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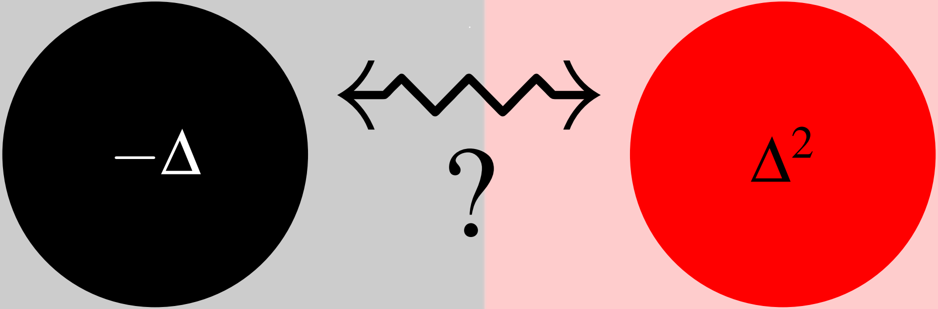
On fractional higher-order powers of the Laplacian.

(joint work with Nicola Abatangelo and Sven Jarohs)

Satellite Conference on Nonlinear Partial Differential Equations,
Fortaleza - Brazil | July 26, 2018.

Alberto Saldaña

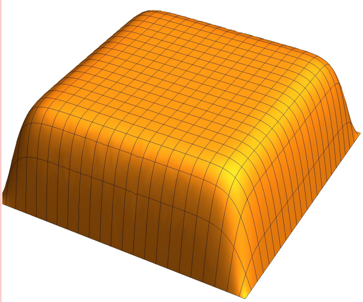

$-\Delta$
?
Δ^2

Consider the problem

$$\begin{cases} -\Delta u = u - u^3 & \textit{in } Q := [0,10]\times[0,10], \\ u = 0 & \textit{on } \partial Q. \end{cases}$$

A positive solution

- *exists,*
- *is unique,*
- *is bounded by 1,*
- *is stable,*
- *is symmetric,*
- *is monotone,*
- *achieves its maximum at the origin.*

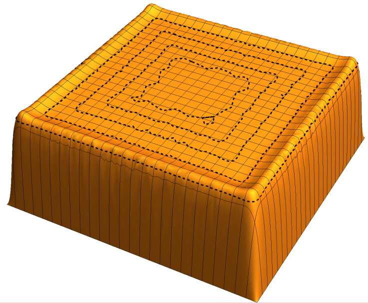


On the other hand, consider the problem

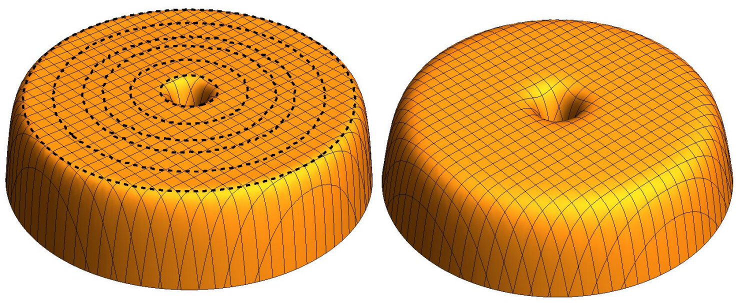
$$\begin{cases} \Delta^2 u = u - u^3 & \text{in } Q := [0,10]\times[0,10], \\ \Delta u = u = 0 \ \text{ or } \ \partial_\nu u = u = 0 & \text{on } \partial Q. \end{cases}$$

Positive solutions

- *can be approximated numerically,*
- *uniqueness is not known,*
- *are not bounded by 1,*
- *stability is not known,*
- *symmetry probably holds, but is not clear due to the many oscillations at* $\{u = 1\}$*,*
- *are not monotone,*
- *achieve their maximum close to the corners and not at the origin.*

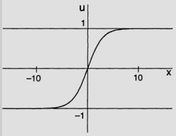


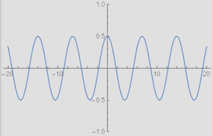
Similar things happen in annuli.



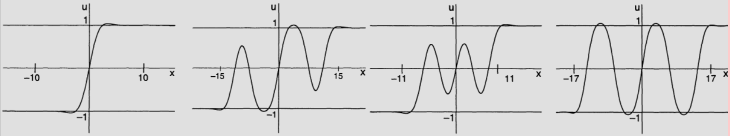
$$-u'' = u - u^3 \quad vs \quad u'''' = u - u^3 \quad in\ \mathbb{R}$$

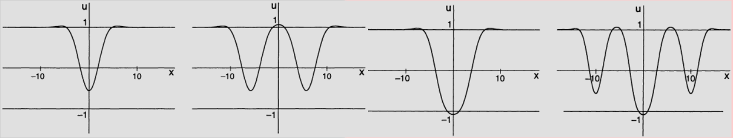
$-\Delta$

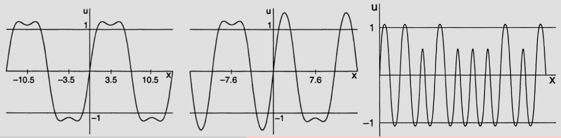




Δ^2





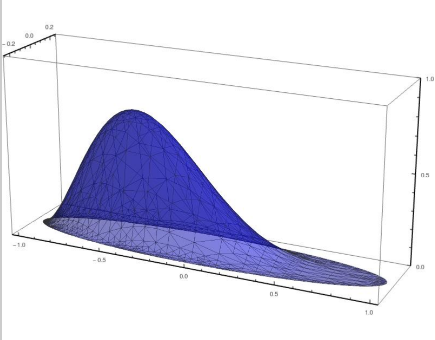


Pictures taken from: L.A. Peletier and W.C. Troy. Spatial patterns.

Even on linear problems there are important differences

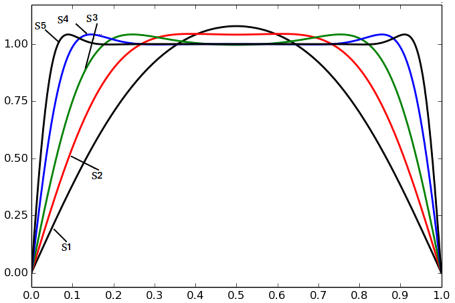
$$\begin{cases} -\Delta u = f > 0 & \text{in } E := \{x^2 + 25y^2 < 1\}, \\ u = 0 & \text{on } \partial E. \end{cases} \implies u > 0 \text{ in } E,$$

$$\begin{cases} (-\Delta)^2 u = f > 0 & \text{in } E := \{x^2 + 25y^2 < 1\}, \\ \partial_\nu u = u = 0 & \text{on } \partial E. \end{cases} \implies u \text{ changes sign in } E.$$



(Shapiro-Tegmark, 1994)

How does this happen?



Let us look at the intermediate powers of the Laplacian: $(-\Delta)^s$ *for* $s \in (1,2)$.

Some fractional questions:

- *If* u_s *is the solution of* $(-\Delta)^s u_s = f$ *+ b.c., is* $s \mapsto u_s$ *continuous in some sense?*
- *If* $(-\Delta)^2$ *requires two boundary conditions for well-posedness and* $-\Delta$ *only one, what happens with the extra boundary condition as* $s \searrow 1$*?*
- *Are solutions increasingly "bad"as* $s \nearrow 2$ *or is it essentially the same for any* $s > 1$*?*
- *Is the Green function for* $(-\Delta)^s$ *in the ellipse* $E = \{x^2 + 25y^2 < 1\}$ *sign-changing for all* $s > 1$*?*

Pointwise evaluation of $(-\Delta)^s$ *for* $s \in (1,2)$

Let $N \in \mathbb{N}$, $s \in (0,2)$, *then*

$$(-\Delta)^s u(x) := c_{N,s} \int_{\mathbb{R}^N} \frac{u(x+2y) - 4u(x+y) + 6u(x) - 4u(x-y) + u(x-2y)}{|y|^{N+2s}}\, dy.$$

Here $c_{N,s} > 0$ *is a normalization constant such that* $\mathscr{F}((-\Delta)^s u) = |\xi|^{2s} \mathscr{F} u$.

In particular, for a suitable (fixed) function $u : \mathbb{R}^N \to \mathbb{R}$,

$$\lim_{s \nearrow 2} (-\Delta)^s u(x) = (-\Delta)^2 u(x) \quad \textit{and} \quad \lim_{s \searrow 1} (-\Delta)^s u(x) = -\Delta u(x)$$

Intuition:

$$\lim_{t \to 0} \frac{f(0) - f(t)}{t} = f'(0), \quad \lim_{t \to 0} \frac{f(2t) - 4f(t) + 6f(0) - 4f(-t) + f(-2t)}{t^4} = f''''(0)$$

Nonhomogeneous linear problems in a ball $B := \{|x| < 1\}$

Let $s \in (1,2)$,

$$(-\Delta)^s u(x) := c_{N,s} \int_{\mathbb{R}^N} \frac{u(x+2y) - 4u(x+y) + 6u(x) - 4u(x-y) + u(x-2y)}{|y|^{N+2s}}\, dy.$$

$$\begin{cases} -\Delta u = f & \text{in } B, \\ u = g & \text{on } \partial B. \end{cases}$$

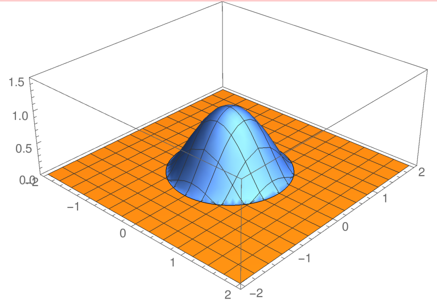
$$\begin{cases} \Delta^2 u = f & \text{in } B, \\ u = h & \text{on } \partial B, \\ -\partial_\nu u = g & \text{on } \partial B. \end{cases}$$

Note that: $2 - s \in (0,1)$,

$$\begin{aligned} (-\Delta)^s u &= f && \text{in } B, \\ u &= v && \text{on } \mathbb{R}^N \setminus B, \\ \lim_{x\to z} (1 - |x|^2)^{2-s} u(x) &= h(z) && \text{for } z \in \partial B, \\ \lim_{x\to z} -\partial_\nu [((1 - |x|^2)^{2-s} u(x))] &= g(z) && \text{for } z \in \partial B. \end{aligned}$$

The Green function, $s \in (1,2)$

$$
\begin{aligned}
(-\Delta)^s u = f \quad & \textit{in } B,\\
u = 0 \quad & \textit{on } \mathbb{R}^N \setminus B,\\
\lim_{x\to z} (1-|x|^2)^{2-s} u(x) = 0 \quad & \textit{for } z \in \partial B,\\
\lim_{x\to z} -\partial_\nu((1-|x|^2)^{2-s} u(x)) = 0 \quad & \textit{for } z \in \partial B.
\end{aligned}
$$



The unique solution is

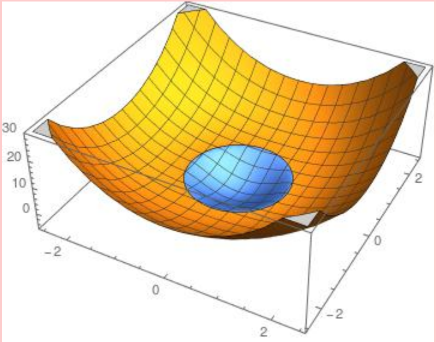
$$u(x) = \int_B G_s(x,y) f(y)\, dy \qquad \textit{(note that } u > 0 \textit{ if } f > 0\textit{)},$$

where (Dipierro-Grunau, 2016), (Abatangelo, Jarohs, S., 2016)

$$G_s(x,y) \;=\; k_{N,s}|x-y|^{2s-N} \int_0^{\frac{(1-|x|^2)_+(1-|y|^2)_+}{|x-y|^2}} \frac{t^{s-1}}{(t+1)^{\frac{N}{2}}}\, dt, \qquad x,y \in \mathbb{R}^N, \quad x \neq y.$$

The nonlocal Poisson kernel, $s \in (1,2)$

$$(-\Delta)^s u = 0 \quad \text{in } B,$$
$$u = v \quad \text{on } \mathbb{R}^N \setminus B,$$
$$\lim_{x\to z}(1-|x|^2)^{2-s}u(x) = 0 \quad \text{for } z \in \partial B,$$
$$\lim_{x\to z} -\partial_\nu((1-|x|^2)^{2-s}u(x)) = 0 \quad \text{for } z \in \partial B.$$



The unique solution is

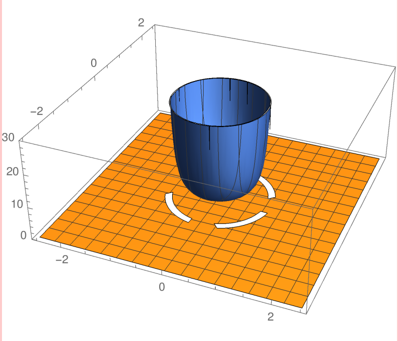
$$u(x) = \int_{\mathbb{R}^N\setminus B} \Gamma_s(x,y)v(y)\,dy \qquad \text{(note that } u < 0 \text{ if } v > 0\text{)},$$

where (Abatangelo, Jarohs, S. 2017)

$$\Gamma_s(x,y) = (-1)\gamma_{N,s}\frac{(1-|x|^2)_+^s}{(|y|^2-1)^s|x-y|^N} \qquad \text{for } x \in \mathbb{R}^N,\ y \in \mathbb{R}^N\setminus\overline{B}.$$

The boundary Poisson kernels, $s \in (1,2)$

$$
\begin{aligned}
(-\Delta)^s u &= 0 \quad \textit{in } B,\\
u &= 0 \quad \textit{on } \mathbb{R}^N \setminus B,\\
\lim_{x \to z} (1-|x|^2)^{2-s} u(x) &= h(z) \quad \textit{for } z \in \partial B,\\
\lim_{x \to z} -\partial_\nu ((1-|x|^2)^{2-s} u(x)) &= 0 \quad \textit{for } z \in \partial B.
\end{aligned}
$$



The unique solution is

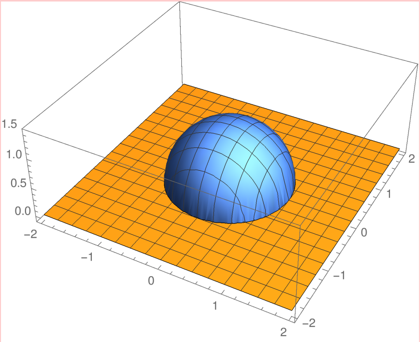
$$
u(x) = \int_{\partial B} E_{0,s}(x,z) h(z)\, dz \qquad \textit{(note that } u > 0 \textit{ if } h > 0 \textit{ and } N \leq 4\textit{)},
$$

where (Abatangelo, Jarohs, S. 2017)

$$
E_{0,s}(x,z) = \frac{1}{4\omega_N} \frac{(1-|x|^2)^s}{|x-z|^{N+2}} (N(1-|x|^2) + (4-N)|x-z|^2) \qquad \textit{for } x \in \mathbb{R}^N,\ y \in \partial B.
$$

The boundary Poisson kernels, $s \in (1,2)$

$$\begin{aligned}(-\Delta)^s u &= 0 \quad \textit{in } B,\\ u &= 0 \quad \textit{on } \mathbb{R}^N \setminus B,\\ \lim_{x\to z}(1-|x|^2)^{2-s}u(x) &= 0 \quad \textit{for } z\in\partial B,\\ \lim_{x\to z} -\partial_\nu((1-|x|^2)^{2-s}u(x)) &= g(z) \quad \textit{for } z\in\partial B.\end{aligned}$$



The unique solution is

$$u(x) = \int_{\partial B} E_{1,s}(x,z)g(z)\,dz$$

(note that $u > 0$ if $g > 0$),

where (Abatangelo, Jarohs, S. 2017)

$$E_{1,s}(x,z) = \frac{1}{\omega_N}\frac{(1-|x|^2)^s}{|x-z|^N}$$

for $x \in \mathbb{R}^N$, $y \in \partial B$.

Some previous results

$s \in \mathbb{N}$:

Green function: Green (1828), Boggio (1905).

Poisson kernels: Poisson (1820), Lauricella-Volterra (1896), Edenhofer (1974).

$s \in (0, 1)$:

Green function: Blumental-Getoor-Ray (1961).

Nonlocal Poisson kernel: Riesz (1937).

Boundary Poisson kernel: Bogdan (1999).

Applications: Representation formulas, $s \in (1,2)$

Theorem (Abatangelo, Jarohs, S. 2017)

Let $r > 1$ and $u \in \mathscr{L}^1_s \cap C^{2s+\alpha}(B)$ ($2s+\alpha \notin \mathbb{N}$) be such that

$$(1-|x|^2)^{2-s}u \in C^{1+\alpha}(\overline{B}), \qquad (-\Delta)^s u \in C^{\alpha}(\overline{B}), \qquad \text{and} \qquad u = 0 \quad \text{in } B_r \backslash \overline{B}.$$

Then, for $x \in B$,

$$u(x) = \int_B G_s(x,y)(-\Delta)^s u(y)\, dy + \int_{\mathbb{R}^N \backslash \overline{B}} \Gamma_s(x,y) u(y)\, dy + \int_{\partial B} E_{0,s}(x,\theta) D^{s-2} u(\theta)\, d\theta + \int_{\partial B} E_{1,s}(x,\theta) D^{s-1} u(\theta)\, d\theta.$$

$$D^{s-2}u(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u(x), \qquad D^{s-1}u(z) := \lim_{x\to z} -\partial_\nu[(1-|x|^2)^{2-s}u(x)]$$

$$\mathscr{L}^1_s := \left\{ u \in L^1_{loc}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\, dx < \infty \right\}.$$

Applications: Characterization of s-harmonic functions, $s \in (1,2)$

A function is s-harmonic in B if $(-\Delta)^s u = 0$ in B.

Theorem (Abatangelo, Jarohs, S. 2017)

If $u \in C^{2s+\alpha}(B)$ is s-harmonic in B, $(1-|x|^2)^{2-s}u \in C^{m+\alpha}(\overline{B})$, and $u = 0$ in $\mathbb{R}^N \setminus \overline{B}$, then

$$u(x) = \int_{\partial B} \frac{(1-|x|^2)_+^{s-1}}{|x-\theta|^N}\, g_0(\theta) + \frac{(1-|x|^2)_+^{s}}{|x-\theta|^N}\, g_1(\theta)\, d\theta$$

for some uniquely determined functions $g_0, g_1 \in C(\partial B)$. In particular,

$$(-\Delta)^s u = 0 \quad \text{in } B \qquad \text{implies that} \qquad (-\Delta)^{s+t}((1-|x|^2)^t u) = 0 \quad \text{in } B.$$

for all $t > 1-s$.

Applications: Higher-order fractional Hopf Lemma, $s \in (1,2)$

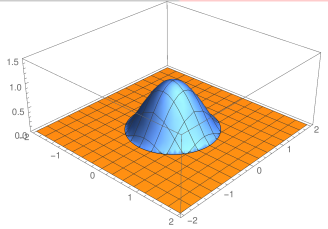
Theorem (Abatangelo, Jarohs, S. 2017)

Let $f \in C^{\alpha}(\overline{B})\setminus\{0\}$, $\alpha \in (0,1)$ *be nonnegative and let* $u \in \mathcal{H}_0^s(B)$ *be the unique weak solution of*

$$(-\Delta)^s u = f \gneq 0 \quad \text{in } B \qquad \text{with} \qquad u = 0 \quad \text{on } \mathbb{R}^N \setminus \overline{B}.$$

Then,

$$\lim_{x \to z} \frac{u(x)}{(1-|x|^2)^s} = m_s \int_B \frac{(1-|y|^2)^s}{|y-z|^N} f(y)\, dy > 0 \qquad \text{for all } z \in \partial B.$$

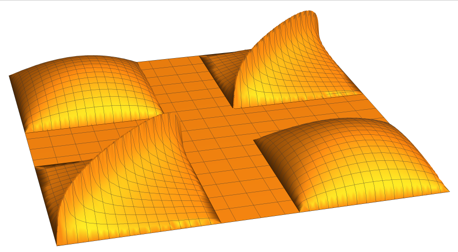


A sign-changing Green function for $s \in (1,2)$

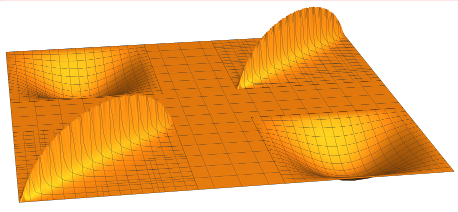
Theorem (Abatangelo, Jarohs, S. 2017)

Let Ω *be the union of two disjoint open balls. If* G^s_Ω *denotes the Green function of* $(-\Delta)^s$ *in* Ω*, then* G^s_Ω *changes sign.*

In 1D: $\Omega = (0,1) \cup (2,3)$, $G^s_\Omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,



for $s \in (0,1)$



for $s \in (1,2)$

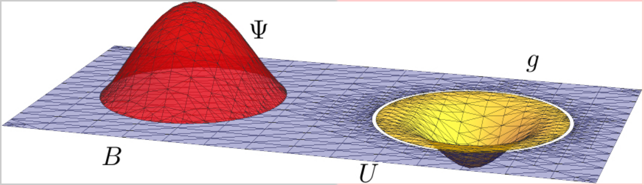
Sign-changing solutions for $s \in (1,2)$

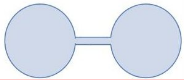
Theorem (Abatangelo, Jarohs, S. 2016)

Let Ω *be the union of two disjoint balls. There is a positive* $f \in C^\infty(\Omega)\setminus\{0\}$ *such that the unique solution of*

$$(-\Delta)^s u = f \gneq 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}\setminus\Omega, \qquad D^{s-2}u = D^{s-1}u = 0 \quad \text{on } \partial\Omega$$

changes sign in Ω.





Note: If $N \geq 2$, *the two balls can be joined by a thin tube.*

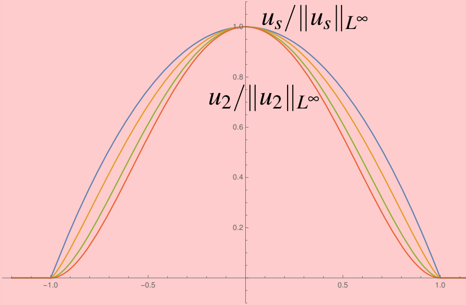
What if $s \nearrow 2$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x),$$

$$D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu\left[(1-|x|^2)^{2-s}u_s(x)\right]$$

$$\begin{cases} (-\Delta)^s u_s = 1 & \text{in } B,\\ u_s = 0 & \text{on } \mathbb{R}^N\setminus B,\\ D^{s-1}u_s(z) = 0 & \text{for } z\in\partial B,\\ D^{s-2}u_s(z) = 0 & \text{for } z\in\partial B.\end{cases}$$

$$\begin{cases} \Delta^2 u_2 = 1 & \text{in } B,\\ -\partial_\nu u_2 = 0 & \text{on } \partial B,\\ u_2 = 0 & \text{on } \partial B.\end{cases}$$



- the (normalized) u_s approximates the (normalized) u_2 "from above".

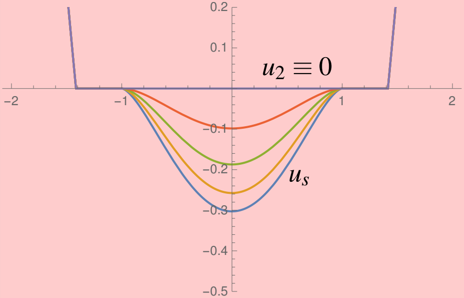
What if $s \nearrow 2$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x),$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = v & \text{on } \mathbb{R}^N\setminus B, \\ D^{s-1}u_s(z) = 0 & \text{for } z\in\partial B, \\ D^{s-2}u_s(z) = 0 & \text{for } z\in\partial B. \end{cases}$$

$$\begin{cases} \Delta^2 u_2 = 0 & \text{in } B, \\ -\partial_\nu u_2 = 0 & \text{on } \partial B, \\ u_2 = 0 & \text{on } \partial B. \end{cases}$$

$$D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu[(1-|x|^2)^{2-s}u_s(x)]$$



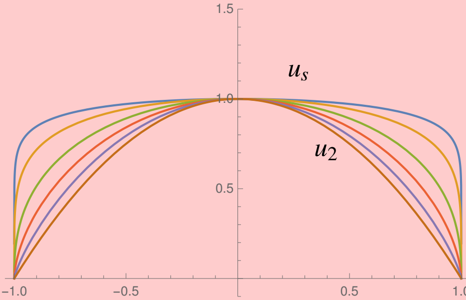
- u_s goes uniformly to 0 (note that $v = 0$ close to ∂B).

What if $s \nearrow 2$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x), \qquad D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu[(1-|x|^2)^{2-s}u_s(x)]$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = 0 & \text{on } \mathbb{R}^N \setminus B, \\ D^{s-1}u_s(z) = 1 & \text{for } z \in \partial B, \\ D^{s-2}u_s(z) = 0 & \text{for } z \in \partial B. \end{cases}$$

$$\begin{cases} \Delta^2 u_2 = 0 & \text{in } B, \\ -\partial_\nu u_2 = 1 & \text{on } \partial B, \\ u_2 = 0 & \text{on } \partial B. \end{cases}$$



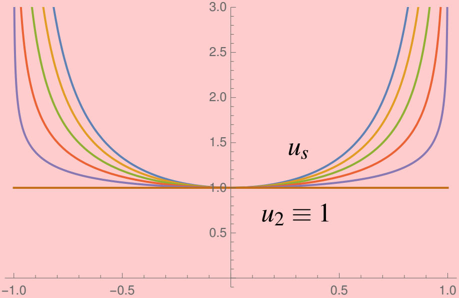
- u_s *goes uniformly to* u_2 *"from above"*..

What if $s \nearrow 2$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x), \qquad D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu[(1-|x|^2)^{2-s}u_s(x)]$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = 0 & \text{on } \mathbb{R}^N \setminus B, \\ D^{s-1}u_s(z) = 0 & \text{for } z \in \partial B, \\ D^{s-2}u_s(z) = 1 & \text{for } z \in \partial B. \end{cases}$$

$$\begin{cases} \Delta^2 u_2 = 0 & \text{in } B, \\ -\partial_\nu u_2 = 0 & \text{on } \partial B, \\ u_2 = 1 & \text{on } \partial B. \end{cases}$$



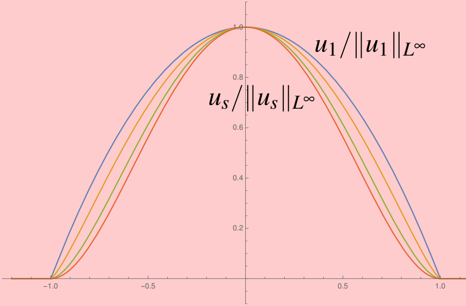
- u_s goes uniformly to 1 in compact sets of $(-1,1)$ "from above".

What if $s \searrow 1$ *?*

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x), \qquad D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu\left[(1-|x|^2)^{2-s}u_s(x)\right]$$

$$\begin{cases} (-\Delta)^s u_s = 1 & \text{in } B,\\ u_s = 0 & \text{on } \mathbb{R}^N\setminus B,\\ D^{s-1}u_s(z) = 0 & \text{for } z\in\partial B,\\ D^{s-2}u_s(z) = 0 & \text{for } z\in\partial B. \end{cases}$$

$$\begin{cases} -\Delta u_1 = 1 & \text{in } B,\\ u_1 = 0 & \text{on } \partial B. \end{cases}$$



$u_1/\|u_1\|_{L^\infty}$

$u_s/\|u_s\|_{L^\infty}$

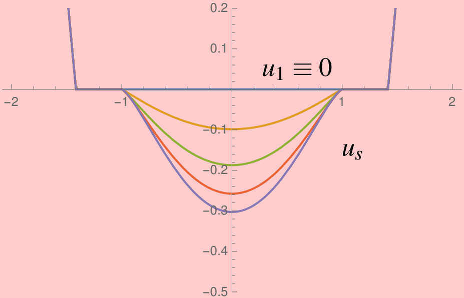
- *The (normalized)* u_s *approximates the (normalized)* u_1 *"from below".*

What if $s \searrow 1$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x), \qquad D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu\left[(1-|x|^2)^{2-s}u_s(x)\right]$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = v & \text{on } \mathbb{R}^N \setminus B, \\ D^{s-1}u_s(z) = 0 & \text{for } z \in \partial B, \\ D^{s-2}u_s(z) = 0 & \text{for } z \in \partial B. \end{cases}$$

$$\begin{cases} -\Delta u_1 = 0 & \text{in } B, \\ u_1 = 0 & \text{on } \partial B. \end{cases}$$



- u_s goes uniformly to 0.

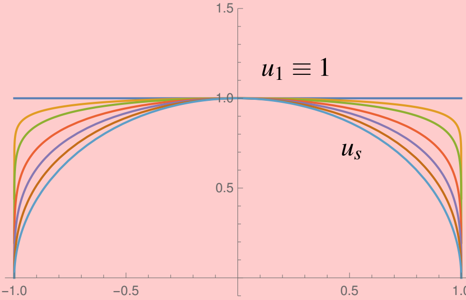
What if $s \searrow 1$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x),$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = 0 & \text{on } \mathbb{R}^N \setminus B, \\ D^{s-1}u_s(z) = 1 & \text{for } z \in \partial B, \\ D^{s-2}u_s(z) = 0 & \text{for } z \in \partial B. \end{cases}$$

$$\begin{cases} -\Delta u_1 = 0 & \text{in } B, \\ u_1 = 1 & \text{on } \partial B. \end{cases}$$

$$D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu[(1-|x|^2)^{2-s}u_s(x)]$$



- u_s approximates u_1 "from below".

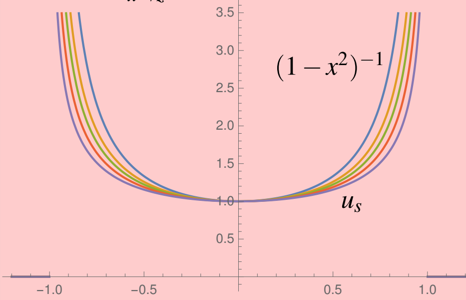
What if $s \searrow 1$?

$$D^{s-2}u_s(z) := \lim_{x\to z}(1-|x|^2)^{2-s}u_s(x), \qquad D^{s-1}u_s(z) := \lim_{x\to z} -\partial_\nu\left[(1-|x|^2)^{2-s}u_s(x)\right]$$

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } B, \\ u_s = 0 & \text{on } \mathbb{R}^N \setminus B, \\ D^{s-1}u_s(z) = 0 & \text{for } z \in \partial B, \\ D^{s-2}u_s(z) = 1 & \text{for } z \in \partial B. \end{cases}$$

$u_s(x) = (1-|x|^2)^{s-2} \to (1-|x|^2)^{-1}$.
But, for $N = 1$,

$$\partial_{xx}(1-x^2)^{-1} = \frac{1}{(x+1)^3} - \frac{1}{(x-1)^3} \neq 0.$$



u_s *converges to* $(1-x^2)^{-1}$, *which is*

- *not integrable,*
- *not harmonic.*

Some fractional questions (revisited):

- *If* u_s *is the solution of* $(-\Delta)^s u_s = f$ *+ b.c., is* $s \mapsto u_s$ *continuous in some sense?*
- *If* $(-\Delta)^2$ *requires two boundary conditions for well-posedness and* $-\Delta$ *only one, what happens with the extra boundary condition as* $s \searrow 1$*?*
- *Are solutions increasingly "bad" as* $s \nearrow 2$ *or is it essentially the same for any* $s > 1$*?*
- *Is the Green function for* $(-\Delta)^s$ *in the ellipse* $E = \{x^2 + 25y^2 < 1\}$ *sign-changing for all* $s > 1$*?*

Some extensions: General $s > 0$

Let $s \in (0,n)$, $n \in \mathbb{N}$, *then*

$$(-\Delta)^s u(x) := c_{N,s,n} \int_{\mathbb{R}^N} \frac{\sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k} u(x+ky)}{|y|^{N+2s}} \, dy$$

In the ball, the Green function G_s *and the nonlocal Poisson kernel* Γ_s *have the same formula, whereas*

$$E_{k,s}(x,\theta) = \frac{1}{\omega_N}(1-|x|^2)_+^s D^{m-k} \zeta_x(\theta), \qquad \text{where} \qquad \zeta_x(y) := \frac{|y|^{N-2}}{|x-y|^N}$$

and

$$D^{k+\sigma-1}u(z) := \frac{(-1)^k}{k!} \lim_{x\to z} \frac{\partial^k}{\partial(|x|^2)^k}[(1-|x|^2)^{1-\sigma}u(x)] \qquad \text{for } z \in \partial B. \qquad (1)$$

Some extensions: The halfspace

Let

$$\kappa x := 2\frac{x+e_1}{|x+e_1|^2} - e_1, \qquad K_s u(x) := |x+e_1|^{2s-N} u(\kappa x)$$

Then

$$(-\Delta)^s (K_s u)(x) = 2^{2s} \frac{K_s((-\Delta)^s u)(x)}{|x+e_1|^{4s}}, \quad \textit{i.e.,} \quad (-\Delta)^s \left(\frac{u \circ \kappa(x)}{|x+e_1|^{N-2s}} \right) = 2^{2s} \frac{(-\Delta)^s u(\kappa x)}{|x+e_1|^{N+2s}}.$$

Then $\kappa(B) = \mathbb{R}^N_+ := \{x_1 > 0\}$ *and*

$$u \text{ is } s\text{-harmonic in } B \text{ if and only if } K_s u \text{ is } s\text{-harmonic in } \mathbb{R}^N_+.$$

Some extensions: The halfspace

For $s = m + \sigma$:

$$D^{k+\sigma-1}u(z) := \frac{1}{k!}\lim_{x_1\to 0^+}\partial_1^k[x_1^{1-\sigma}u(x)], \quad k \in \{0,1,\dots,m\},$$

$$\Gamma_s(x,y) := (-1)^m\gamma_{N,s}\frac{(x_1)_+^s}{(-y_1)^s|x-y|^N},$$

$$G_s(x,y) := k_{N,s}|x-y|^{2s-N}\int_0^{\frac{4x_1^+y_1^+}{|x-y|^2}}\frac{v^{s-1}}{(v+1)^{\frac{N}{2}}}\,dv,$$

$$E_{k,s}(x,y) := \sum_{i=0}^{\lfloor\frac{m-k}{2}\rfloor}\alpha_{N,s,i}\frac{(x_1)_+^{s+m-k-2i}}{|y-x|^{N+2(m-k-i)}}$$

For $k \in \{0,\dots,2m+1\}$, $x_1^{k+\sigma-1}$ *is* s*-harmonic in* $\mathbb{R}^N_+$ *(note that* $x_1^{\sigma-1}$ *is singular).*

(joint work with N. Abatangelo, S. Dipierro, M.M. Fall, and S. Jarohs)

$$\left(\textit{Muito obrigado!} \right)^s$$

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- N. Abatangelo, S. Dipierro, M.M. Fall, S. Jarohs, A.S., Positive powers of the Laplacian in the half-space under Dirichlet boundary conditions, submitted, available on arXiv.