B)$, whenever $f \in L^p(B)$, $p > \frac{N}{s}$, and $s \in (0,1)$, then Theorem 3.7 would also hold for $f \in L^p(B)$ with $p > \frac{N}{s}$ with a very similar proof.
2. Arguing as in the proof of Theorem 3.7 one can show that $u_s(x) := (1-|x|^2)_+^s$, $x \in \mathbb{R}^N$, belongs to $\mathscr{H}_0^s(B)$. Indeed, for $m \in \mathbb{N}_0$, $\sigma \in (0,1]$, and $s = m+\sigma$, we have that $u_s \in H^{m+1}(B) \subset H^s(B)$ if $\sigma > \frac{1}{2}$ and $u_s \in W^{m,p}(B) \cap W^{m+1,q}(B) \subset H^s(B)$ if $\sigma \le \frac{1}{2}$, where p and q are as in (3.11). But then $u_s \in \mathscr{H}_0^s(B)$, by (3.6).

4 The Martin kernel and s-harmonic functions

For $s > 0$ we define M_s the *s-Martin kernel for the ball* by (see for example [1,6])

$$M_s(x,\theta) := \lim_{z\to\theta, z\in B} \frac{\mathscr{G}_s(x,z)}{(1-|z|^2)^s} \qquad \text{for } x\in B,\ \theta\in\partial B.$$

The next Lemma provides an explicit formula for M_s.

Lemma 4.1. *Let $s > 0$ and $N \geq 1$. Then*

$$M_s(x,\theta) = \frac{k_{N,s}}{s}\frac{(1-|x|^2)_+^s}{|\theta-x|^N} \qquad \textit{for } x\in B,\ \theta\in\partial B,$$

where $k_{N,s}$ is as in (1.2).

Proof. For $x,z\in\mathbb{R}^N$ with $x\neq z$ and $\rho(x,z) = (1-|x|^2)_+(1-|z|^2)_+|x-z|^{-2}$ let $t = \rho(x,z)$, then

$$\mathscr{G}_s(x,z) = k_{N,s}(1-|x|^2)_+^s(1-|z|^2)_+^s \int_0^1 \frac{t^{s-1}}{((1-|x|^2)_+(1-|z|^2)_+t+|x-z|^2)^{\frac{N}{2}}}\,dt.$$

Hence, for $\theta\in\partial B$ and $x\in B$, it follows that

$$\begin{aligned} M_s(x,\theta) &= k_{N,s}(1-|x|^2)^s \lim_{z\to\theta,z\in B}\int_0^1 \frac{t^{s-1}}{((1-|x|^2)_+(1-|z|^2)_+t+|x-z|^2)^{\frac{N}{2}}}\,dt \\ &= k_{N,s}\frac{(1-|x|^2)^s}{|x-\theta|^N}\int_0^1 t^{s-1}\,dt = \frac{k_{N,s}}{s}\frac{(1-|x|^2)^s}{|x-\theta|^N} \end{aligned}$$

□

Martin kernels provide a useful characterization of some s-harmonic functions.

Lemma 4.2. *Let $s > 0$ and assume*

$$\int_B \mathscr{G}_s(x,y)(-\Delta)^s\psi(y)\,dy = \psi(x) \quad \textit{for all } x\in B \textit{ and } \psi\in C_c^\infty(B). \tag{4.1}$$

If $\mu\in\mathscr{M}(\partial B)$ is a finite Radon measure, then the function $\mathbb{R}^N\ni x\mapsto u(x) := \int_{\partial B} M_s(x,z)\,d\mu(z)$ is s-harmonic in B.

Proof. We first show that $u\in L^1(B)$. Indeed,

$$\int_B |u(x)|\,dx \leq \int_{\partial B}\int_B M_s(x,z)\,dx\,d|\mu|(z) \leq 2^s k_{N,s}\int_{\partial B}\int_B |x-z|^{s-N}\,dxd|\mu|(z) < +\infty.$$

Since $u=0$ in $\mathbb{R}^N\setminus B$, then $u\in L^1(B)$. Let $\psi\in C_c^\infty(B)$ and note that $u\in C^\infty(B)$. Then $(-\Delta)^s u(x)$ exists for all $x\in B$ and, by 4.1,

$$\begin{aligned}\langle(-\Delta)^s u,\psi\rangle &= \int_B u(x)(-\Delta)^s\psi(x)\,dx = \int_B\int_{\partial B} M_s(x,\theta)\,d\mu(\theta)(-\Delta)^s\psi(x)\,dx\\ &= \int_B\int_{\partial B}\lim_{z\to\theta,z\in B}\frac{\mathscr{G}_s(x,z)}{(1-|z|^2)^s}\,d\mu(\theta)(-\Delta)^s\psi(x)\,dx\\ &= \int_{\partial B}\lim_{z\to\theta,z\in B}\frac{1}{(1-|z|^2)^s}\int_B \mathscr{G}_s(x,z)(-\Delta)^s\psi(x)\,dx\,d\mu(\theta)\\ &= \int_{\partial B}\lim_{z\to\theta,z\in B}\frac{\psi(z)}{(1-|z|^2)^s}\,d\mu(\theta)=0,\end{aligned}$$

since ψ has compact support in B. Therefore u is s-harmonic in the sense of distributions and, since u is clearly in $C^\infty(B)\cap L^1(B)$, we have, by [2, Lemma 1.5], that u is pointwisely s-harmonic in B. □

Remark 4.3. We assume (4.1) as part of our iteration argument, but once Theorem 1.1 is proved then (4.1) holds for all $s>0$.

We now show the relationship between P_{s-1} from Lemma 3.1 and M_s.

Lemma 4.4. *Let $s>1$ and $y\in B$. Then*

$$P_{s-1}(x,y)=\frac{2k_{N,1}(s-1)s}{k_{N,s-1}k_{N,s}}\int_{\partial B} M_{s-1}(x,\theta)M_s(y,\theta)\,d\theta\qquad \textit{for } x\in B.$$

Proof. Fix $y\in B$ and let $v(x):=\frac{(1-|x|^2|y|^2)}{(1-|y|^2)[x,y]^N}$ for $x\in B$. Note that $-\Delta v=0$ in B and $v(\theta)=|\theta-y|^{-N}$ for $\theta\in\partial B$. Indeed, if $y=0$ then $v\equiv 1$ and if $y\in B\setminus\{0\}$ then $v(x)=\frac{|\eta|^N}{|\eta|^2-1}\frac{|\eta|^2-|x|^2}{|x-\eta|^N}$ with $\eta:=\frac{y}{|y|^2}$, and $-\Delta v=0$ follows by a simple calculation. Then, by uniqueness and using the Poisson kernel for the Laplacian,

$$\frac{(1-|x|^2|y|^2)}{(1-|y|^2)[x,y]^N}=v(x)=2k_{N,1}\int_{\partial B}\frac{1-|x|^2}{|x-\theta|^N[\theta,y]^N}\,d\theta.$$

Therefore,

$$\begin{aligned}P_{s-1}(x,y)&=(1-|x|^2)^{s-2}(1-|y|^2)^s\frac{(1-|x|^2|y|^2)}{(1-|y|^2)[x,y]^N}\\ &=2k_{N,1}(1-|x|^2)^{s-2}(1-|y|^2)^s\int_{\partial B}\frac{1-|x|^2}{|x-\theta|^N[\theta,y]^N}\,d\theta=2k_{N,1}\int_{\partial B}\frac{(1-|x|^2)^{s-1}}{|x-\theta|^N}\frac{(1-|y|^2)^s}{|\theta-y|^N}\,d\theta\\ &=\frac{2k_{N,1}(s-1)s}{k_{N,s-1}k_{N,s}}\int_{\partial B}M_{s-1}(x,\theta)M_s(y,\theta)\,d\theta,\end{aligned}$$

by Lemma 4.1, as claimed. □

Corollary 4.5. *Let $y \in B$ and $s > 1$. If (4.1) holds, then $P_{s-1}(\cdot, y)$ is $(s-1)$-harmonic in B.*

Proof. Combine Lemma 4.4 and Lemma 4.2. ☐

Remark 4.6.

1. As mentioned before, the Martin kernel M_s provides a useful characterization of some s-harmonic functions. This characterization is new for $s > 1$ and may be of independent interest. Namely, if $s > 0$ and $g \in C(\partial B)$, then $v(x) := \int_{\partial B} M_s(x,\theta) g(\theta)\, d\theta$ for $x \in B$, is s-harmonic.

2. Arguing as in [1], it is possible to prove that if $g \in C(\partial B)$, then

$$\lim_{z\to\tilde{\theta}, z\in B} \frac{\int_{\partial B} M_s(z,\theta) g(\theta)\, d\theta}{(1-|z|^2)^{s-1}} = \frac{k_{N,s}}{2k_{N,1}s} g(\tilde{\theta}) \qquad \text{for } \tilde{\theta} \in \partial B.$$

   Therefore, if $v = \int_{\partial B} M_s(\cdot,\theta) g(\theta)\, d\theta$, then $g(\theta) = 2k_{N,1} k_{N,s}^{-1} s \lim_{z\to\theta,\, z\in B} v(z)(1-|z|^2)^{1-s}$.

3. If $\varphi \in C^2(B) \cap C(\overline{B})$ is harmonic, *i.e.* $-\Delta\varphi = 0$ in B, then $u(x) := (1-|x|^2)_+^{s-1} \varphi(x)$, $x \in \mathbb{R}^N$ is s-harmonic in B. Indeed, using the Poisson kernel representation and Lemma 4.1 we have that

$$u(x) = 2k_{N,1}(1-|x|^2)^{s-1} \int_{\partial B} \frac{1-|x|^2}{|x-\theta|^N} \varphi(\theta)\, d\theta = \frac{2k_{N,1}s}{k_{N,s}} \int_{\partial B} M_s(x,\theta)\varphi(\theta)\, d\theta,$$

   and then $(-\Delta)^s u = 0$ in B, by the first Remark.

4. If a function u is s-harmonic in B, then u is $(s+1)$-harmonic. Indeed, $\int_{\mathbb{R}^N} u(-\Delta)^{s+1}\varphi\, dx = \int_{\mathbb{R}^N} u(-\Delta)^s[-\Delta\varphi]\, dx = 0$ for any $\varphi \in C_c^\infty(B)$. Thus, for $j \in (0,s) \cap \mathbb{N}$ functions of the type $\int_{\partial B} M_{s-j}(x,\theta) g(\theta)\, d\theta$ are also s-harmonic.

4.1 Proof of Theorem 1.1 and consequences

Proof of Theorem 1.1. Let $f \in C^\alpha(\overline{B})$ for some $\alpha \in (0,1)$ and u as in (1.4). The claim is known for $s \in (0,1]$, see [5, 8, 14]. Let $s > 1$ and assume that the statement holds for $s-1$. Then $u \in C^{2s+\alpha}(B) \cap C_0^s(B) \cap \mathcal{H}_0^s(B)$, by Theorem 3.7. Furthermore, by Lemmas 3.1, 4.5, B.4, and the induction hypothesis,

$$\langle (-\Delta)^s u\, ,\, \varphi \rangle = \int_B u(-\Delta)^s \varphi\, dx = \int_B -\Delta u(-\Delta)^{s-1}\varphi\, dx$$

$$= \langle \int_B \mathcal{G}_{s-1}(\cdot,y) f(y)\, dy\, ,\, (-\Delta)^{s-1}\varphi \rangle - 4k_{N,s}(s-1) \int_B f(y) \langle P_{s-1}(\cdot,y)\, ,\, (-\Delta)^{s-1}\varphi \rangle\, dy = \langle f\, ,\, \varphi \rangle$$

for all $\varphi \in C_c^\infty(B)$, in particular,

$$\int_B \int_B \mathcal{G}_s(x,y)(-\Delta)^s \varphi(y)\, dy\, f(x)\, dx = \int_B u(x)(-\Delta)^s\varphi(x)\, dx = \int_B f(x)\varphi(x)\, dx.$$

for any $\varphi \in C_c^\infty(B)$. Since $f \in C^\alpha(\overline{B})$ is arbitrary, we obtain that $\int_B \mathscr{G}_s(x,y)(-\Delta)^s\varphi(y)\,dy = \varphi(x)$ for every $x \in B$ and thus $\mathscr{G}_s(\cdot,y)$ is a distributional solution of $(-\Delta)^s v = \delta_y$. Note that $u \in \mathscr{H}_0^s(B)$ is a distributional solution of $(-\Delta)^s u = f$ with $f \in L^2(B)$, thus u is also a weak solution (see [4] and Lemma B.4), since $C_c^\infty(B)$ is dense in $\mathscr{H}_0^s(B)$ (see *e.g.* [16, Theorem 1.4.2.2]). In particular, u solves $(-\Delta)^s u = f$ uniquely in $\mathscr{H}_0^s(B)$ (see [4, Corollary 3.6]). Moreover, u satisfies $(-\Delta)^m(-\Delta)^\sigma u(x) = f(x)$ pointwise for every $x \in B$, by Lemmas B.4 and [4, Corollary 3.6] and the decay (1.6) follows from Proposition 3.5. □

Proof of Proposition 1.2. The Proposition follows from Theorem 1.1 and Lemma 4.2. □

Proof of Corollary 1.3. Let $j \in \mathbb{N}$ and $s > j$. For any $\varphi \in C_c^\infty(B)$ we have that $(-\Delta)^j\varphi \in C_c^\infty(B)$ and thus, for $x \in B$,

$$\int_B \mathscr{G}_{s-j}(x,y)(-\Delta)^s\varphi(y)\,dy \;=\; \int_B \mathscr{G}_{s-j}(x,y)(-\Delta)^{s-j}(-\Delta)^j\varphi(y)\,dy \;=\; (-\Delta)^j\varphi(x),$$

by Proposition B.2 and Theorem 1.1, using that $(-\Delta)^{s-j}v = (-\Delta)^j\varphi$ in B has a unique solution in $\mathscr{H}_0^{s-j}(B)$, by [4, Corollary 3.6]. Let μ be a finite Radon measure and

$$u_j(x) = \int_B \mathscr{G}_{s-j}(x,y)\int_B \mathscr{G}_j(y,z)\,d\mu(z)dy \quad \text{for } x \in \mathbb{R}^N,$$

then

$$\begin{aligned}
\int_B u_j(-\Delta)^s\varphi\,dx &= \int_B\int_B \mathscr{G}_{s-j}(x,y)\int_B \mathscr{G}_j(y,z)\,d\mu(z)\,dy\,(-\Delta)^s\varphi(x)\,dx \\
&= \int_B\int_B \mathscr{G}_j(y,z)\int_B \mathscr{G}_{s-j}(x,y)(-\Delta)^s\varphi(x)\,dx\,dy\,d\mu(z) \\
&= \int_B\int_B \mathscr{G}_j(y,z)(-\Delta)^j\varphi(y)\,dy\,d\mu(z) = \int_B \varphi(z)\,d\mu(z).
\end{aligned}$$

In particular, if $d\mu(z) = f(z)\,dz$ for some $f \in C^\alpha(\overline{B})$, then, by Theorem 3.7,

$$y \mapsto \int_B \mathscr{G}_j(y,z)f(z)\,dz \in C^\alpha(\overline{B}) \quad \text{and} \quad x \mapsto \int_B \mathscr{G}_{s-j}(x,y)\int_B \mathscr{G}_j(y,z)f(z)\,dzdy \in C_0^{s-j}(B).$$

□

A Differential recurrence equation

Proof of Lemma 3.1. Let $s > 1$, $y \in B$, $x \in \mathbb{R}^N$, and $x \neq y$, and ρ as in 1.2. In the following, differentiation is always w.r.t. x. To simplify notation we write $F_s := |x-y|^{2s-N}$ and $V_s(v) := v^{s-1}(v+1)^{-\frac{N}{2}}$.

We consider first the case $2s \neq N$. Note that

$$\nabla F_s = (2s-N)F_{s-1}(x-y) = (2s-N)F_s \frac{x-y}{|x-y|^2} \quad \text{and} \quad -\Delta F_s = (N-2s)2(s-1)F_{s-1},$$

hence

$$-\Delta \mathscr{G}_s(x,y) = -k_{N,s}(\Delta F_s \int_0^\rho V_s(v)\, dv + 2V_s(\rho)\nabla F_s \cdot \nabla \rho + V_s'(\rho) F_s |\nabla \rho|^2 + F_s V_s(\rho)\Delta\rho). \quad \text{(A.1)}$$

Note that, for $a \geq 0$,

$$\int_0^a V_s(v)\, dv = \frac{2}{2s-N} \frac{a^{s-1}}{(a+1)^{\frac{N}{2}-1}} - \frac{2(s-1)}{2s-N} \int_0^a V_{s-1}(v)\, dv. \quad \text{(A.2)}$$

Thus, using (A.2), we obtain

$$-k_{N,s}\Delta F_s \int_0^\rho V_s(v)\, dv = \mathscr{G}_{s-1}(x,y) - k_{N,s} 4(s-1) \frac{F_s}{|x-y|^2} \frac{\rho^{s-1}}{(\rho+1)^{\frac{N}{2}-1}}.$$

Then, $-\Delta \mathscr{G}_s = \mathscr{G}_{s-1} - k_{N,s}4(s-1)P$, where

$$P := \frac{F_s}{|x-y|^2} \frac{\rho^{s-1}}{(\rho+1)^{\frac{N}{2}-1}} + \frac{2V_s(\rho)\nabla F_s \cdot \nabla\rho + F_s V_s'(\rho)|\nabla\rho|^2 + F_s V_s(\rho)\Delta\rho}{4(s-1)}.$$

It suffices to show that $P = P_{s-1}$, with P_{s-1} given by (3.1). Note that

$$\begin{aligned} 4(s-1)P &= 4(s-1)\frac{F_s}{|x-y|^2} \frac{\rho^{s-1}}{(\rho+1)^{\frac{N}{2}-1}} + 2V_s(\rho)\nabla F_s \cdot \nabla\rho + F_s V_s'(\rho)|\nabla\rho|^2 + F_s V_s(\rho)\Delta\rho \\ &= F_s\Big[\frac{V_s(\rho)(4(s-1)(\rho+1) + 2(2s-N)(x-y)\cdot\nabla\rho + |x-y|^2\Delta\rho)}{|x-y|^2} + V_s'(\rho)|\nabla\rho|^2\Big]. \end{aligned} \quad \text{(A.3)}$$

To simplify this expression we use

$$V_s'(v) = (s-1)\frac{v^{s-2}}{(v+1)^{\frac{N}{2}}} - \frac{N}{2}\frac{v^{s-1}}{(v+1)^{\frac{N}{2}+1}} = V_s(v)\frac{(s-1)(v+1) - \frac{N}{2}v}{v(v+1)}$$

so that

$$\begin{aligned} 4(s-1)P &= F_s V_s(\rho)\Big[\frac{4(s-1)(\rho+1) + 2(2s-N)(x-y)\cdot\nabla\rho + |x-y|^2\Delta\rho}{|x-y|^2} \\ &\qquad + \frac{(s-1)(\rho+1) - \frac{N}{2}\rho}{\rho(\rho+1)}|\nabla\rho|^2\Big] \\ &= F_{s-1}V_s(\rho)\Big[\frac{4(s-1)((1-|x|^2)(1-|y|^2) + |x-y|^2)}{|x-y|^2} + 2(2s-N)(x-y)\cdot\nabla\rho + |x-y|^2\Delta\rho \end{aligned}$$

$$+\frac{(s-1-\frac{N}{2})(1-|x|^2)(1-|y|^2)+(s-1)|x-y|^2}{(1-|x|^2)^2(1-|y|^2)^2+(1-|x|^2)(1-|y|^2)|x-y|^2}|x-y|^4|\nabla\rho|^2\Big]. \tag{A.4}$$

Direct calculations yield that

$$\Delta\rho = \frac{2(1-|y|^2)}{|x-y|^4}\left(-N(|y|^2-2x\cdot y+1)+4(1-x\cdot y)\right),$$
$$(x-y)\cdot\nabla\rho = -2\frac{1-|y|^2}{|x-y|^2}(|x|^2-x\cdot y+1-|x|^2) = -2\frac{(1-|y|^2)(1-x\cdot y)}{|x-y|^2}.$$

Hence, the first three terms in (A.4) reduces to

$$\frac{4}{|x-y|^2}[(s-1)(1-2x\cdot y+|x|^2|y|^2)-(1-|y|^2)(\frac{N}{2}(|y|^2-2x\cdot y+1)+(2s-2-N)(1-x\cdot y))] \tag{A.5}$$

and the last term in (A.4) reduces to

$$4(1-|y|^2)\frac{(s-1-\frac{N}{2})(1-|x|^2)(1-|y|^2)+(s-1)|x-y|^2}{(1-|x|^2)|x-y|^2}. \tag{A.6}$$

Combining (A.5), (A.6) with (A.4) we find

$$\begin{aligned}4(s-1)P = \frac{4F_{s-1}V_s(\rho)}{(1-|x|^2)|x-y|^2}\Big[&(s-1)(1-2x\cdot y+|x|^2|y|^2)(1-|x|^2)\\ &+(1-|y|^2)\Big(-\frac{N}{2}(|y|^2-2x\cdot y+1)(1-|x|^2)+(s-1)|x-y|^2\\ &-(2s-2-N)(1-x\cdot y)(1-|x|^2)+(s-1-\frac{N}{2})(1-|y|^2)(1-|x|^2)\Big)\Big].\end{aligned} \tag{A.7}$$

Note that the bracket in (A.7) reduces to

$$(s-1)(|x-y|^2-|x|^2|y|^2(|x|^2-2x\cdot y+|y|^2)) = (s-1)|x-y|^2(1-|x|^2|y|^2). \tag{A.8}$$

We conclude that

$$P = \frac{V_s(\rho)}{(1-|x|^2)}\frac{1-|x|^2|y|^2}{|x-y|^{2+N-2s}} = \frac{(1-|x|^2)^{s-2}(1-|y|^2)^{s-1}(1-|x|^2|y|^2)}{\left|x|y|-\frac{y}{|y|}\right|^N} = P_{s-1}(x,y), \tag{A.9}$$

as claimed.

We now consider the case $2s=N$. Since $s>1$ then $N\geq 3$. Note that $k_{N,s-1}=4(s-1)^2k_{N,s}$ and

$$\mathscr{G}_{s-1}(x,y) = k_{N,s-1}|x-y|^{-2}\int_0^\rho \frac{v^{\frac{N}{2}-2}}{(v+1)^{\frac{N}{2}}}\,dv = 4(s-1)k_{N,s}\frac{\rho^{s-1}}{(\rho+1)^{s-1}|x-y|^2}.$$

On the other hand,

$$(-\Delta)\mathscr{G}_s(x,y) = -k_{N,s}\Delta\left(\int_0^\rho \frac{v^{\frac{N-2}{2}}}{(v+1)^{\frac{N}{2}}}\,dv\right)$$
$$= 4(s-1)k_{N,s}\frac{\rho^{s-1}(1-|y|^2)}{(\rho+1)^s|x-y|^4}\left[|y|^2-2x\cdot y+1-\frac{|x-y|^2}{1-|x|^2}\right].$$

Hence,

$$(-\Delta)\mathscr{G}_{\frac{N}{2}}(x,y) = \mathscr{G}_{\frac{N-2}{2}}(x,y)$$
$$+4(s-1)k_{N,s}\frac{\rho^{s-1}}{(\rho+1)^s|x-y|^4}\left[(1-|y|^2)\Big[|y|^2-2x\cdot y+1-\frac{(1-|y|^2)}{\rho}\Big]-(\rho+1)|x-y|^2\right],$$

where,

$$(1-|y|^2)\Big[|y|^2-2x\cdot y+1-\frac{(1-|y|^2)}{\rho}\Big]-(\rho+1)|x-y|^2$$
$$= -|y|(|y|^2-2x\cdot y+|x|^2)-\frac{1-|y|^2}{1-|x|^2}|x-y|^2 = -|x-y|^2\Big(|y|^2+\frac{1-|y|^2}{1-|x|^2}\Big).$$

Since $\rho+1=[x,y]^2|x-y|^{-2}$ we obtain that $-\Delta\,\mathscr{G}_s = \mathscr{G}_{s-1}-k_{N,s}4(s-1)P_{s-1}$ with P_{s-1} as given by (3.1) and the proof is finished. □

B Interchange of derivatives

The next proposition provides conditions to allow the interchange between derivatives and fractional Laplacians. The main difficulty in the proof relies on the fact that u is allowed to have unbounded or discontinuous derivatives outside a domain Ω. For this we use the following space given for $s>0$ (see *e.g.* [13, 23] for $s\in(0,1)$) by

$$\mathscr{L}^1_s := \left\{u\in L^1_{loc}(\mathbb{R}^N)\ :\ \|u\|_{\mathscr{L}^1_s}<\infty\right\},\qquad \|u\|_{\mathscr{L}^1_s} := \int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2s}}\,dx.$$

We start with the following estimate, where H_u denotes the Hessian of u.

Lemma B.1. *Let $V\subset\mathbb{R}^N$ open, $u:V\to\mathbb{R}^N$ such that $\|u\|_{C^2(V)}<\infty$, and $w:V\times\mathbb{R}^N\to\mathbb{R}$, $w(x,y):=2u(x)-u(x+y)-u(x-y)$. Then*

$$w(x,y) = -\left[\int_0^1\int_0^1 H_u(x+(\tau-t)y)\,d\tau dt\right]y\cdot y\quad \text{for all } x\in V,\ y\in\mathbb{R}^N,\ x\pm y\in V.$$

In particular, $|w(x,y)|\le\|u\|_{C^2(V)}|y|^2$ for all $x\in V$ and $y\in\mathbb{R}^N$ such that $x\pm y\in V$.

Proof. Since $w(x,y) = u(x) - u(x+y) - (u(x) - u(x-y))$ we have by the Mean Value Theorem that $w(x,y) = \int_0^1 [\nabla u(x+y-ty) - \nabla u(x-ty)]\, dt \cdot (-y)$. A second application of the Mean Value Theorem yields the result. □

Proposition B.2. *Let $\Omega \subset \mathbb{R}^N$ open, $\sigma \in (0,1)$, and $u \in C^3(\Omega) \cap \mathscr{L}^1_\sigma \cap W^{1,1}_{loc}(\mathbb{R}^N)$. If $\partial_1 u \in \mathscr{L}^1_\sigma$, then $\partial_1(-\Delta)^\sigma u(x) = (-\Delta)^\sigma \partial_1 u(x)$ pointwise for all $x \in \Omega$. In particular, if $m \in \mathbb{N}_0$, $u \in C^{2m+2}(\Omega) \cap \mathscr{L}^1_\sigma \cap W^{2m,1}_{loc}(\mathbb{R}^N)$, and $\partial^\alpha u \in \mathscr{L}^1_\sigma$ for all $|\alpha| \leq 2m$, then*

$$(-\Delta)^{m+\sigma} u(x) = (-\Delta)^\sigma [(-\Delta)^m u(x)] = (-\Delta)^m [(-\Delta)^\sigma u(x)] \qquad \textit{for all } x \in \Omega.$$

Proof. Let $u \in C^3(\Omega) \cap \mathscr{L}^1_\sigma \cap W^{1,1}_{loc}(\mathbb{R}^N)$ and $\partial_1 u \in C^2(\Omega) \cap \mathscr{L}^1_\sigma$. In the following all derivatives ∂_1 are taken with respect to x. By [12, Lemma 2.1] we have that

$$(-\Delta)^\sigma u(x) = c_{N,\sigma} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\sigma}}\, dxdy = c_{N,\sigma} \int_{\mathbb{R}^N} \frac{2u(x)-u(x-y)-u(x+y)}{|y|^{N+2\sigma}}\, dxdy,$$

where the integral on the right does not have a principal value (*cf.* [9, Lemma 3.2]). Let $H : \Omega \times \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ and $h_t : \Omega \times \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be given by

$$H(x,y) := \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2\sigma}}, \quad h_t(x,y) := \frac{H(x+te_1,y)-H(x,y)}{t}, \quad t \in \mathbb{R}\setminus\{0\}.$$

Fix $x \in \Omega$ and V an open set with $\overline{V} \subset \Omega$ and $x \in V$. Let $T, \varepsilon \in (0,1)$ such that $x+y+te_1 \in V$ for all $0 < |t| < T$ and $|y| < \varepsilon$. Set $U := B_\varepsilon(0)$. We show separately that

$$\lim_{t\to 0} \int_U h_t(x,y)\, dy = \int_U \partial_1 H(x,y)\, dy \quad \text{and} \tag{B.1}$$

$$\lim_{t\to 0} \int_{\mathbb{R}^N\setminus U} h_t(x,y)\, dy = \int_{\mathbb{R}^N\setminus U} \partial_1 H(x,y)\, dy. \tag{B.2}$$

By the Mean Value Theorem, for every $0 < |t| < T$ there is $|t_0| < t$ and $\xi := x + t_0 e_1 \in V$ such that $h_t(x,y) = \partial_1 H(\xi, y)$ for $y \in U$. Then, by the mean value theorem Lemma B.1, $|\partial_1 H(\xi,y)| \leq \|u\|_{C^3(V)} |y|^{-2\sigma-N+2} \in L^1(U)$. Thus, by the Dominated Convergence Theorem, $\partial_1 H(x,\cdot) \in L^1(U)$ and (B.1) holds.

Moreover, if $A := \{|y - ste_1 - x| \geq \varepsilon\}$, then

$$\left| \frac{\partial_1 u(y)}{|y-ste_1-x|^{N+2\sigma}} 1_A(y) \right| \leq \frac{|\partial_1 u(y)|}{1+|y|^{N+2\sigma}} \frac{1+|y|^{N+2\sigma}}{|y-ste_1-x|^{N+2\sigma}} 1_A(y) \leq K \frac{|\partial_1 u(y)|}{1+|y|^{N+2\sigma}} =: f(y),$$

where $K > 0$ is a constant depending only on V, N, ε, and σ. Since $f \in L^1(\mathbb{R}^N)$ then, by the Dominated Convergence Theorem,

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \int_0^1 \frac{\partial_1 u(y)}{|y-ste_1-x|^{N+2\sigma}} 1_{\{|y-ste_1-x|\geq\varepsilon\}}(y)\, dsdy = \int_{\mathbb{R}^N} \frac{\partial_1 u(y)}{|y-x|^{N+2\sigma}} 1_{\{|y-x|\geq\varepsilon\}}(y)\, dy$$

or equivalently,

$$\int_{\mathbb{R}^N} \frac{\partial_1 u(x\pm y)}{|y|^{N+2\sigma}} 1_{\{|y|\geq\varepsilon\}}\, dy = \lim_{t\to 0} \int_{\mathbb{R}^N} \int_0^1 \frac{\partial_1 u(ste_1+x\pm y)}{|y|^{N+2\sigma}} 1_{\{|y|\geq\varepsilon\}}\, dsdy$$
$$= \lim_{t\to 0} \int_{\mathbb{R}^N} \frac{u(x+te_1\pm y)-u(x\pm y)}{|y|^{N+2\sigma}} 1_{\{|y|\geq\varepsilon\}}\, dy. \tag{B.3}$$

Since it trivially holds that

$$\lim_{t\to 0}\frac{1}{t} \int_{\mathbb{R}^N\setminus U} \frac{u(x+te_1)-u(x)}{|y|^{N+2\sigma}}\, dy = \int_{\mathbb{R}^N\setminus U} \frac{\partial_1 u(x)}{|y|^{N+2\sigma}}\, dy, \tag{B.4}$$

then (B.2) follows from (B.4) and (B.3). □

To perform the integration by parts we use the following standard regularity result.

Lemma B.3. *Let $\Omega\subset\mathbb{R}^N$ open, $m\in\mathbb{N}$, $\sigma\in(0,1)$, $s=m+\sigma$, and let $u\in C^{2s+\alpha}(\Omega)\cap C^s(\mathbb{R}^N)\cap \mathscr{L}^1_s$ for some $\alpha>0$. Then $(-\Delta)^\sigma u\in C^{2m}(\Omega)\cap C^{m-\sigma}(\mathbb{R}^N)$.*

The proof can be done by arguing as in the proof of [23, Propositions 2.6 and 2.7] and hence we omit it.

Lemma B.4. *Let $\sigma\in(0,1)$, $m\in\mathbb{N}$, and $s=m+\sigma>1$. If $u\in W^{2,1}(B)$ satisfies $u=\nabla u=0$ on ∂B in the trace sense, then*

$$\int_B u\,(-\Delta)^s\varphi\, dx = \int_B -\Delta u\,(-\Delta)^{s-1}\varphi\, dx \qquad \text{for all } \varphi\in C_c^\infty(B). \tag{B.5}$$

This is in particular the case if $u\in W^{2,1}(\mathbb{R}^N)$ with supp $u\subset\overline{B}$. *If $u\in C^{2s+\alpha}(B)\cap C^s_0(B)$ for some $\alpha\in(0,1)$, then*

$$\int_{\mathbb{R}^N} u\,(-\Delta)^s\varphi\, dx = \int_{\mathbb{R}^N} (-\Delta)^m(-\Delta)^\sigma u\,\varphi\, dx \qquad \text{for all } \varphi\in C_c^\infty(B), \tag{B.6}$$

and if $u\in\mathscr{H}^s_0(B)$ then $\int_{\mathbb{R}^N} u\,(-\Delta)^s\varphi\, dx=\mathscr{E}_s(u,\varphi)$ for all $\varphi\in C_c^\infty(B)$, where $\mathscr{E}_s$ is the scalar product in $\mathscr{H}^s_0(B)$ (see [4]).

Proof. Equality (B.5) follows from two integrations by parts, since $u\equiv 0$ in $\mathbb{R}^N\setminus B$ and $\nabla u=0$ on ∂B. For (B.6), note that $u\in C^{2s+\alpha}(B)\cap C^s_0(B)$ implies that $(-\Delta)^\sigma u\in C^{2m}(B)\cap C^{m-\sigma}(\mathbb{R}^N)$ by Lemma B.3, since $s>1$. Moreover, since $u\equiv 0$ in $\mathbb{R}^N\setminus B$, there is $C>0$ such that $|\Delta^\sigma u(x)|\leq C(1+|x|^{N+2\sigma})^{-1}$ for all $x\in\mathbb{R}^N$. In particular, $(-\Delta)^\sigma u\in L^2(\mathbb{R}^N)$. Using Fourier transform, integration by parts, and the fact that $\varphi\in C_c^\infty(B)$, we obtain

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^s\varphi(x)\, dx = \int_{\mathbb{R}^N} (-\Delta)^\sigma u(x)(-\Delta)^m\varphi(x)\, dx = \int_{\mathbb{R}^N} (-\Delta)^m(-\Delta)^\sigma u(x)\varphi(x)\, dx.$$

The last claim follows from [4, Lemma 4.2]. □

For $s>0$ and $k\in\mathbb{N}$ denote (*cf.* [12, Section 2])

$$S^k_s := \{\varphi\in C^k(\mathbb{R}^N)\ :\ \sup_{x\in\mathbb{R}^N}(1+|x|^{N+2s})\sum_{|\alpha|\le k}|\partial^\alpha\varphi(x)|<\infty\} \tag{B.7}$$

endowed with the norm $\|\varphi\|_{k,s}:=\sup_{x\in\mathbb{R}^N}(1+|x|^{N+2s})\sum_{|\alpha|\le k}|\partial^\alpha\varphi(x)|$.

Lemma B.5. *Let $\sigma\in(0,1]$, $m\in\mathbb{N}_0$, and $s=m+\sigma$. There is $C=C(N,m,\sigma)>0$ such that*

$$|(-\Delta)^s f(x)|\le C\frac{\|f\|_{2m+2,s}}{1+|x|^{N+2s}}\qquad \text{for every } f\in S^{2m+2}_s \text{ and for all } x\in\mathbb{R}^N. \tag{B.8}$$

Proof. If $\sigma=1$, then (B.8) follows by definition with $C=1$. For the rest of the proof, we denote by $C>0$ possibly different constants depending only on N, m, and σ. Let $\sigma\in(0,1)$ and note that $(-\Delta)^{m+\sigma}f=(-\Delta)^\sigma(-\Delta)^m f$. To simplify the notation let $\varphi:=(-\Delta)^m f$ and recall that $B:=B_1(0)$. For $x\in\mathbb{R}^N$ we have, by the Mean Value Theorem (see Lemma B.1),

$$\begin{aligned}|(-\Delta)^{\sigma+m}f(x)| &= \frac{c_{N,\sigma}}{2}\left|\int_{\mathbb{R}^N}\frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2\sigma}}\,dy\right|\\ &\le C\int_B\int_0^1\int_0^1\frac{|H_\varphi(x+(t-\tau)y)|}{|y|^{N+2\sigma-2}}\,d\tau dt dy+\left|\int_{\mathbb{R}^N\setminus B}\frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2\sigma}}\,dy\right| =: f_1+f_2.\end{aligned} \tag{B.9}$$

Note that

$$f_1\le C\|f\|_{2m+2,s}\int_B\int_0^1\int_0^1\frac{|y|^{-N-2\sigma+2}}{1+|x+(t-\tau)y|^{N+2s}}\,d\tau dt dy\le C\frac{\|f\|_{2m+2,s}}{1+|x|^{N+2s}}, \tag{B.10}$$

$$f_2\le 2\int_{\mathbb{R}^N\setminus B}\frac{|\varphi(x)|}{|y|^{N+2\sigma}}\,dy+2\left|\int_{\mathbb{R}^N\setminus B}\frac{\varphi(x+y)}{|y|^{N+2\sigma}}\,dy\right|\le C\frac{\|f\|_{2m+2,s}}{1+|x|^{N+2s}}+2\left|\int_{\mathbb{R}^N\setminus B}\frac{\varphi(x+y)}{|y|^{N+2\sigma}}\,dy\right|. \tag{B.11}$$

Using integration by parts $m-$times we obtain

$$\left|\int_{\mathbb{R}^N\setminus B}\frac{\varphi(x+y)}{|y|^{N+2\sigma}}\,dy\right|=\left|\int_{\mathbb{R}^N\setminus B}\frac{(-\Delta)^m f(x+y)}{|y|^{N+2\sigma}}\,dy\right|\le C\frac{\|f\|_{2m+2,s}}{1+|x|^{N+2s}}+C\int_{\mathbb{R}^N\setminus B}\frac{|f(x+y)|}{|y|^{N+2\sigma+2m}}\,dy. \tag{B.12}$$

Moreover,

$$\int_{\mathbb{R}^N\setminus B}\frac{|f(x+y)|}{|y|^{N+2\sigma+2m}}\,dy\le\frac{\|f\|_{2m+2,s}}{1+|x|^{N+2s}}\int_{\mathbb{R}^N\setminus B}\frac{1+|x|^{N+2s}}{(1+|x+y|^{N+2s})|y|^{N+2s}}\,dy \tag{B.13}$$

By (B.9)-(B.13) it suffices to show that there is $C>0$ depending only on N, m, and σ such that

$$\int_{\mathbb{R}^N\setminus B} \frac{1+|x|^{N+2s}}{(1+|x+y|^{N+2s})|y|^{N+2s}}\,dy < C \tag{B.14}$$

for all $x\in\mathbb{R}^N$. If $|x|<2$ then (B.14) follows by taking the maximum over $x\in 2B$. We now argue as in [12, Lemma 2.1]. Fix $|x|\geq 2$ and let $U:=\{y\in\mathbb{R}^N\setminus B : |x+y|\geq \frac{|x|}{2}\}$. If $y\in U$ then $1+|x|^{N+2s}\leq C(1+|x+y|^{N+2s})$ and if $y\in\mathbb{R}^N\setminus U$ then $|y|>\frac{|x|}{2}$. Thus,

$$\int_U \frac{1+|x|^{N+2s}}{(1+|x+y|^{N+2s})|y|^{N+2s}}\,dy \leq C\int_{\mathbb{R}^N\setminus B} |y|^{-N-2s}\,dy < C,$$

$$\int_{\mathbb{R}^N\setminus U} \frac{1+|x|^{N+2s}}{(1+|x+y|^{N+2s})|y|^{N+2s}}\,dy \leq C\frac{1+|x|^{N+2s}}{|x|^{N+2s}}\int_{\mathbb{R}^N} (1+|x+y|^{N+2s})^{-1}\,dy < C.$$

This implies (B.14) and finishes the proof. □

Lemma B.6. *Let $s>1$ and $u\in H^2_{loc}(\mathbb{R}^N)$ such that $\Delta u\in\mathscr{L}^1_{s-1}$. Then,*

$$\int_{\mathbb{R}^N} u\,(-\Delta)^s\varphi\,dx = \int_{\mathbb{R}^N} -\Delta u\,(-\Delta)^{s-1}\varphi\,dx \qquad \textit{for all } \varphi\in C_c^\infty(\mathbb{R}^N). \tag{B.15}$$

Proof. Fix $\psi:=(-\Delta)^{s-1}\varphi$. Then $\psi\in C^\infty(\mathbb{R}^N)$ (see by [23, Proposition 2.7]) and, by Lemma B.5 and Proposition B.2, there is $K=K(\varphi,N,s)>0$ such that

$$|\psi(x)|+|\nabla\psi(x)| \leq \frac{K}{1+|x|^{N+2(s-1)}} \qquad \text{for all } x\in\mathbb{R}^N. \tag{B.16}$$

Let $(\eta_n)_{n\in\mathbb{N}}\subset C^\infty(\mathbb{R}^N)$ satisfy

$$0\leq\eta_n\leq 1,\quad \eta_n\equiv 1 \quad \text{in } B_n(0),\quad \eta_n\equiv 0 \quad \text{in } \mathbb{R}^N\setminus B_{n+1}(0),\quad \|\eta_n\|_{C^2(\mathbb{R}^N)}<C \tag{B.17}$$

for some $C>0$ independent of n, and set $\psi_n:=\eta_n\psi\in C_c^\infty(\mathbb{R}^N)$. Then $\psi_n\to\psi$ in $L^2(\mathbb{R}^N)$ and $-\Delta\psi_n=-\Delta\psi\eta_n-\nabla\eta_n\nabla\psi_n-\psi\Delta\eta_n\to-\Delta\psi=(-\Delta)^s\varphi$ in $L^2(\mathbb{R}^N)$, by (B.17), (B.16), and Proposition B.2. Therefore,

$$\int_{\mathbb{R}^N} u\,(-\Delta)^s\varphi\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} u\,(-\Delta)\psi_n\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} -\Delta u\,\psi_n\,dx = \int_{\mathbb{R}^N} -\Delta u\,(-\Delta)^{s-1}\varphi\,dx,$$

as claimed. □

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