Sharp concentration estimates near criticality for Dirichlet and Neumann problems.

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(joint work with M. Grossi and H. Tavares)

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Model problem

Let $n \geq 3$, $p \in (0, \frac{n+2}{n-2})$, and u be a solution of

$$\begin{cases} -\Delta u = |u|^{p-1}u, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases}$$

where $B = \{|x| < 1\} \subset \mathbb{R}^n$.

The Lane-Emden equation (1870)

If $n = 3$, $p \in (0, 5)$, and u is radially symmetric and positive, then $\theta(|x|) = u(x)$, solves the *Lane-Emden equation of index p*,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) + \theta(r)^p = 0, \qquad \theta(0) = 1,$$

which models spheres in polytropic-convective equilibrium

- $\dfrac{dM(r)}{dr} = Cr^2\rho(r)$ (mass continuity),
- $\dfrac{dP(r)}{dr} = -\dfrac{G}{r^2}M(r)\rho(r)$ (hydrostatic equilibrium),
- $P(r) = K\rho(r)^{\frac{p+1}{p}}$ (polytropic fluid),
- $\theta(r) = Q\rho(r)^{\frac{1}{p}}$ (change of variables),

where M is the mass inside S_r, ρ is the density, and P is the pressure.

The Lane-Emden equation

If $n = 3$, $p \in (0,5)$, and u is radially symmetric and positive, then $\theta(|x|) = u(x)$, solves the *Lane-Emden equation of index p*,

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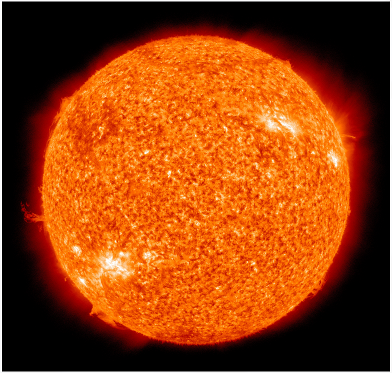


Up to constants:

- θ is the temperature.
- θ^{p+1} is the pressure.
- The first root r_1 of θ is the star's radius.
- $\int_0^{r_1} \theta(r)^p r^2\, dr$ is the total mass.

The Lane-Emden equation

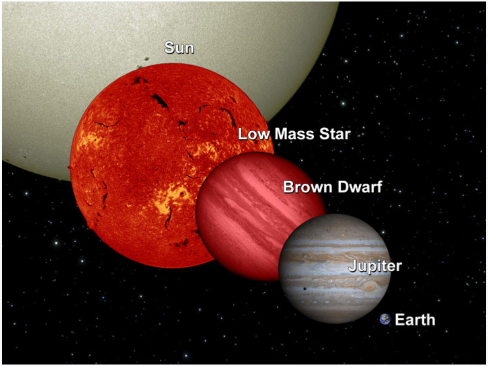
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right)+\theta(r)^p=0.$$

Some values of the polytropic index p are of particular physical importance: $p \in (0.5, 1)$ yields a good approximation for a neutron star.

The Lane-Emden equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right)+\theta(r)^p=0.$$

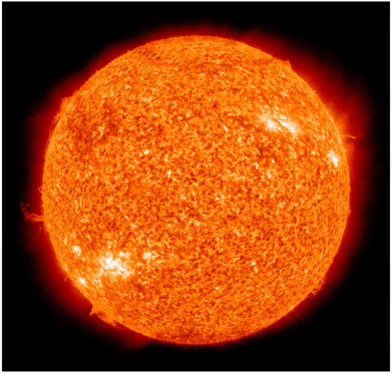
Some values of the polytropic index are of particular physical importance: $p \in (1, 1.5)$ yield good approximations for late-type stars (brown dwarfs).



The Lane-Emden equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right)+\theta(r)^p=0.$$

Some values of the polytropic index are of particular physical importance:



- $p = 3$ yields a good approximations for the Sun.
- As $p \to 5$, the radius of the star (or the temperature at its core) tends to infinity.

The critical exponent

The problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{n-2}-\varepsilon} u, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases}$$

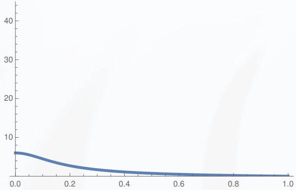
has no non-trivial smooth solutions if $\varepsilon \leq 0$ (due to the Pohozahev identity).

The positive solution

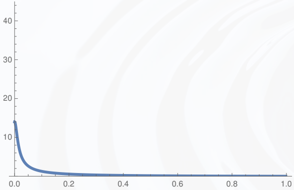
For $\varepsilon > 0$, the problem

$$\begin{cases} -\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon, & \text{in } B, \\ u_\varepsilon > 0, & \text{in } B, \\ u_\varepsilon = 0, & \text{on } \partial B, \end{cases}$$

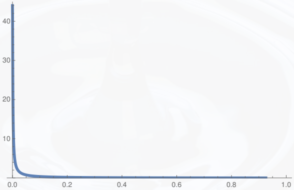
has a solution u_ε which concentrates at the origin as $\varepsilon \to 0$.



$\varepsilon = 1$



$\varepsilon = 0.1$



$\varepsilon = 0.01$

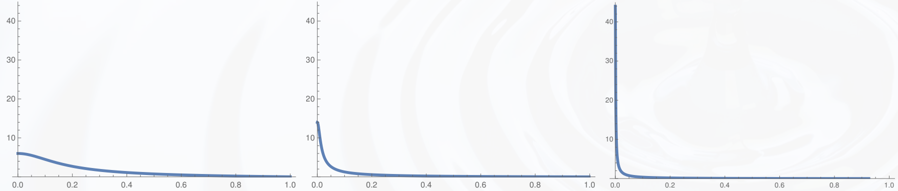
The precise rate ($n = 3$)

In fact, one can compute the *precise rates*

(Atkinson-Peletier, 1987)

$$\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon^2(0) = \frac{32\sqrt{3}}{\pi}, \qquad (u_\varepsilon(0) \sim \varepsilon^{-\frac{1}{2}})$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = \frac{\sqrt[4]{3}\sqrt{\pi}}{4\sqrt{2}}(|x|^{2-n} - 1) \qquad \text{for } |x| \in (0,1] \qquad (u_\varepsilon(x) \sim \varepsilon^{\frac{1}{2}}).$$



$\varepsilon = 1$ $\qquad$ $\varepsilon = 0.1$ $\qquad$ $\varepsilon = 0.01$

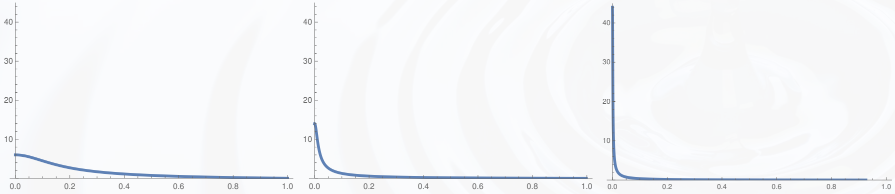
The precise rate (general $n \geq 3$)

In fact, one can compute the *precise rates*

(Atkinson-Peletier, 1987)

$$\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon^2(0) = (n(n-2))^{\frac{n-2}{2}} \frac{4\Gamma(n)}{(n-2)\Gamma(\frac{n}{2})^2}, \qquad (u_\varepsilon(0) \sim \varepsilon^{-\frac{1}{2}})$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{(n-2)\Gamma(\frac{n}{2})^2}{4\Gamma(n)} \right)^{\frac{1}{2}} (|x|^{2-n} - 1) \qquad \text{for } |x| \in (0,1].$$



$\varepsilon = 1$ $\qquad$ $\varepsilon = 0.1$ $\qquad$ $\varepsilon = 0.01$

The precise rate (revisited)

Two years later:

(Brezis-Peletier, 1989) $\longleftrightarrow -\Delta - \lambda$ for $\lambda \geq 0$ using PDE tools

(Pohozahev identity, Green functions, elliptic regularity,...)

And conjectured that something similar should happen in general (starshaped) bounded domains.

$$\begin{cases} -\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon, & \text{in } \Omega, \\ u_\varepsilon > 0, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof of the conjecture

$$\begin{cases} -\Delta u_\varepsilon = n(n-2)|u_\varepsilon|^{\frac{4}{n-2}-\varepsilon}u_\varepsilon, & \text{in } \Omega, \\ u_\varepsilon > 0, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

(Rey, 1989), (Han, 1991)

Let u_ε be a solution s.t. $\frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{L^{p+1}}} = S_n + o(1)$ as $\varepsilon \to 0$, then

(*i*) $\exists\ x_0 \in \Omega$ s.t. $u \to 0$ in $C^1(\Omega \backslash \{x_0\})$,

(*ii*) $\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)} = 2|\partial B|^2 \left(\frac{n(n-2)}{S_n}\right)^{\frac{n}{2}} |g(x_0, x_0)|$.

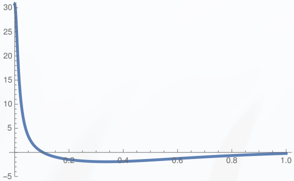
(*iii*) $\frac{u_\varepsilon(x)}{\sqrt{\varepsilon}} \to \left(\frac{n(n-2)}{S_n}\right)^{\frac{n}{4}} \frac{(n-2)}{\sqrt{2|g(x_0,x_0)|}} G(x, x_0)$ as $\varepsilon \to 0$.

Here $g(x, y)$ is the regular part of the Green function of Ω.

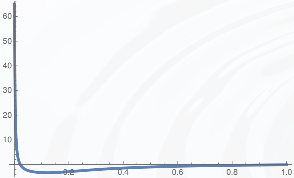
What about radial sign-changing solutions?

$$\begin{cases} -\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon, & \text{in } B, \\ u_\varepsilon = 0, & \text{on } \partial B. \end{cases}$$

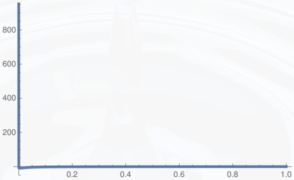
There are infinitely many nodal radial solutions which concentrate as $\varepsilon \to 0$.



$\varepsilon = 1$



$\varepsilon = 0.6$

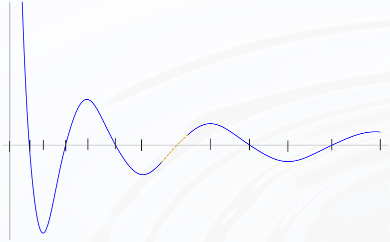


$\varepsilon = 0.1$

Question 1: Can we also obtain sharp estimates for radial nodal solutions?

... or radial Neumann solutions?

$$\begin{cases} -\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon, & \text{in } B, \\ \partial_\nu u_\varepsilon = 0, & \text{on } \partial B. \end{cases}$$



Question 2: Can we also obtain sharp estimates for radial Neumann solutions?

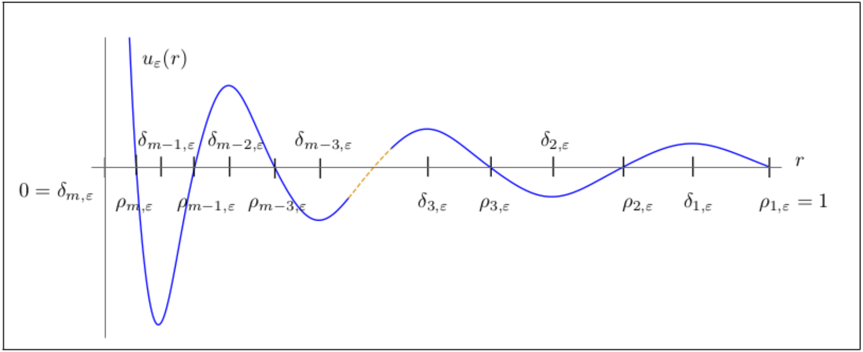
Note that all (nontrivial) Neumann solutions are sign-changing because

$$\int_B |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon = -\int_B \Delta u_\varepsilon = \int_B \nabla u_\varepsilon \nabla 1 = 0.$$

Theorem (Grossi-S.-Tavares, 2018)

If u_ε is a Dirichlet radial solution with $m-1$ interior zeros, then

$$u_\varepsilon(\delta_{k,\varepsilon}) \sim \varepsilon^{-\frac{1}{2}-(k-1)} \textit{(maxima and minima)}, \;\; \delta_{k,\varepsilon} \sim \varepsilon^{\frac{2(n-1)}{n(n-2)}+\frac{2}{n-2}(k-1)} \textit{(critical points)},$$
$$u'_\varepsilon(\rho_{k,\varepsilon}) \sim \varepsilon^{\frac{1}{2}-\frac{n}{n-2}(k-1)} \textit{(derivatives at roots)}, \;\; \rho_{k,\varepsilon} \sim \varepsilon^{\frac{2}{n-2}(k-1)} \textit{(roots)}.$$

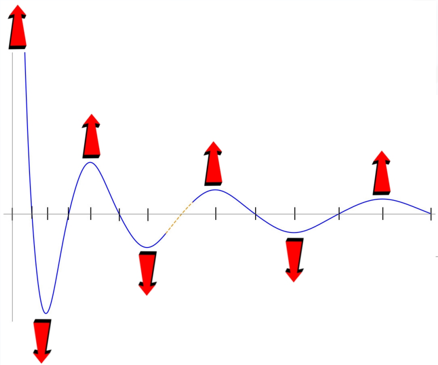


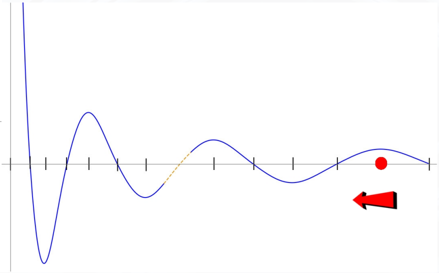
Theorem (Grossi-S.-Tavares, 2018)

If u_ε is a **Dirichlet** *radial solution with $m-1$ interior zeros, then*

$$u_\varepsilon(\delta_{k,\varepsilon}) \sim \varepsilon^{-\frac{1}{2}-(k-1)} \text{(maxima and minima)}, \quad \delta_{k,\varepsilon} \sim \varepsilon^{\frac{2(n-1)}{n(n-2)}+\frac{2}{n-2}(k-1)} \text{(critical points)},$$

$$u_\varepsilon'(\rho_{k,\varepsilon}) \sim \varepsilon^{\frac{1}{2}-\frac{n}{n-2}(k-1)} \text{(derivatives at roots)}, \quad \rho_{k,\varepsilon} \sim \varepsilon^{\frac{2}{n-2}(k-1)} \text{(roots)}.$$





Theorem (Grossi-S.-Tavares, 2018)

$$\lim_{\varepsilon\to 0} |u_\varepsilon(\delta_{k,\varepsilon})| \ (\kappa_n\varepsilon)^{\frac{2k-1}{2}} = D(k,m) \qquad \textit{for } k\in\{1,\dots,m\},$$

$$\lim_{\varepsilon\to 0} \delta_{k,\varepsilon} \ (\kappa_n\varepsilon)^{-\frac{2(kn-1)}{n(n-2)}} = d(k,m) \qquad \textit{for } k\in\{1,\dots,m-1\} \textit{ if } m\geq 2,$$

$$\lim_{\varepsilon\to 0} |u_\varepsilon'(\rho_{k,\varepsilon})| \ (\kappa_n\varepsilon)^{\frac{2kn-3n+2}{2(n-2)}} = Z(k,m) \qquad \textit{for } k\in\{1,\dots,m\},$$

$$\lim_{\varepsilon\to 0} \rho_{k,\varepsilon} \ (\kappa_n\varepsilon)^{-\frac{2(k-1)}{n-2}} = z(k,m) \qquad \textit{for } k\in\{2,\dots,m\} \textit{ if } m\geq 2;$$

where $\kappa_n = (n-2)\Gamma(\frac{n}{2})^2/(4\Gamma(n))$ *and*

$$D(k,m) = (n(n-2))^{\frac{n-2}{4}} \frac{\Gamma(m-k+1)}{m^{\frac{1}{2}}\Gamma(m)},$$

$$d(k,m) = (m-k)^{\frac{1}{n}} \left(\frac{m^{\frac{1}{2}}\Gamma(m)}{\Gamma(m-k+1)}\right)^{\frac{2}{n-2}},$$

$$Z(k,m) = n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}(m-k+1)^{\frac{n-1}{n-2}} \left(\frac{\Gamma(m-k+1)}{m^{\frac{1}{2}}\Gamma(m)}\right)^{\frac{n}{n-2}},$$

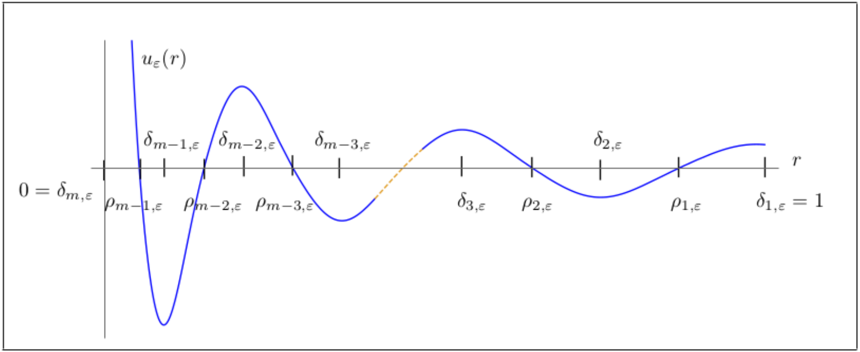
$$z(k,m) = (m-k+1)^{-\frac{1}{n-2}} \left(\frac{m^{\frac{1}{2}}\Gamma(m)}{\Gamma(m-k+1)}\right)^{\frac{2}{n-2}}.$$

Theorem (Grossi-S.-Tavares, 2018)

If u_ε is a **Neumann** *radial solution with $m-1$ interior zeros, then*

$$u_\varepsilon(\delta_{k,\varepsilon}) \sim \varepsilon^{\frac{n-2}{2n}-(k-1)} \text{(maxima and minima)}, \;\; \delta_{k,\varepsilon} \sim \varepsilon^{\frac{2(k-1)}{n-2}} \text{(critical points)},$$

$$u_\varepsilon'(\rho_{k,\varepsilon}) \sim \varepsilon^{\frac{n-4}{2(n-2)}-\frac{n(k-1)}{n-2}} \text{(derivatives at roots)}, \;\; \rho_{k,\varepsilon} \sim \varepsilon^{\frac{2}{n(n-2)}+\frac{2(k-1)}{n-2}} \text{(roots)}.$$



Theorem (Grossi-S.-Tavares, 2018)

$$\lim_{\varepsilon\to 0} |u_\varepsilon(\delta_{k,\varepsilon})| \, (\kappa_n \varepsilon)^{\frac{2kn-3n+2}{2n}} = \widetilde{D}(k,m) \qquad \textit{for } k \in \{1,\dots,m\},$$

$$\lim_{\varepsilon\to 0} \delta_{k,\varepsilon} \, (\kappa_n \varepsilon)^{-\frac{2(k-1)}{n-2}} = \widetilde{d}(k,m) \qquad \textit{for } k \in \{2,\dots,m-1\},$$

$$\lim_{\varepsilon\to 0} |u_\varepsilon'(\rho_{k,\varepsilon})| \, (\kappa_n \varepsilon)^{\frac{2kn-3n+4}{2(n-2)}} = \widetilde{Z}(k,m) \qquad \textit{for } k \in \{1,\dots,m-1\},$$

$$\lim_{\varepsilon\to 0} \rho_{k,\varepsilon} \, (\kappa_n \varepsilon)^{-\frac{2kn-2n+2}{n(n-2)}} = \widetilde{z}(k,m) \qquad \textit{for } k \in \{1,\dots,m-1\};$$

where $\kappa_n = (n-2)\Gamma(\frac{n}{2})^2/(4\Gamma(n))$ *and*

$$\widetilde{D}(k,m) = (n(n-2))^{\frac{n-2}{4}} (m-1)^{\frac{1}{2}-\frac{1}{n}} \frac{\Gamma(m-k+1)}{\Gamma(m)},$$

$$\widetilde{d}(k,m) = (m-1)^{-\frac{1}{n}} (m-k)^{\frac{1}{n}} \left(\frac{\Gamma(m)}{\Gamma(m-k+1)} \right)^{\frac{2}{n-2}},$$

$$\widetilde{Z}(k,m) = n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}} (m-1)^{\frac{1}{2}} (m-k)^{-\frac{1}{n-2}} \left(\frac{\Gamma(m-k+1)}{\Gamma(m)} \right)^{\frac{n}{n-2}},$$

$$\widetilde{z}(k,m) = (m-1)^{-\frac{1}{n}} (m-k)^{\frac{1}{n-2}} \left(\frac{\Gamma(m)}{\Gamma(m-k+1)} \right)^{\frac{2}{n-2}}.$$

Some remarks

Dirichlet:

$$u_\varepsilon(\delta_{k,\varepsilon}) \sim \varepsilon^{-\frac{1}{2}-(k-1)}(\text{max. \& min.}),\ \delta_{k,\varepsilon} \sim \varepsilon^{\frac{2(n-1)}{n(n-2)}+\frac{2(k-1)}{n-2}}(\text{critical points}),$$
$$u'_\varepsilon(\rho_{k,\varepsilon}) \sim \varepsilon^{\frac{1}{2}-\frac{n(k-1)}{n-2}}(\text{derivatives at roots}),\ \rho_{k,\varepsilon} \sim \varepsilon^{\frac{2(k-1)}{n-2}}(\text{roots}).$$

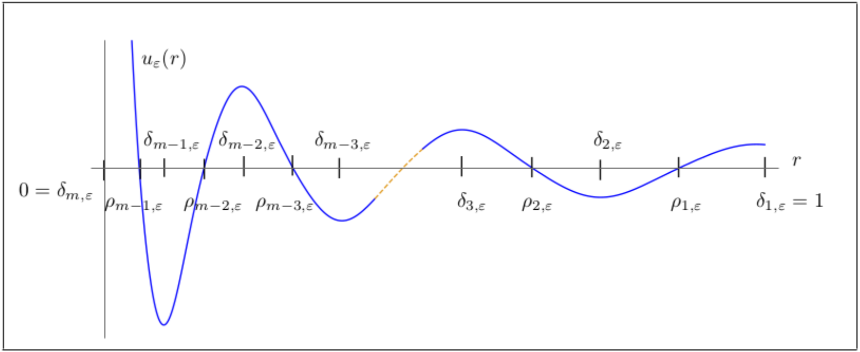
Neumann:

$$u_\varepsilon(\delta_{k,\varepsilon}) \sim \varepsilon^{\frac{n-2}{2n}-(k-1)}(\text{max. \& min.}),\ \delta_{k,\varepsilon} \sim \varepsilon^{\frac{2(k-1)}{n-2}}(\text{critical points}),$$
$$u'_\varepsilon(\rho_{k,\varepsilon}) \sim \varepsilon^{\frac{n-4}{2(n-2)}-\frac{n(k-1)}{n-2}}(\text{derivatives at roots}),\ \rho_{k,\varepsilon} \sim \varepsilon^{\frac{2}{n(n-2)}+\frac{2(k-1)}{n-2}}(\text{roots}).$$

Some remarks

In the Neumann case,

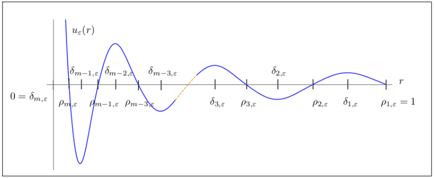
$$\text{for } n = 3,\ |u'_\varepsilon(\rho_{1,\varepsilon})| \to \infty;$$
$$\text{for } n = 4,\ |u'_\varepsilon(\rho_{1,\varepsilon})| \to 4\sqrt{2};$$
$$\text{for } n \geq 5,\ u'_\varepsilon(\rho_{1,\varepsilon}) \to 0.$$

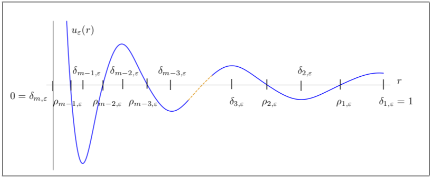


Some remarks

Other differences between the Dirichlet $u_{D,\varepsilon}$ and the Neumann $u_{N,\varepsilon}$ solutions:

$$|u_{D,\varepsilon}(\delta_{1,\varepsilon})| \to \infty, \qquad u_{N,\varepsilon}(\delta_{1,\varepsilon}) = u_{N,\varepsilon}(1) \to 0$$





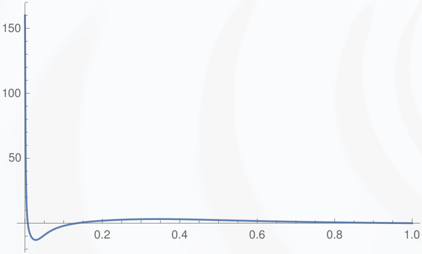
Some consequences

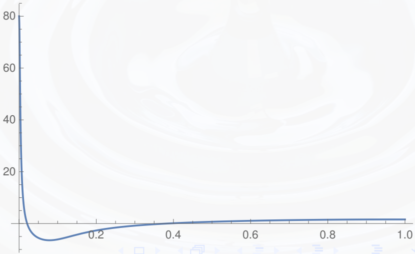
Theorem (Grossi-S.-Tavares, 2018)

If $u_{D,\varepsilon}$ (resp $u_{N,\varepsilon}$) is a Dirichlet (resp. Neumann) radial solution with $m-1$ interior zeros, then

$$|u_{D,\varepsilon}(x)|(\kappa_n\varepsilon)^{-\frac{1}{2}} = (n(n-2))^{\frac{n-2}{4}} m^{\frac{1}{2}}(|x|^{2-n}-1)+o(1),$$
$$|u_{N,\varepsilon}(x)|(\kappa_n\varepsilon)^{-\frac{n-2}{2n}} = (n(n-2))^{\frac{n-2}{4}}(m-1)^{\frac{n-2}{2n}}+o(1),$$

where $o(1)\to 0$ as $\varepsilon\to 0$, uniformly in compact subsets of $\overline{B}\backslash\{0\}$. In particular, solutions converge uniformly to zero in compact subsets of $\overline{B}\backslash\{0\}$.





Dirichlet tower of bubbles

Theorem (Pistoia-Weth (2005), Contreras-Del Pino (2006))

For $\varepsilon > 0$ small there are $\alpha_k > 0$, $f_\varepsilon \in L^\infty(B)$, and u_ε and with $m-1$ inner nodal spheres such that $-\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon$ in B, $u_\varepsilon = 0$ on ∂B, and

$$u_\varepsilon(y) = \gamma_n \sum_{k=1}^{m} (-1)^{k+1} \left(\frac{1}{1 + [\alpha_k \varepsilon^{\frac{1}{2}-k}]^{\frac{4}{n-2}} |y|^2} \right)^{\frac{n-2}{2}} \alpha_k \varepsilon^{\frac{1}{2}-k} - f_\varepsilon(y) \varepsilon^{\frac{1}{2}},$$

for $y \in B$.

$u(0) \sim \varepsilon^{\frac{1}{2}-m}$
$u(y) \sim \varepsilon^{\frac{1}{2}}$, $y \neq 0$

Dirichlet tower of bubbles

Theorem (Pistoia-Weth (2005), Contreras-Del Pino (2006))

For $\varepsilon > 0$ small there are $\alpha_k > 0$, $f_\varepsilon \in L^\infty(B)$, and u_ε and with $m-1$ inner nodal spheres such that $-\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon$ in B, $u_\varepsilon = 0$ on ∂B, and

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for $y \in B$.

Theorem (Grossi-S.-Tavares, 2018)

$$\alpha_k = \frac{\Gamma(m-k+1)}{m^{\frac{1}{2}} \Gamma(m)} \left(\frac{(n-2)\Gamma\left(\frac{n}{2}\right)^2}{4\Gamma(n)} \right)^{\frac{1}{2}-k},$$

$$\lim_{\varepsilon \to 0} \| f_\varepsilon - (n(n-2))^{\frac{n-2}{4}} \alpha_0 \|_{L^\infty(K)} = 0.$$

Neumann tower of bubbles

Theorem (Grossi-S.-Tavares, 2018)

For $\varepsilon > 0$ small there are $\alpha_k > 0$, $g_\varepsilon \in L^\infty(B)$, and u_ε and with $m-1$ inner nodal spheres such that $-\Delta u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2}-\varepsilon} u_\varepsilon$ in B, $\partial_\nu u_\varepsilon = 0$ on ∂B, and

$$u_\varepsilon(y) = \gamma_n \sum_{k=1}^{m} (-1)^{k+1} \left(\frac{1}{1 + [\beta_{k,\varepsilon}\, \varepsilon^{\frac{n-2}{2n}-(k-1)}]^{\frac{4}{n-2}} |y|^2} \right)^{\frac{n-2}{2}} \beta_{k,\varepsilon}\, \varepsilon^{\frac{n-2}{2n}-(k-1)} + g_\varepsilon(y) \varepsilon^{1+\frac{n-2}{2n}},$$

for $y \in B$, where $\beta_{k,\varepsilon} \to \beta_k$ with

$$\beta_k = (m-1)^{\frac{n-2}{2n}} \frac{\Gamma(m-k+1)}{\Gamma(m)} \left(\frac{(n-2)}{4} \frac{\Gamma(\frac{n}{2})^2}{\Gamma(n)} \right)^{\frac{n-2}{2n}-(k-1)}$$

and g_ε is a function which is uniformly bounded in B.

The problem in the whole space

Theorem (Grossi-S.-Tavares, 2018)

Let $w_\varepsilon \in C^2([0,\infty))$ *be the radial solution of*

$$-\Delta w_\varepsilon = |w_\varepsilon|^{\frac{4}{n-2}-\varepsilon} w_\varepsilon \quad \text{in } \mathbb{R}^n, \qquad w_\varepsilon(0) = 1.$$

Moreover, let $(r_{i,\varepsilon})_{i=1}^{\infty}$ *and* $(s_{i,\varepsilon})_{i=1}^{\infty}$ *be respectively the (divergent) increasing sequences of all zeros and critical points of* w_ε*, such that*

$$0 = s_{1,\varepsilon} < r_{1,\varepsilon} < s_{2,\varepsilon} < r_{2,\varepsilon} < \ldots < s_{i,\varepsilon} < r_{i,\varepsilon} < \ldots$$

Then,

$$\lim_{\varepsilon\to 0} r_{m,\varepsilon}(\kappa_n\varepsilon)^{\frac{2m-1}{n-2}} = (n(n-2))^{\frac{1}{2}} m^{\frac{1}{2-n}}\ \Gamma(m)^{-\frac{2}{n-2}} \qquad \text{for } m \geq 1,$$

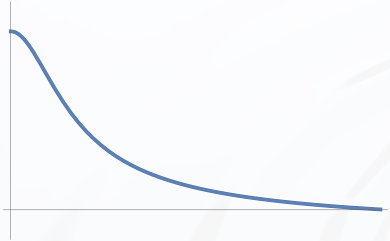
$$\lim_{\varepsilon\to 0} |w_\varepsilon'(r_{m,\varepsilon})|(\kappa_n\varepsilon)^{\frac{1-mn}{n-2}} = \left((n-2)n^{-1}\right)^{\frac{1}{2}} m^{\frac{n-1}{n-2}} \Gamma(m)^{\frac{n}{n-2}} \qquad \text{for } m \geq 1,$$

$$\lim_{\varepsilon\to 0} s_{m,\varepsilon}\,(\kappa_n\varepsilon)^{\frac{2mn-3n+2}{n(n-2)}} = (n(n-2))^{\frac{1}{2}} (m-1)^{\frac{1}{n}} \Gamma(m)^{-\frac{2}{n-2}} \qquad \text{for } m \geq 2,$$

$$\lim_{\varepsilon\to 0} |w_\varepsilon(s_{m,\varepsilon})|\,(\kappa_n\varepsilon)^{1-m} = \Gamma(m) \qquad \text{for } m \geq 2.$$

The proof (via induction).

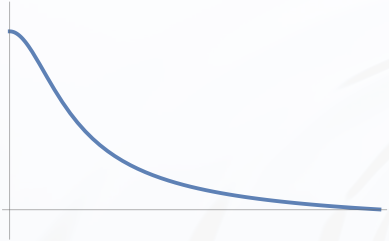
Positive Dirichlet solution: $v_{1,\varepsilon}$



(Atkinson-Peletier, 1987)+(Han,1991)

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{2}} v_{1,\varepsilon}(0) = D(1,1),$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}} v'_{1,\varepsilon}(1) = R(1,1).$$

Positive Dirichlet solution: $v_{1,\varepsilon}$

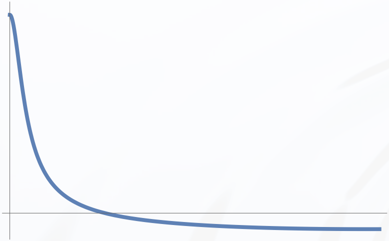


(Atkinson-Peletier, 1987)+(Han,1991)

$$\varepsilon^{\frac{1}{2}} v_{1,\varepsilon}(0) = D(1,1) + o(1),$$
$$\varepsilon^{-\frac{1}{2}} v'_{1,\varepsilon}(1) = R(1,1) + o(1).$$

We have 2 unknowns and 2 equations.

Neumann solution with 1 interior zero: $v_{2,\varepsilon}$

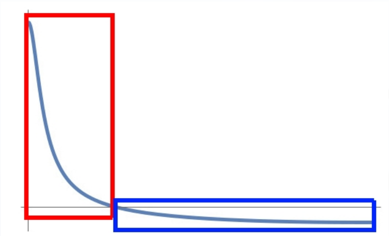


Unknowns:

$$v_{2,\varepsilon}(0), \quad v_{2,\varepsilon}(1), \quad \rho_{1,\varepsilon}, \quad v'_{2,\varepsilon}(\rho_{1,\varepsilon}),$$

We need: Four equations

Neumann solution with 1 interior zero



Rescaling:

$v_{1,\varepsilon}(x) = \rho_{1,\varepsilon}^{\frac{2(n-2)}{4-\varepsilon(n-2)}} v_{2,\varepsilon}(x\rho_{1,\varepsilon})$

(we obtain 2 equations)

Normalization $\left(\frac{v_{2,\varepsilon}(x)}{v_{2,\varepsilon}(1)} \to 1\right)$
+ Energy estimates

(we obtain the other 2 equations)

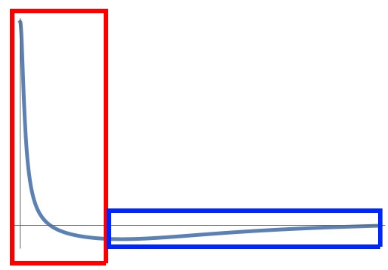
Dirichlet solution with 1 interior zero: $v_{3,\varepsilon}$

Unknowns:

$$v_{3,\varepsilon}(0), \quad v'_{3,\varepsilon}(1), \quad \rho_{1,\varepsilon}, \quad v'_{3,\varepsilon}(\rho_{1,\varepsilon}), \quad \delta_{1,\varepsilon}, \quad v_{3,\varepsilon}(\delta_{1,\varepsilon})$$

We need: Six equations

Dirichlet solution with 1 interior zero: $v_{3,\varepsilon}$



Rescaling:

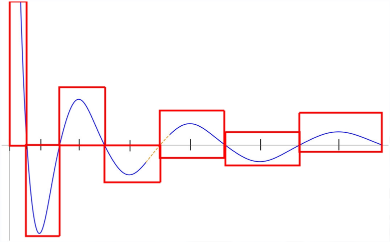
$$v_{2,\varepsilon}(x) = \delta_{1,\varepsilon}^{\frac{2(n-2)}{4-\varepsilon(n-2)}} v_{3,\varepsilon}(x\delta_{1,\varepsilon})$$

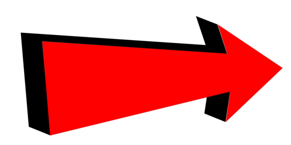
(we obtain 4 equations)

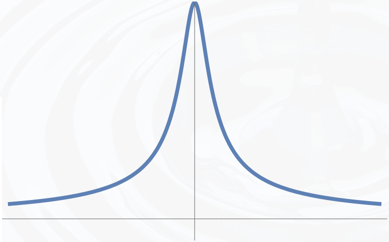
Normalization
+ Pohozahev identity
+ Energy estimates

(we obtain the other 2 equations)

Normalization (De Marchis-Ianni-Pacella, 2017)

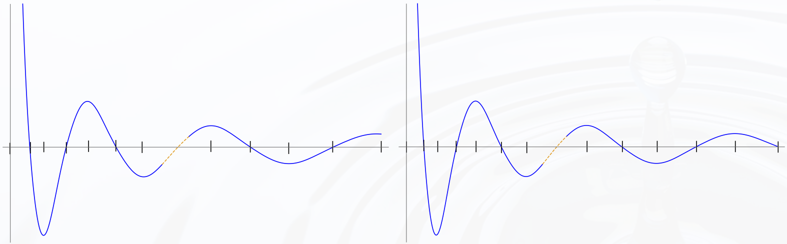






$$\frac{v_{3,\varepsilon}\left(u(\delta_{1,\varepsilon})^{\frac{\varepsilon}{2}-\frac{2}{n-2}}x\right)}{v_{3,\varepsilon}(\delta_{1,\varepsilon})} \to U(x) = \left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{2-n}{2}}$$

Continue inductively: $v_{4,\varepsilon}$, $v_{5,\varepsilon}$, $v_{6,\varepsilon}$,...



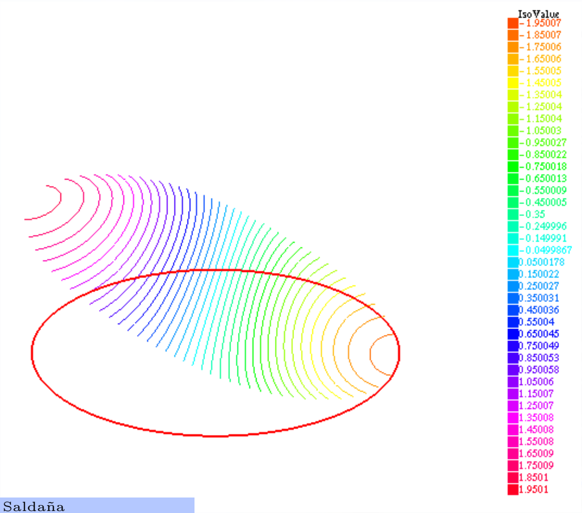
It sounds simple... but it is a computational challenge

The actual computation of the rates and constants, even for $v_{6,\varepsilon}$, requires hundreds (if not thousands) of basic algebraic manipulations.

To avoid any calculation mistake, we developed an algorithm based on a symbolic calculus software (Mathematica 11.1.1.0) to compute some of the rates and constants, from which general formulas could be deduced, and ultimately shown by induction.

And what about non-radial Neumann solutions?

$$\begin{cases} -\Delta u = |u|^{p-1}u, & \text{in } B, \\ \partial_\nu u = 0, & \text{on } \partial B. \end{cases}$$



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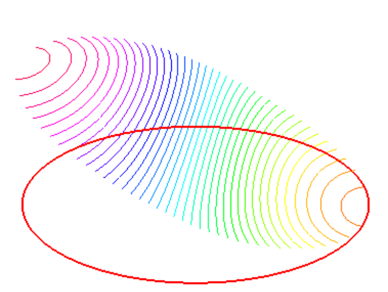

Existence of (least-energy) solutions and symmetry breaking:

- Sublinear case: $p \in (0,1)$ (Parini-Weth, 2015)
- Superlinear case: $p \in (1, \frac{n+2}{n-2})$ (S.-Tavares, 2017)
- Critical case: $p = \frac{n+2}{n-2}$ (Comte-Knaap, 1990)

Thank you!

The solution u_ε satisfies the so-called (pointwise) radial Pohozaev identity:

$$\Big(\frac{r^n}{2n}(u_\varepsilon'(r))^2+\frac{n-2}{2n}r^{n-1}u_\varepsilon'(r)u(r)+\frac{r^n}{n(p_\varepsilon+1)}|u_\varepsilon(r)|^{p_\varepsilon+1}\Big)' = r^{n-1}|u_\varepsilon(r)|^{p_\varepsilon+1}\Big(\frac{1}{p_\varepsilon+1}-\frac{n-2}{2n}\Big)$$