Addendum to the paper:

M. Clapp, V. Hernández-Santamaría, and A. Saldaña, Positive and nodal limiting profiles for a semilinear elliptic equation with a shrinking region of attraction. Nonlinear Analysis, Volume 251, Februray 2025, 113680.

- In the proof of **Theorem 1.4 (ii)** there is a gap in the case when Ω has more than one connected component. Here we give a full proof filling this gap, which actually is shorter than the original.

We thank Cristian Morales for this remark.

Quotes to previous results and references are with respect to the published version of the paper.

Theorem 0.1. *(ii) Let $N \geq 2$ and let Ω be a radially symmetric open bounded subset of $\mathbb{R}^N$ with smooth boundary. Then, any least energy solution and any least energy nodal solution of*

$$-\Delta w = Q(x)|w|^{p-2}w, \qquad w \in E, \tag{0.1}$$

is foliated Schwarz symmetric in $\mathbb{R}^N$.

Proof. First, we adapt the strategy from [1,Lemma 2.5] to our setting. Let Ω be radially symmetric and let u be a least energy (positive or nodal) solution of the limit problem (0.1). Let $e \in \mathbb{S}^{N-1}$ and set $H := H(e)$. By Lemma 6.2, u_H is a least energy (positive or nodal) solution of (0.1). Observe that $|u - u_e| = 2u_H - (u + u_e)$ in H and $-|u - u_e| = 2u_H - (u + u_e)$ in $\mathbb{R}^N \smallsetminus H$. By Lemma 3.3,

$$u, u_H \in W^{2,s}_{loc}(\mathbb{R}^N) \cap \mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N) \text{ for all } s \in [1,\infty) \text{ and } \alpha \in (0,1). \tag{0.2}$$

Then $|u - u_e| \in W^{2,N}_{loc}(\mathbb{R}^N)$ and

$$\begin{aligned}-\Delta|u - u_e| &= 2Q(x)|u_H|^{p-2}u_H - \left[Q(x)|u|^{p-2}u + Q(x)|u_e|^{p-2}u_e\right] \\ &= Q(x)\left([|u_H|^{p-2}u_H - |u|^{p-2}u] + [|u_H|^{p-2}u_H - |u_e|^{p-2}u_e]\right) \qquad \text{in} \quad H.\end{aligned}$$

By (0.2), we have that $-\Delta|u - u_e| \in L^s_{loc}(\mathbb{R}^N)$ for all $s \geq 1$. By elliptic interior regularity,

$$w_e := |u - u_e| \in W^{2,s}_{loc}(H) \cap \mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N) \qquad \text{for all } s \geq 1,\ \alpha \in (0,1). \tag{0.3}$$

Moreover, $w_e \geq 0$ in H,

$$-\Delta w_e \leq 0 \ \text{ in } \ H \cap (\mathbb{R}^N \smallsetminus \Omega), \quad \text{ and } \quad -\Delta w_e \geq 0 \ \text{ in } \ H \cap \Omega \quad \text{for every } \ e \in \mathbb{S}^{N-1}. \tag{0.4}$$

By the maximum principle for strong solutions [8,Theorem 9.6], either $w_e \equiv 0$ or $w_e > 0$ in each connected component of $H \cap \Omega$. We claim that, if $w_e > 0$ in some connected component $A \subset \Omega$, then $w_e > 0$ in H. Indeed, let B be the connected component of the positive nodal set $\{x \in \mathbb{R}^n \ : \ w_e > 0\}$ which contains A and assume, by contradiction, that there is $x_0 \in H$ such that $w_e(x_0) = 0$. Then there is $y_0 \in \partial B$ and a small open ball U with $y_0 \in \partial U$ which is contained either in $B \cap \Omega \cap H$ or in $B \cap (\mathbb{R}^N \backslash \Omega) \cap H$. If $U \subset B \cap \Omega \cap H$, then

$$-\Delta w_e \geq 0 \ \text{ in } \ U, \qquad w_e > 0 \ \text{ in } \ U, \qquad w_e(y_0) = 0,$$

and, by Hopf Lemma, $\nabla w_e(y_0) \neq 0$. However, by (0.3) and a Taylor expansion around y_0, $0 \leq |u - u_e|(x) = w_e(x) = \nabla w_e(y_0) \cdot (x - y_0) + o(|x - y_0|)$ as $|x - y_0| \to 0$. But this implies that $\nabla w_e(y_0) = 0$, which yields a contradiction.

Nos suppose that $U \subset B \cap (\mathbb{R}^N \backslash \Omega) \cap H$. Since $w_e > 0$ in U we have that $u > u_e$ in U. Hence $u^H = \max\{u, u_e\} = u$ and therefore

$$c := \frac{[|u_H|^{p-2}u_H - |u|^{p-2}u] + [|u_H|^{p-2}u_H - |u_e|^{p-2}u_e]}{u - u_e} = \frac{|u|^{p-2}u - |u_e|^{p-2}u_e}{u - u_e} \in L^\infty(U).$$

Then $-\Delta w_e + cw_e = 0$ in U, $w_e > 0$ in U, $w_e(y_0) = 0$, and we can reach a contradiction as before using Hopf Lemma. As a consequence, either $u \geq u_e$ or $u \leq u_e$ in H and the claim now follows from Lemma 6.3. □