On the shape of solutions to the Extended Fisher-Kolmogorov equation

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Karlsruhe, December 1 2015

Introduction

Consider the Allen-Cahn equation

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}$$

- This is a model for phase-transition.
- Solutions to the 1-D problem are fully classified in terms of the energy $E = -\frac{1}{2}(u')^2 + \frac{1}{4}(1-u^2)^2$.

The Extended Fisher-Kolmogorov equation

Consider the (EFK) equation

$$\begin{aligned} \Delta^2 u - \beta \Delta u = u - u^3 & \qquad \text{in } \Omega, \\ u = \Delta u = 0 & \qquad \text{on } \partial\Omega. \end{aligned}$$

Solutions of (EFK) are critical points of the energy

$$J(u) = \int_\Omega \frac{|\Delta u|^2}{2} + \beta \frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2}\, dx$$

The Extended Fisher-Kolmogorov equation

Theorem (Existence)

Let $\beta > 0$ and $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a smooth bounded domain or a hyperrectangle. If $\lambda_1^2 + \beta\lambda_1 \geq 1$, then $u \equiv 0$ is the unique weak solution of (EFK). If

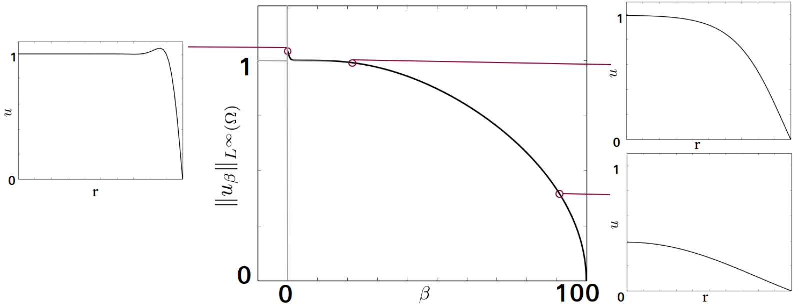
$$\lambda_1^2 + \beta\lambda_1 < 1\,, \tag{1}$$

then,

1. *there is $\varepsilon > 0$ such that (EFK) admits for each $\beta \in (\bar{\beta} - \varepsilon, \bar{\beta})$ a positive classical solution u, where $\bar{\beta} = \frac{1-\lambda_1^2}{\lambda_1}$.*
2. *For each $\beta \geq \frac{\sqrt{8}}{(\sqrt{27}-2)^{1/2}}$ there is a positive classical solution u of (EFK) such that $\|u\|_{L^\infty(\Omega)} \leq \frac{1}{\beta^2}(\frac{4+\beta^2}{3})^{\frac{3}{2}}$ and $\Delta u < \frac{\beta}{2}u$ in Ω.*
3. *For every $\beta \geq \sqrt{8}$ there exists a unique positive solution u of (EFK). Moreover, this solution is strictly stable and satisfies $\|u\|_{L^\infty(\Omega)} \leq 1$.*

Bifurcation branch

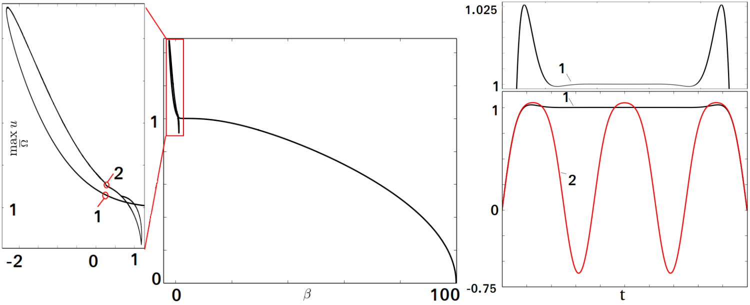
Radial solutions of (EFK) in $\Omega = B_{240.483}(0) \subset \mathbb{R}^2$.



- Solutions are bounded by 1 only for $\beta >> 0$
- Solutions are monotone only for $\beta >> 0$.
- For $\beta \approx 0$, the maximum is close to the boundary.

Bifurcation branch

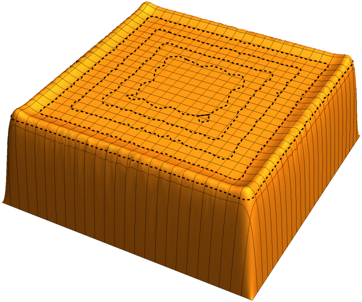
Continuation of the branch.



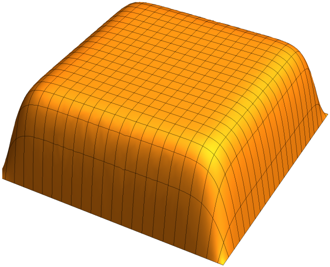
- Positivity is lost along the branch.
- The branch goes back to positive values of β.
- For $\beta \approx 0$, the solution oscillates around 1.

Other bifurcation branch for $\Omega = (0, 2\pi)$

Numerical approximation in squares

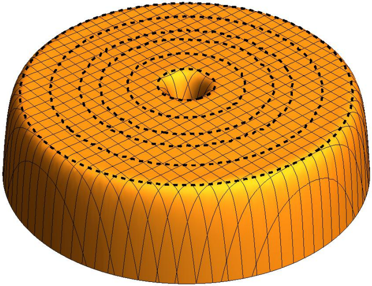


β small

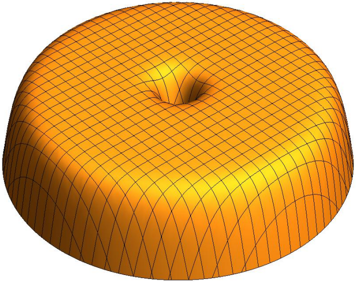


β large.

Numerical approximation in annuli



β small



β large.

The Extended Fisher-Kolmogorov equation

Theorem (Stability of positive solutions)

Let $\partial\Omega$ *be of class* $C^{1,1}$ *and* $\beta \geq \sqrt{8}$. *Then any positive solution of* (3) *is strictly stable.*

Theorem (Symmetry of stable solutions)

Let Ω *be a ball or an annulus and let* u *be a stable solution of* (3) *with* $\beta > \sqrt{12} - 2\lambda_1$ *such that* $\|u\|_{L^\infty(\Omega)} \leq 1$. *Then* u *is a radial function.*

Proposition (Symmetry of positive solutions)

Let $\beta \geq \sqrt{8}$ *and let* $\Omega \subset \mathbb{R}^N$ *be a hyperrectangle or a bounded smooth domain which is convex and symmetric in the* e_1-*direction. Then, any positive solution of* (3) *satisfies*

$$u(x_1, x_2, \ldots, x_N) = u(-x_1, x_2, \ldots, x_N) \qquad \text{for all } x = (x_1, \ldots, x_N) \in \Omega,$$

$$\partial_{x_1} u(x) < 0 \qquad \text{for all } x \in \Omega \text{ such that } x_1 > 0.$$

The Extended Fisher-Kolmogorov equation

GLOBAL MINIMIZERS

Theorem (Positivity and symmetry of some minimizers)

Let Ω be a ball or an annulus, (1) *hold, and let $u \in H$ be a global radial minimizer of* (**??**) *with $\|u\|_{L^\infty(\Omega)} \leq 1$. Then $\partial_r u$ does not change sign if Ω is a ball and $\partial_r u$ changes sign exactly once if Ω is an annulus.*

Corollary

Let Ω be a ball or an annulus, $\beta > \sqrt{12} - 2\lambda_1$, (1) *hold, and let u be a global minimizer of* (**??**) *in H. Then u is radial and does not change sign in Ω. Moreover, $\partial_r u$ does not change sign if Ω is a ball while $\partial_r u$ changes sign exactly once if Ω is an annulus.*

The Extended Fisher-Kolmogorov equation

$$\begin{aligned} \gamma\Delta^2 u - \Delta u &= u - u^3 && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{2}$$

when $\gamma \to 0$.

Theorem (Convergence to second order equation)

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with the first Dirichlet eigenvalue $\lambda_1 < 1$. Let $\gamma \geq 0$ and u_γ be a global minimizer in H of

$$\int_\Omega \left(\gamma \frac{|\Delta u|^2}{2} + \frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2} \right) dx. \tag{3}$$

There is an open neighborhood of 0 such that, for all $\gamma \in I \cap [0, 1]$: u_γ is the unique global minimizer in H and $u_\gamma > 0$ in Ω. Moreover, the function $I \to C^4(\Omega)$; $\gamma \mapsto u_\gamma$ is continuous and u_0 is the global minimizer of (3) in $H^1_0(\Omega)$ with $\gamma = 0$.

The Extended Fisher-Kolmogorov equation

Lemma (A priori bounds for classical solutions)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta > 0$, $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$, and let $u \in C^4(\Omega) \cap C_0(\bar{\Omega})$ be a solution of $\Delta^2 u - \beta \Delta u = f(u)$ in Ω such that $\Delta u \in C_0(\bar{\Omega})$. Set $\overline{u} := \max_{\overline{\Omega}} u$, $\underline{u} := \min_{\overline{\Omega}} u$, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(s) := \frac{4}{\beta^2} f(s) + s$. Then

$$\overline{u} \le \max_{[\underline{u},\overline{u}]} g \qquad \text{and} \qquad \underline{u} \ge \min_{[\underline{u},\overline{u}]} g. \tag{4}$$

Moreover,

1. *If $\overline{u} \le \max_{[0,\overline{u}]} g$ and $f < 0$ in $(1, \infty)$, then $\overline{u} \le \max_{[0,1]} g$.*
2. *If $\underline{u} \ge \min_{[\underline{u},0]} g$ and $f > 0$ in $(-\infty, -1)$, then $\underline{u} \ge \min_{[-1,0]} g$.*

The Extended Fisher-Kolmogorov equation

Lemma (Truncation Lemma)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta > 0, f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(s) := \begin{cases} -\frac{\beta^2}{4}s & \text{if } s < 0, \\ s - s^3 & \text{if } s \in [0, C_\beta], \\ C_\beta - C_\beta^3 = -\frac{\beta^2}{4}C_\beta & \text{if } s > C_\beta, \end{cases} \tag{5}$$

and let u be a classical solution of

$$\begin{aligned} \Delta^2 u - \beta\Delta u = f(u) \qquad & \text{in } \Omega, \\ u = \Delta u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{6}$$

Then $0 \leq u \leq M_\beta$. Moreover, if $\beta \geq \sqrt{8}$, then $u \leq 1$. In particular, if $\beta \geq K_0$, then u is a classical solution of (3).

The Extended Fisher-Kolmogorov equation

Theorem (Variational Argument)

Let $\beta > 0$ and $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a smooth bounded domain or a hyperrectangle. If $\lambda_1^2 + \beta\lambda_1 \geq 1$, then $u \equiv 0$ is the unique weak solution of (3). *If $\lambda_1^2 + \beta\lambda_1 < 1$, then for $\beta \geq \frac{\sqrt{8}}{(\sqrt{27}-2)^{1/2}}$ there is a positive classical solution u of* (3) *such that $\|u\|_{L^\infty(\Omega)} \leq \frac{1}{\beta^2}\left(\frac{4+\beta^2}{3}\right)^{\frac{3}{2}}$ and $\Delta u < \frac{\beta}{2}u$ in Ω. Additionally, if $\beta \geq \sqrt{8}$ then $\|u\|_{L^\infty(\Omega)} \leq 1$ and u is the unique positive solution of* (3).

The Extended Fisher-Kolmogorov equation

Lemma (A priori estimates for classical solutions)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta > 0$, and let u be a classical solution of (3).

i) *If u is nonnegative in Ω, then $\|u\|_{L^\infty(\Omega)} \leq M_\beta$.*

ii) *If $\beta \geq \sqrt{8}$ and u is nonnegative in Ω, then $\|u\|_{L^\infty(\Omega)} \leq 1$.*

iii) *If $\beta \geq \sqrt{8}$ and $\|u\|_{L^\infty(\Omega)} < \sqrt{\frac{\beta^2}{2} + 1}$, then $\|u\|_{L^\infty(\Omega)} \leq 1$.*

The Extended Fisher-Kolmogorov equation

Proposition (A priori estimates for minimizers)

Let v be a global minimizer of (3) *in H. Then*

$$\|v\|_{L^\infty(\Omega)} \leq \frac{\sqrt{4+\beta^2}}{2} \quad \text{if } \beta \geq \sqrt{\frac{8}{\sqrt{27}-2}} \quad \text{and}$$

$$\|v\|_{L^\infty(\Omega)} \leq 1 \qquad \text{if } \beta \geq \sqrt{8}.$$

The Extended Fisher-Kolmogorov equation

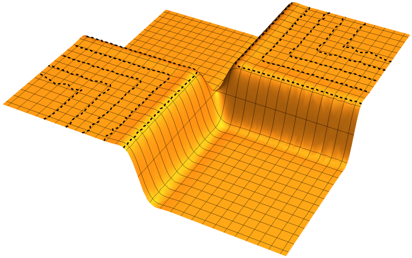
Theorem (Bifurcation argument)

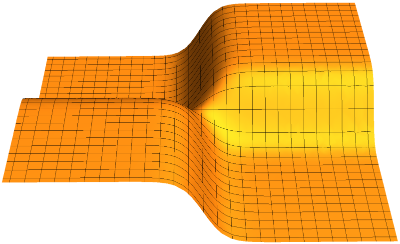
Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ *be a smooth bounded domain or a hyperrectangle. If the first Dirichlet eigenvalue* $\lambda_1 < 1$, *then there is* $\varepsilon > 0$ *such that* (3) *admits a positive solution* $u_\beta \in C^{4,\alpha}(\Omega)$ *for all* $\beta \in (\bar{\beta} - \varepsilon, \bar{\beta})$, *where*

$$\bar{\beta} = \frac{1 - \lambda_1^2}{\lambda_1}.$$

Additionally, if $\bar{\beta} \geq \sqrt{8}$ *and* Ω *is of class* C^1, *then* (3) *admits a unique positive solution* u_β *such that* $\|u_\beta\|_{L^\infty(\Omega)} \leq 1$ *for all* $\beta \in [\sqrt{8}, \bar{\beta})$.

A saddle-solution in $\mathbb{R}^2$ for the Extended Allen-Cahn





Theorem (Saddle-solution)

If $\beta \geq \sqrt{\frac{8}{\sqrt{27}-2}}$, there exists a saddle-solution in the plane, i.e. u s.t.

$$u(x,y)xy \geq 0.$$

Moreover $\|u\|_{L^\infty(\Omega)} \leq 1$ if $\beta > 2\sqrt{2}$.

A saddle-solution in $\mathbb{R}^2$ for the Extended Allen-Cahn

- we use the global minimizers on a sequence of squares and let (a quarter of) the squares go to $\mathbb{R}^+ \times \mathbb{R}^+$.
- we have a control from below using the minimality so that the sequence of solutions do not converge to 0.
- when $\beta \geq \sqrt{8}$, the profile is monotone (in each quarter) and the limit is ± 1 on the diagonals.
- for smaller value of β, the profile starts monotonically and the limit is ± 1 on the diagonals.