

A glimpse into partial differential equations.

Alberto Saldaña.

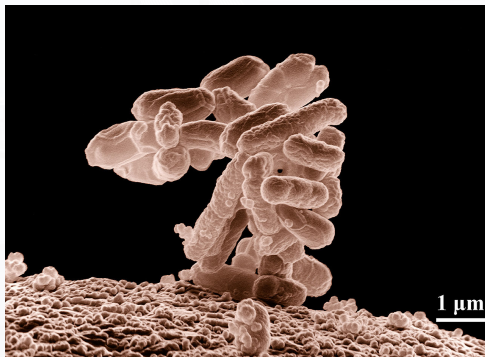
Humboldt Regional Group Karlsruhe-Pforzheim e.V. Lecture |
International Scholars and Welcome Office

November 30, 2017.

Partial differential equations (PDEs) are one of the most successful mathematical objects to model real-world phenomena.

PDEs capture the essence of **change**,
either **physical**, **chemical**, **biological**,
social, or **anything else**!

There are models for population dynamics...



$$\partial_t u - \Delta u = \lambda u - u^2$$

...for heat diffusion...



$$\partial_t \phi - \Delta \phi = f(\phi, t)$$

...for soap bubbles...



$$\Delta p = \gamma \frac{2}{R}$$

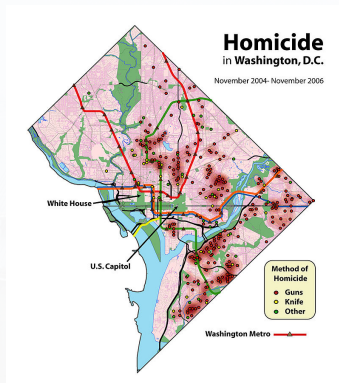
...for cell motility...



$$\partial_t u = \nabla(D_1 \nabla u - \chi u \nabla v) + f$$

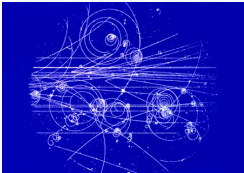
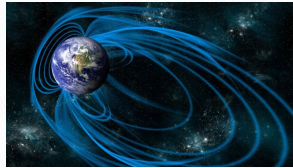
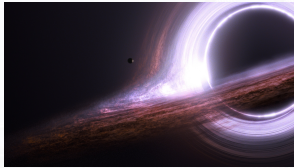
$$\partial_t v = D_2 \Delta v + g - h$$

...for crime hotspots...



$$\begin{aligned}\partial_t s = & \Delta s(x, t) - s(x, t) + s_b(x) \\ & + (\rho(x) - c(x, t))u(x, t)\end{aligned}$$

... and many many more models exist!



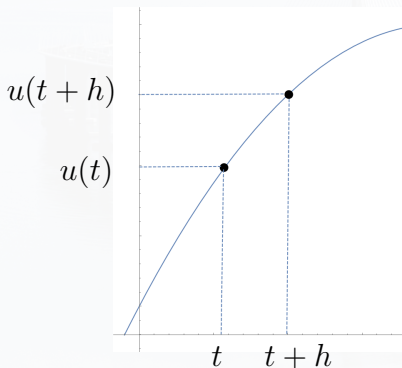
*“The purpose of [mathematical] models is
not to fit the data but to **sharpen** the
questions.”*

—Samuel Karlin.

To get a feeling, let's recall first some of the basics

The derivative of a function $u : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$u'(t) = \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{(t+h) - t} = \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}.$$



The derivative $u'(t)$ contains the following information:

- ✓ The (instantaneous) rate of change of u at point t .
- ✓ If u is increasing ($u'(t) > 0$) or decreasing ($u'(t) < 0$).

Derivatives can be used to translate into a **mathematical language** the essential **laws or principles** of a given phenomenon.

For example, the principle

*“the number of **new bacteria per unit time** is **proportional** to the **present population**”*

can be written as

$$\begin{aligned}u'(t) &= a u(t) && \text{for } t > 0, \\u(0) &= b,\end{aligned}$$

where

- ✓ $u(t)$: population density at time t .
- ✓ $a > 0$: birth rate.
- ✓ $b > 0$: initial amount of bacteria.

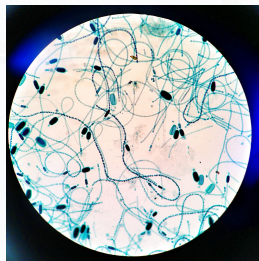
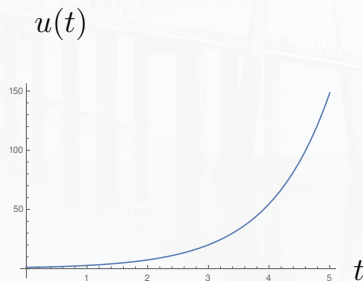
For example, cyanobacteria...

The equation (with initial condition)

$$\begin{aligned}u'(t) &= a u(t) \quad \text{for } t > 0, \\u(0) &= b\end{aligned}$$

has the solution $u(t) = be^{at}$, that is:

bacteria grow exponentially fast (at least at the beginning).



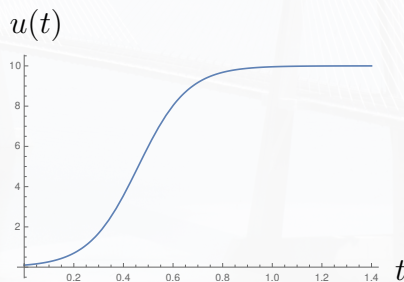
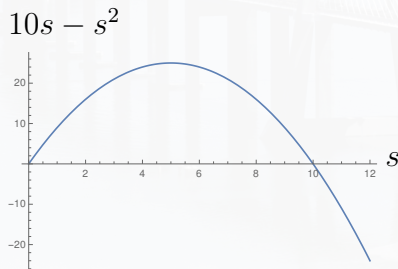
Cyanobacteria can double their population four times a day.

However...

Exponential growth cannot continue indefinitely (because of space, lack of nutrients, wastes,...). Therefore, we may include a **concentration** term:

$$\begin{aligned}u'(t) &= a u(t) - u(t)^2 \quad \text{for } t > 0, \\u(0) &= b.\end{aligned}$$

For example, if $a = 10$ and $b = 0.1$,

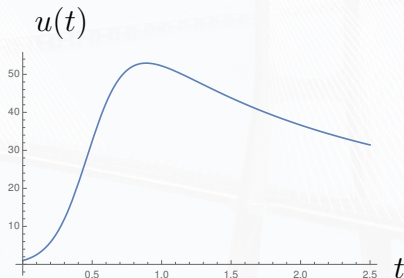


And if the nutrients decrease through time...

The consumption of (finite) nutrients can be translated into a **time-dependent** decreasing birth rate:

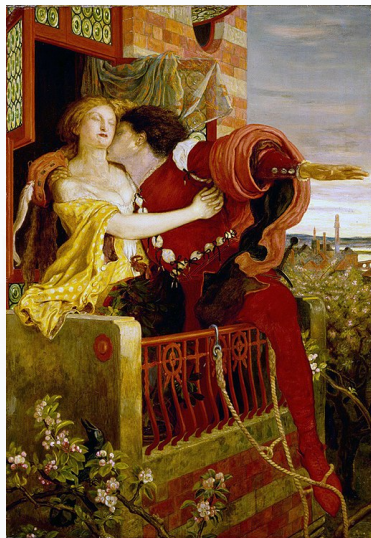
$$u'(t) = \frac{a}{(1+t)} u(t) - u(t)^2,$$

$$u(0) = b.$$



Modeling bacteria is all fine and good,
but... can differential equations model
love and romance?

The Romeo and Juliet model



Let $a, b, c, d, e, f \in \mathbb{R}$ and

$$r'(t) = a r(t) + b j(t),$$

$$j'(t) = c r(t) + d j(t),$$

$$r(0) = e, \quad j(0) = f.$$

- ✓ $r(t)$: Romeo's **love** (or discomfort if negative) for Juliet at time t .
- ✓ $j(t)$: Juliet's **love** (or discomfort if negative) for Romeo at time t .

The Romeo and Juliet model

$$\begin{aligned}r'(t) &= a r(t) + b j(t), \\j'(t) &= c r(t) + d j(t), \\r(0) &= e, \quad j(0) = f.\end{aligned}$$

The parameters a and b specify Romeo's **romantic style**:

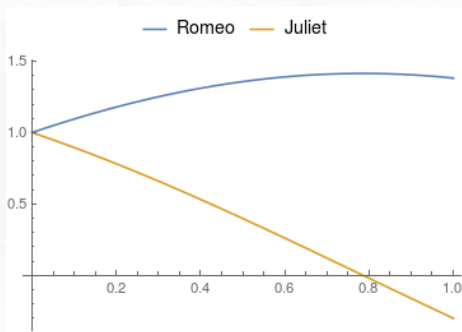
- ✓ $a > 0$ & $b > 0$: *Eager lover.*
- ✓ $a > 0$ & $b \leq 0$: *Narcissistic.*
- ✓ $a \leq 0$ & $b > 0$: *Cautious or secure.*
- ✓ $a \leq 0$ & $b \leq 0$: *Hermit.*

Similarly, c and d specify Juliet's **romantic style**, whereas e and f describe the **initial interest** for each other at time $t = 0$.

A simple model

$$\begin{aligned}r'(t) &= j(t), \\ r(0) &= 1,\end{aligned}$$

$$\begin{aligned}j'(t) &= -r(t), \\ j(0) &= 1.\end{aligned}$$



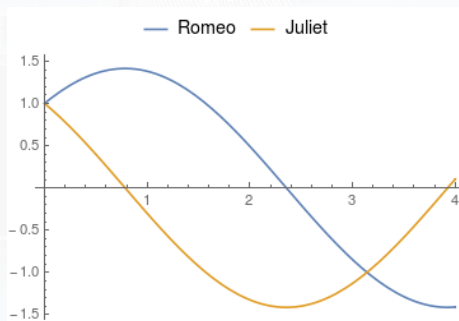
A simple model

$$r'(t) = j(t),$$

$$r(0) = 1,$$

$$j'(t) = -r(t),$$

$$j(0) = 1.$$



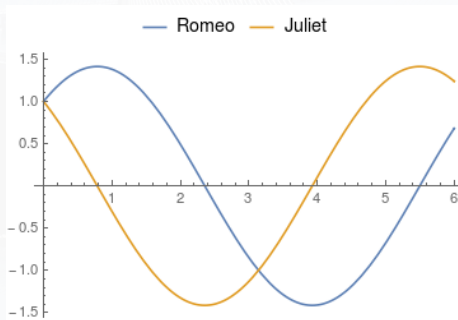
A simple model

$$r'(t) = j(t),$$

$$r(0) = 1,$$

$$j'(t) = -r(t),$$

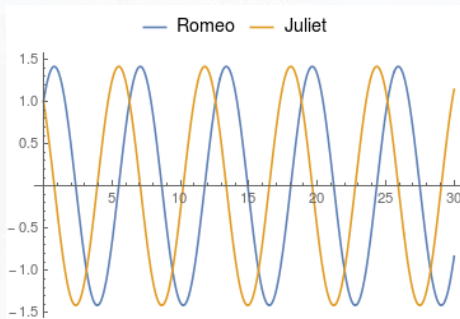
$$j(0) = 1.$$



A simple model

$$\begin{aligned}r'(t) &= j(t), \\ r(0) &= 1,\end{aligned}$$

$$\begin{aligned}j'(t) &= -r(t), \\ j(0) &= 1.\end{aligned}$$



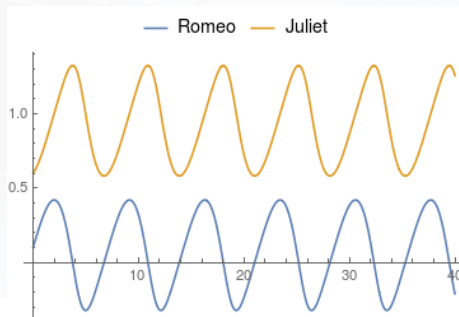
A (nonlinear) concentration term

The function

$$s \mapsto s(1 - s) = s - s^2$$

is also known as the “repair nonlinearity”.

$$\begin{aligned} r'(t) &= j(t)(1 - j(t)), & j'(t) &= r(t)(1 - r(t)), \\ r(0) &= 0.1, & j(0) &= 0.6 \end{aligned}$$



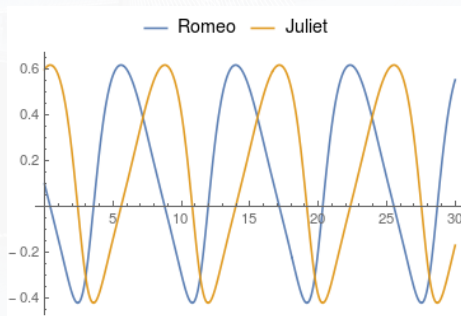
A (nonlinear) concentration term

$$r'(t) = j(t)(j(t) - 1),$$

$$r(0) = 0.1,$$

$$j'(t) = r(t)(1 - r(t)),$$

$$j(0) = 0.6$$



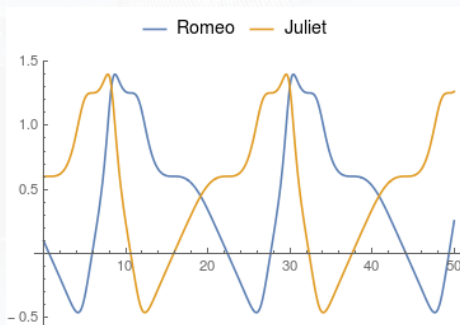
A (nonlinear) concentration term

$$r'(t) = j(t)^2(j(t) - 1),$$

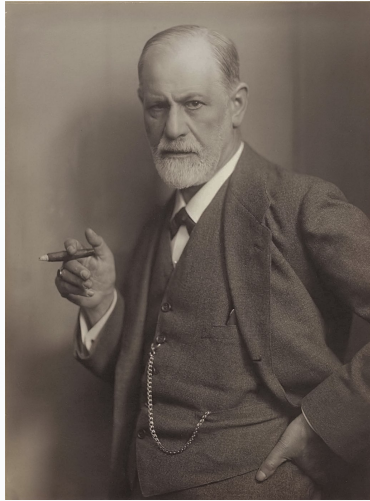
$$r(0) = 0.1,$$

$$j'(t) = r(t)^2(1 - r(t)),$$

$$j(0) = 0.6$$

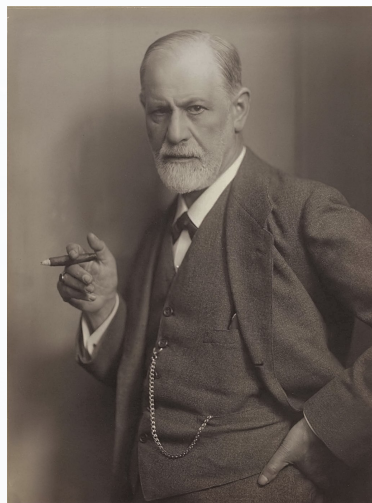


Maybe some professional help?



Alberto Saldaña

Maybe some professional help?

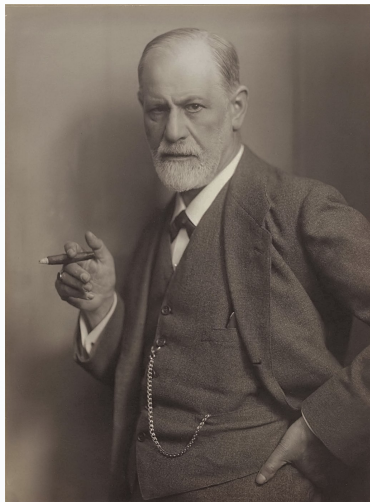


To Romeo:

- ✓ Be a little on the **safe side** with your feelings.
- ✓ About Juliet, **love her** when she excels at showing her interest for you, but show her your **discomfort** if she is not cheerful enough.

$$r'(t) = -2r(t) + 2j(t)(j(t) - 1)$$

Maybe some professional help?



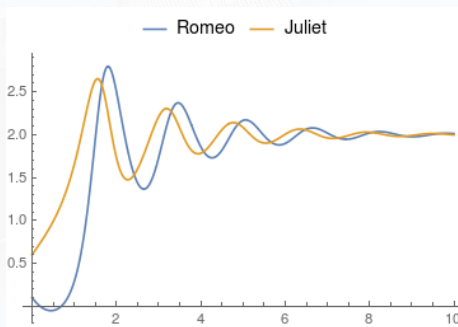
To Juliet:

- ✓ Don't hold back your feelings, **be happy to be in love.**
- ✓ About Romeo, **show him that you care when he is not happy enough**, but don't be afraid of **asking for your own space** if you feel overwhelmed.

$$j'(t) = j(t) + r(t)(1 - r(t))$$

A happy equilibrium

$$\begin{aligned}r'(t) &= -2j(t) + 2j(t)(j(t) - 1), & j'(t) &= j(t) + r(t)(r(t) - 1), \\ r(0) &= 0.1, & j(0) &= 0.6\end{aligned}$$



Now that we know how to read
differential equations...

A model for democracy

$$G'(t) = 1 - G(t),$$

$$S'(t) = G(t)S(t)(1 - S(t)),$$

$$D'(t) = \left(S(t)(S(t) - D(t)) - D(t) \right) (1 - D(t)).$$

- ✓ G: Gross domestic product per capita.
- ✓ D: Democracy.
- ✓ S: Self-expressive values (social toleration, life satisfaction, public expression, aspiration to liberty).

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The Dynamics of Democracy, Development and Cultural Values



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Abstract

Over the past decades many countries have experienced rapid changes in their economies, their democratic institutions and the values of their citizens. Comprehensive data measuring these changes across very different countries has recently become openly available. Between country similarities suggest common underlying dynamics in how countries develop in terms of economy, democracy and cultural values. We apply a novel Bayesian dynamical systems approach to identify the model which best captures the complex, mainly non-linear dynamics that underlie these changes. We show that the level of Human Development Index (HDI) in a country drives first democracy and then higher emancipation of citizens. This change occurs once the countries pass a certain threshold in HDI. The data also suggests that there is a limit to the growth of wealth, set by higher emancipation. Having reached a high level of democracy and emancipation, societies tend towards equilibrium that does not support further economic growth. Our findings give strong empirical evidence against a popular political science theory, known as the Human Development Sequence. Contrary to this theory, we find that implementation of human-rights and democratisation precede increases in emancipative values.

A model for democracy

$$G'(t) = 1 - G(t),$$

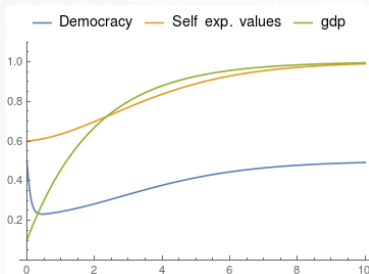
$$S'(t) = G(t)S(t)(1 - S(t)),$$

$$D'(t) = \left(S(t)(S(t) - D(t)) - D(t) \right) (1 - D(t)).$$

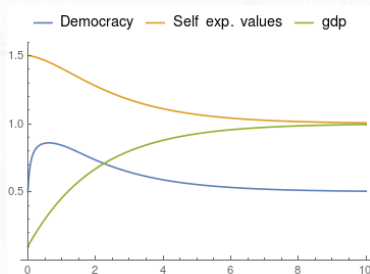
✓ $S(t) - D(t)$: S is a threshold for D .

✓ $-D(t)$: D is not self-reinforcing.

✓ $(1 - D(t))$: Concentration term.



$G[0] = 0.1, D[0] = 0.5, S[0] = 0.6$



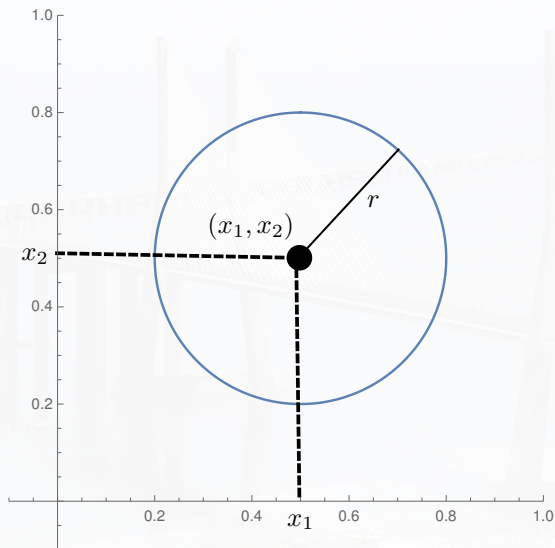
$G[0] = 0.1, D[0] = 0.5, S[0] = 1.5$

We are ready to introduce a new
dimension...

... and a higher order...

Two-dimensional space variables

Let $S_r(x)$ denote a circle of radius r centered at $x = (x_1, x_2)$, that is,



The Laplacian in dimension two

The **Laplacian** is

$$\Delta u(x_1, x_2, t) = \partial_{x_1, x_1} u(x_1, x_2, t) + \partial_{x_2, x_2} u(x_1, x_2, t),$$

where

$$\partial_{x_1} u(x_1, x_2, t) = \lim_{h \rightarrow 0} \frac{u(x_1 + h, x_2, t) - u(x_1, x_2, t)}{h}$$

and

$$\partial_{x_1, x_1} u(x_1, x_2, t) = \lim_{h \rightarrow 0} \frac{\partial_{x_1} u(x_1 + h, x_2, t) - \partial_{x_1} u(x_1, x_2, t)}{h}.$$

The Laplacian's defining property

The **mean value properties**:

$$\Delta u(x, t) > 0 \quad \Longrightarrow \quad u(x, t) < \oint_{S_r(x)} u(s, t) \, ds.$$

*If the Laplacian of u is positive at x , then the values of u at the surrounding points are in average **above** $u(x, t)$.*

$$\Delta u(x, t) < 0 \quad \Longrightarrow \quad u(x, t) > \oint_{S_r(x)} u(s, t) \, ds.$$

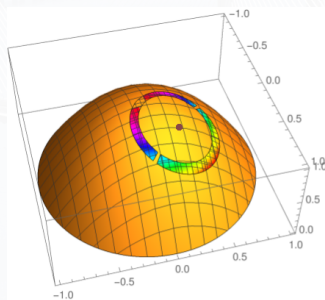
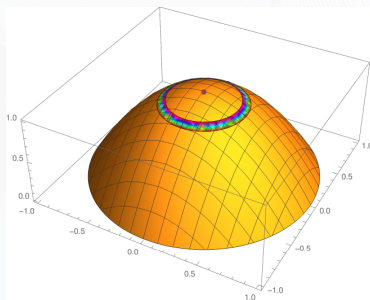
*If the Laplacian of u is negative at x , then the values of u at the surrounding points are in average **below** $u(x, t)$.*

For example

If $u(x_1, x_2) = -x_1^2 - x_2^2 + 1$ (see below), then

$$\Delta u(x_1, x_2) = -4 < 0 \quad \text{for all } x_1, x_2,$$

then for every x , the values of u at the surrounding points are in average *below* $u(x)$.



The function $u(x_1, x_2)$

The heat equation



Heat diffusion principle:

*“When an object is at a different temperature from its surroundings, **heat** flows so that the object and its surroundings reach the same temperature”*

The heat equation

*“When an object is at a different temperature from its surroundings, **heat** flows so that the object and its surroundings reach the same temperature”*

Let $u(x, t)$ denote the **temperature** at point x and at time t .

$$\begin{aligned}\partial_t u(x, t) - \Delta u(x, t) &= 0, \\ u(x, 0) &= b(x).\end{aligned}$$

Observe that

$$\partial_t u(x, t) = \Delta u(x, t) < 0 \iff u(x, t) > \oint_{\partial B_r(x)} u(s, t) \, ds.$$

The temperature around x is (in average) **below** than the temperature at x at time t , thus the temperature at x will **decrease**.

The heat equation

*“When an object is at a different temperature from another body or its surroundings, **heat** flows so that the body or its surroundings reach the same temperature”*

Let $u(x, t)$ denote the **temperature** at point x and at time t .

$$\begin{aligned}\partial_t u(x, t) - \Delta u(x, t) &= 0, \\ u(x, 0) &= b(x).\end{aligned}$$

Observe that

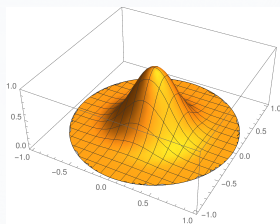
$$\partial_t u(x, t) = \Delta u(x, t) > 0 \quad \Longleftrightarrow \quad u(x, t) < \oint_{\partial B_r(x)} u(s, t) \, ds.$$

The temperature around x is (in average) **above** than the temperature at x at time t , thus the temperature at x will **increase**.

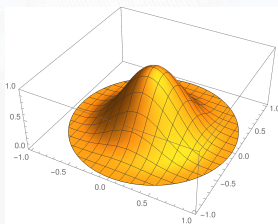
A new look at population dynamics...

The growth with **movement/diffusion** of a population is modeled by

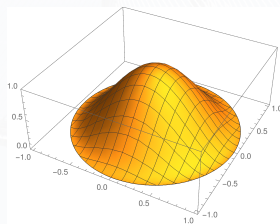
$$\partial_t u(x, t) - \Delta u(x, t) = a u(x, t) - u(x, t)^2$$



$u(x, 0)$



$u(x, 1)$



$u(x, 2)$

$u(x, t)$ is the population density at time t and at point $x \in B$.

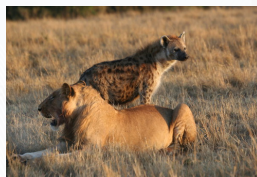
A model for competition, cooperation, or predation

$$\partial_t u - \Delta u = a u - u^2 + \alpha u v$$

$$\partial_t v - \Delta v = b v - v^2 + \beta u v$$



$$\alpha > 0, \beta > 0$$



$$\alpha < 0, \beta < 0$$



$$\alpha > 0, \beta < 0$$

Asymptotic behavior

$$\begin{aligned}\partial_t u - \Delta u &= a u - u^2 + \alpha u v, \\ \partial_t v - \Delta v &= b v - v^2 + \beta u v.\end{aligned}$$

As $t \rightarrow \infty$ there are two possibilities:

✓ **Extinction:**

$$\lim_{t \rightarrow \infty} u(x, t) = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} v(x, t) = 0.$$

✓ **Coexistence:**

$$\lim_{t \rightarrow \infty} u(x, t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} v(x, t) > 0.$$

In each case, what can be said about the asymptotic shape of the population densities?

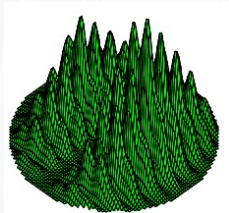
Asymptotic shape

What can be said about the asymptotic shape of the population densities?

Extinction:

If $\lim_{t \rightarrow \infty} u(x, t) > 0$ and $\lim_{t \rightarrow \infty} v(x, t) = 0$,

then u is asymptotically radially symmetric.



The population density $u(x, t)$ as $t \rightarrow \infty$.

Asymptotic shape

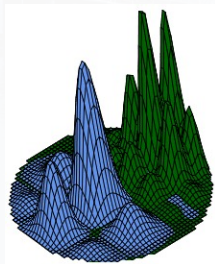
$$\partial_t u - \Delta u = a(t) u - u^2 + \alpha u v,$$

$$\partial_t v - \Delta v = b(t) v - v^2 + \beta u v.$$

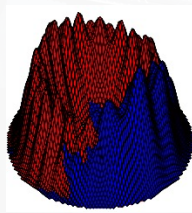
Coexistence:

If $\lim_{t \rightarrow \infty} u(x, t) > 0$ and $\lim_{t \rightarrow \infty} v(x, t) > 0$,

then...



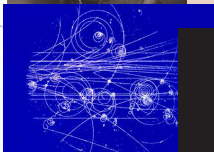
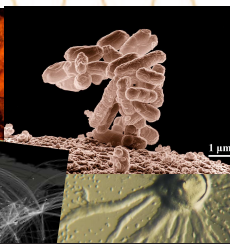
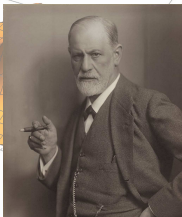
If $\alpha < 0$ and $\beta < 0$.



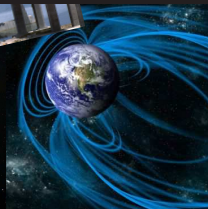
If $\alpha > 0$ and $\beta > 0$.

?

If $\alpha > 0$ and $\beta < 0$.



How would *you* use PDEs
in your own research?





Thank you for your Δ ttention!



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