Foliated Schwarz symmetry in nonautonomous Lotka-Volterra parabolic systems

Alberto Saldaña De Fuentes

Séminaire $A^N{}_{EDP}$
Analyse non linéaire et EDP.

Joint work with Tobias Weth.

September 26, 2014.

Model Problem

Let $B \subset \mathbb{R}^N$, $N \geq 2$, be a ball or an annulus and consider the nonautonomous Lotka-Volterra system

$$\begin{aligned} (u_1)_t - \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 \qquad \text{in } B \times (0,\infty), \\ (u_2)_t - \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 \qquad \text{in } B \times (0,\infty), \end{aligned}$$

where a_i, b_i, α_i $i = 1,2$ are regular bounded nonnegative functions. Assume

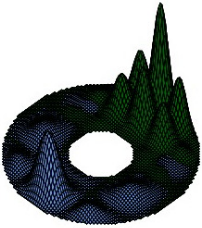
$\partial_\nu u_1 = \partial_\nu u_2 = 0$ on $\partial B \times (0,\infty)$ (Neumann boundary conditions),

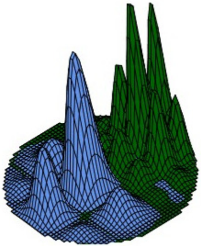
or $u_1 = u_2 = 0$ on $\partial B \times (0,\infty)$ (Dirichlet boundary conditions).

This system is used to model the **competition** between two different species, and takes into account birth, saturation, and aggressiveness (or capacity to carry food) rates, which may change according to the season.

Initial profile and asymptotic behavior

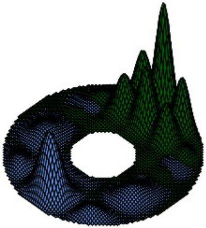
$$
\begin{aligned}
(u_1)_t - \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 && \text{in } B \times (0,\infty),\\
(u_2)_t - \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 && \text{in } B \times (0,\infty),\\
u_1 = u_2 &= 0 && \text{on } \partial B \times (0,\infty),\\
u_1(x,0) = u_{0,1}(x) \geq 0,\ & u_2(x,0) = u_{0,2}(x) \geq 0 && \text{for } x \in B.
\end{aligned}
$$

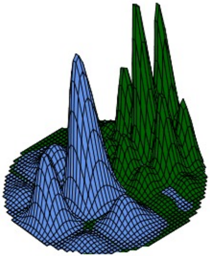




Initial profile and asymptotic behavior

$$
\begin{aligned}
(u_1)_t - \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 && \text{in } B\times(0,\infty),\\
(u_2)_t - \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 && \text{in } B\times(0,\infty),\\
u_1 = u_2 &= 0 && \text{on } \partial B\times(0,\infty),\\
u_1(x,0) = u_{0,1}(x) \geq 0,\ & u_2(x,0) = u_{0,2}(x) \geq 0 && \text{for } x\in B.
\end{aligned}
$$





Can we always expect a reduction of complexity?

Theorem (A.S., T. Weth, 2014)

Let $k \in \mathbb{N}$. *Then there are* $\varepsilon, \lambda > 0$ *satisfying the following. For* $B := \{x \in \mathbb{R}^2 \,:\, 1 - \varepsilon < |x| < 1\} \subset \mathbb{R}^2$, *the system*

$$\begin{aligned} -\Delta u_1 &= \lambda u_1 - u_1 u_2 && \text{in } B, \\ -\Delta u_2 &= \lambda u_2 - u_1 u_2 && \text{in } B, \\ \partial_\nu u_1 &= \partial_\nu u_2 = 0 && \text{on } \partial B, \end{aligned}$$

admits a positive classical solution (u_1, u_2) *such that the angular derivatives* $\frac{\partial u_1}{\partial \theta}$ *and* $\frac{\partial u_2}{\partial \theta}$ *change sign at least* k *times on every circle contained in* $\overline{B}$.

A theorem from P. Polàčik (2007)

Let $B \subset \mathbb{R}^N$ be a ball and let u be a positive bounded classical solution of

$$u_t - \Delta u = f(t,u) \quad \text{in } B \times (0,\infty),$$
$$u(x,0) = u_0(x) \quad \text{for } x \in B,$$

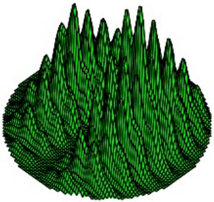
satisfying Dirichlet boundary conditions, where f satisfy some regularity assumptions. Then u is asymptotically radially symmetric and nonincreasing in the radial variable, that is, all the elements of

$$\omega(u) := \{z \in C(\overline{B}) \ : \ \lim_{n\to\infty} \|u(\cdot,t_n) - z\|_{L^\infty(B)} = 0 \text{ for some } t_n \to \infty\}$$

are radially symmetric and nonincreasing in the radial variable.

What about:

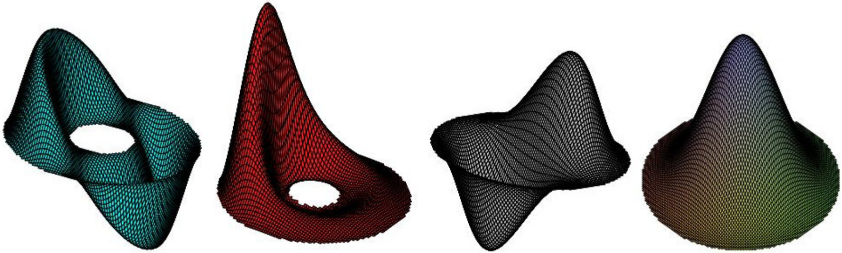
- Neumann boundary conditions.
- Annular domains.
- Sign changing solutions.
- Competitive systems.



We look for a particular partial symmetry.

Definition (Foliated Schwarz symmetry)

We say that a function $u \in C(B)$ is *foliated Schwarz symmetric with respect to some unit vector* $p \in \mathbb{S}^{N-1} := \{e \in \mathbb{R}^N : |e| = 1\}$, if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$.



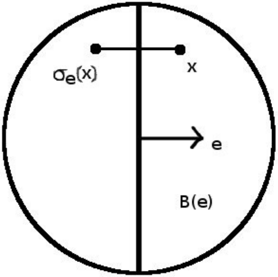
An assumption on the initial profile

For $e \in \mathbb{S}^{N-1}$ define

$B(e) := \{x \in B : x \cdot e > 0\}$,

$\sigma_e : \overline{B} \to \overline{B}$,

$x \mapsto \sigma_e(x) := x - 2(x \cdot e)e$.



$(U0)$ There is $e \in \mathbb{S}^{N-1}$ such that

$$u_{0,1}(x) \geq u_{0,1}(\sigma_e(x)) \quad \text{and} \quad u_{0,2}(x) \leq u_{0,2}(\sigma_e(x)) \quad \text{for } x \in B(e),$$

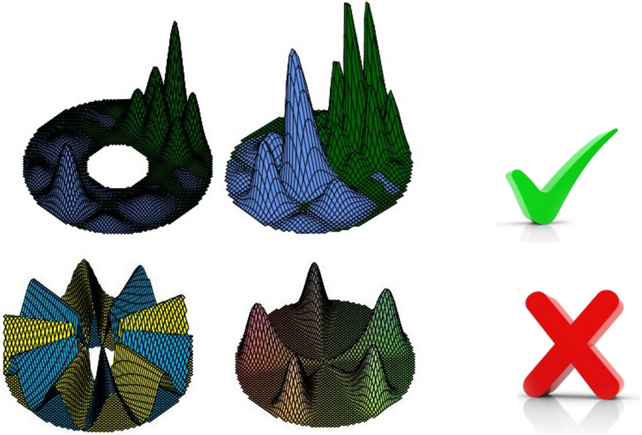
and $u_{0,1}, u_{0,2}$ are not invariant with respect to σ_e, that is $u_{0,1} \not\equiv u_{0,1} \circ \sigma_e$ and $u_{0,2} \not\equiv u_{0,2} \circ \sigma_e$ in B.

An assumption on the initial profile

(U0) There is $e \in \mathbb{S}^{N-1}$ such that

$$u_{0,1}(x) \geq u_{0,1}(\sigma_e(x)) \quad \text{and} \quad u_{0,2}(x) \leq u_{0,2}(\sigma_e(x)) \quad \text{for } x \in B(e),$$

and $u_{0,1}, u_{0,2}$ are not invariant with respect to σ_e.



Omega limit set

We study the asymptotic (in time) symmetries in terms of the **omega limit set**.

$$\omega(u_1, u_2) := \{(z_1, z_2) \in C(\overline{B}) \times C(\overline{B}) : \text{ there is } \ t_n \to \infty \ \text{ with } \\ \|u_1(\cdot, t_n) - z_1\|_{L^\infty(B)} + \|u_2(\cdot, t_n) - z_2\|_{L^\infty(B)} \to 0\}.$$

If (u_1, u_2) is a bounded classical solution of the Lotka-Volterra system (with Dirichlet or Neumann boundary conditions) then the set $\omega(u_1, u_2)$ is nonempty, connected and compact. This is proved using interior and boundary parabolic Hölder estimates in an standard way.

Theorem (A.S., T. Weth, 2014)

Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be bounded nonnegative functions such that (u_1, u_2) solves

$$(u_1)_t - \Delta u_1 = a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 \quad \text{in } B \times (0,\infty),$$
$$(u_2)_t - \Delta u_2 = a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 \quad \text{in } B \times (0,\infty),$$

satisfying Neumann or Dirichlet boundary conditions, where a_i, b_i, α_i are nonnegative uniformly bounded Hölder functions and $\inf_{t>0} \alpha_i(t) > 0$ for $i = 1, 2$. Further, assume that

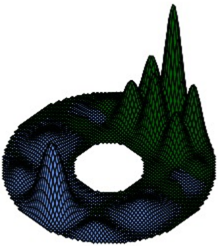
$(U0)$ *there is $e \in \mathbb{S}^{N-1}$ such that*

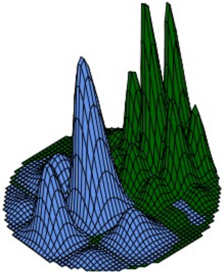
$u_{0,1}(x) \geq u_{0,1}(\sigma_e(x))$, $u_{0,2}(x) \leq u_{0,2}(\sigma_e(x))$ *for all $x \in B(e)$*

and $u_{0,1}$, $u_{0,2}$ are not invariant with respect to σ_e.

Theorem (Continuation...)

Then, there is $p \in \mathbb{S}^{N-1}$ such that every element $(z_1, z_2) \in \omega(u_1, u_2)$ has the property that z_1 is foliated Schwarz symmetric with respect to p and z_2 is foliated Schwarz symmetric with respect to $-p$.





Other competition problems

The previous result extends to these other problems.

- **More general nonlinearities:**

$$(u_1)_t - \mu(|x|,t)\Delta u_1 = f_1(t,|x|,u_1) - \alpha_1(|x|,t)u_1u_2,$$
$$(u_2)_t - \mu(|x|,t)\Delta u_2 = f_2(t,|x|,u_2) - \alpha_2(|x|,t)u_1u_2,$$

- **Systems with cubic coupling:**

$$(u_1)_t - \Delta u_1 = \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_1 u_2^2,$$
$$(u_2)_t - \Delta u_2 = \lambda_1 u_2 + \gamma_1 u_2^3 - \alpha_2 u_1^2 u_2,$$

Cooperative Systems

Theorem (A.S., T. Weth, 2014)

Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ *be bounded nonnegative functions such that* (u_1, u_2) *solves*

$$
\begin{aligned}
(u_1)_t - \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 + \alpha_1(t)u_1u_2 \quad \text{in } B \times (0,\infty),\\
(u_2)_t - \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 + \alpha_2(t)u_1u_2 \quad \text{in } B \times (0,\infty),
\end{aligned}
$$

satisfying Neumann or Dirichlet boundary conditions, where a_i, b_i, α_i *are nonnegative uniformly bounded Hölder functions and* $\inf_{t>0} \alpha_i(t) > 0$ *for* $i = 1, 2$. *Further, assume that*

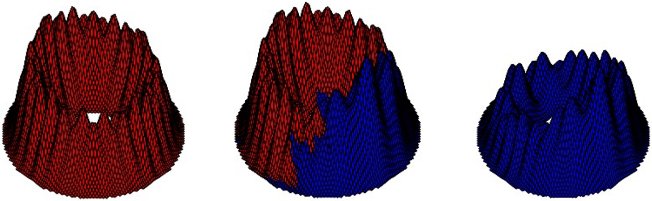
$(U0)'$ *there is* $e \in S$ *such that*

$$u_{0,1}(x) \geq u_{0,1}(\sigma_e(x)), \; u_{0,2}(x) \geq u_{0,2}(\sigma_e(x)) \text{ for all } x \in B(e)$$

and $u_{0,1}$, $u_{0,2}$ *are not invariant with respect to* σ_e.

Theorem (Continuation...)

Then, there is $p \in \mathbb{S}^{N-1}$ such that every element $(z_1, z_2) \in \omega(u_1, u_2)$ has the property that z_1 and z_2 are foliated Schwarz symmetric with respect to p.



This result extends to irreducible cooperative systems of n-equations where also sign changing solutions can be considered.

Proof via a "Parabolic rotating plane method":

The strategy for the proof consists of three main steps:

1. *Symmetry characterization.*
2. *Linearization and initialization of the method.*
3. *Perturbation and conclusion.*

We explain these steps by focusing on bounded classical solutions of the following semilinear problem.

$$\begin{aligned} u_t - \Delta u &= f(t, |x|, u) && \text{in } B \times (0,\infty),\\ \partial_\nu u &= 0 && \text{on } \partial B \times (0,\infty),\\ u(x,0) &= u_0(x) && \text{for } x \in B, \end{aligned}$$

where f is locally Lipschitz continuous in u uniformly with respect to t and $|x|$. We show that all elements of $\omega(u)$ are foliated Schwarz symmetric. This result is also new.

Symmetry characterization

Theorem (A.S., T. Weth, 2012)

Let $u \in C(B \times (0, \infty))$ *and define*

$$\mathcal{N} := \left\{ e \in \mathbb{S}^{N-1} \; : \; \begin{array}{c} \text{there is some } T > 0 \text{ such that} \\ u(x,t) \geq u(\sigma_e(x), t) \text{ for all } x \in B(e), \;\; t \geq T \end{array} \right\}$$

Suppose that

(i) $\mathcal{N}$ *is nonempty and relatively open in* $\mathbb{S}^{N-1}$;

(ii) *for every* $e \in \partial\mathcal{N}$ *we have that*

$$\limsup_{t \to \infty} \| u(\cdot, t) - u(\sigma_e(\cdot), t) \|_{L^\infty(B)} = 0,$$

that is, u *is asymptotically invariant with respect to* σ_e.

Then there is $p \in \mathbb{S}^{N-1}$ *such that every* $z \in \omega(u)$ *is foliated Schwarz symmetric with respect to* p.

Linearization...

For $e \in \mathbb{S}^{N-1}$ define $u^e : \overline{B} \times (0, \infty) \to \mathbb{R}$ by

$$u^e(x,t) := u(x,t) - u(\sigma_e(x), t).$$

Then

$$\begin{aligned} u^e_t - \Delta u^e &= c^e(x,t) u^e && \text{in } B(e) \times (0,\infty), \\ u^e &= 0 && \text{on } [\partial B(e) \backslash \partial B] \times (0,\infty), \\ \partial_\nu u^e &= 0 && \text{on } [\partial B(e) \cap \partial B] \times (0,\infty), \\ u^e(x,0) &= u_0(x) - u_0(\sigma_e(x)) && \text{for } x \in B(e), \end{aligned}$$

where $c^e \in L^\infty(B \times (0,\infty))$ is given by

$$c^e(x,t) := \frac{f(t,|x|,u(x,t)) - f(t,|x|,u(\sigma_e(x),t))}{u^e(x,t)}$$

...and initialization of the method

If $e \in \mathbb{S}^{N-1}$ is such that $u_0 \geq u_0 \circ \sigma_e$ in $B(e)$ and u_0 is not invariant under σ_e —analog of assumption (U0) —, then

$$\begin{aligned} u^e_t - \Delta u^e &= c^e(x,t)u^e && \text{in } B(e) \times (0,\infty), \\ u^e &= 0 && \text{on } [\partial B(e) \backslash \partial B] \times (0,\infty), \\ \partial_\nu u^e &= 0 && \text{on } [\partial B(e) \cap \partial B] \times (0,\infty), \\ u^e(x,0) &\geq 0, \not\equiv 0 && \text{for } x \in B(e), \end{aligned}$$

and therefore, by the maximum principle and the Hopf lemma,

$$u^e > 0 \quad \text{in } B(e) \times (0,\infty).$$

In particular, the set $\mathcal{N}$ is nonempty.

Perturbation & conclusion

Lemma

If $u^e > 0$ in $B(e) \times [T, \infty)$ for some $e \in \mathbb{S}^{N-1}$ and $T > 0$ then

$(P_{e,\varepsilon})$ *there is $\varepsilon > 0$ such that $u^{e'} > 0$ in $B(e') \times [T+1, \infty)$ for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$.*

In particular, $\mathcal{N}$ is relatively open.

Lemma

If $e \in \overline{\mathcal{N}}$ and $\limsup_{t\to\infty} \|u^e(\cdot, t)\|_{L^\infty(B(e))} > 0$ then $(P_{e,\varepsilon})$ holds for some $T > 0$. In particular, for every $e \in \partial\mathcal{N}$ we have that $\limsup_{t\to\infty} \|u^e(\cdot, t)\|_{L^\infty(B(e))} = 0$.

The proof makes use of different forms of maximum principles and a new quantitative Harnack-Hopf type principle.

Harnack-Hopf type principle

Theorem (A.S., T. Weth, 2014)

Let $B_+ := B(e_N)$, $I := (0,1)$, *and* $v \in C^{2,1}(\overline{B_+ \times I})$ *satisfy*

$$\begin{aligned} v_t - \Delta v - cv &\geq 0 && \text{in } B_+ \times I, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } [\partial B_+ \cap \partial B] \times I, \\ v &= 0 && \text{on } [\partial B_+ \setminus \partial B] \times I, \\ v(x,0) &\geq 0 && \text{for } x \in B_+, \end{aligned}$$

where $|c| \leq M$ *for some* $M > 0$. *Then* $v \geq 0$ *in* $B_+ \times I$. *Moreover, if* $v(\cdot, 0) \not\equiv 0$ *in* B_+, *then*

$$v > 0 \text{ in } B_+ \times I \qquad \text{and} \qquad \frac{\partial v}{\partial e_N} > 0 \text{ on } [\partial B_+ \setminus \partial B] \times I.$$

Theorem (Continuation...)

Furthermore, for every $\delta_1 > 0$, $\delta_2 \in (0, \frac{1}{4}]$, *there exist* $\kappa > 0$ *and* $p > 0$ *depending only on* δ_1, δ_2, B, *and* M *such that*

$$v(x,t) \geq x_N \, \kappa \left(\int_{Q(\delta_1,\delta_2)} v^p \; d(x,t) \right)^{\frac{1}{p}}$$

for all $x \in B_+$ *and* $t \in [3\delta_2 \, , \, 4\delta_2]$, *where*

$$Q(\delta_1, \delta_2) := \{(x,t) \; : \; x \in B_+, \; x_N \geq \delta_1, \; t \in [\delta_2, 2\delta_2]\}.$$

Competitive Neumann systems

Let $u_1^e, u_2^e : \overline{B} \times (0, \infty) \to \mathbb{R}$ be given by

$$\begin{aligned} u_1^e(x,t) &:= u_1(x,t) - u_1(\sigma_e(x),t), \\ u_2^e(x,t) &:= u_2(\sigma_e(x),t) - u_2(x,t). \end{aligned}$$

We thus obtain the system

$$\left.\begin{aligned} (u_1^e)_t - \Delta u_1^e - c_1^e u_1^e &= \alpha_1 u_1 u_2^e \\ (u_2^e)_t - \Delta u_2^e - c_2^e u_2^e &= \alpha_2 u_2 u_1^e \end{aligned}\right. \qquad \text{in } B(e) \times (0,\infty)$$

for some coefficients c_1^e, c_2^e, together with the boundary conditions

$$\frac{\partial u_i^e}{\partial \nu} = 0 \text{ on } [B(e) \cap \partial B] \times (0,\infty), \quad u_i^e = 0 \text{ on } [B(e) \setminus \partial B] \times (0,\infty),$$

i=1,2.

Let

$$\mathcal{N} := \left\{ e \in \mathbb{S}^{N-1} : \begin{array}{c} \text{there is some } T > 0 \text{ such that} \\ u_1^e \geq 0, \;\; u_2^e \geq 0 \text{ in } B(e) \times [T, \infty) \end{array} \right\}$$

A further complication appear with the possible occurrence of the so-called semi-trivial limit profiles, that is, elements of $\omega(u_1, u_2)$ of the form $(z, 0)$ or $(0, z)$. This difficulty was circumvented using a new normalization technique in the proof of a perturbation result.

The occurrence and qualitative properties of semi-trivial limit profiles are active research topics.

Competitive Dirichlet systems

The linearization, the statements of the perturbation Lemmas, and the definition of $\mathcal{N}$ are the **same** as in the Neumann case, **but** the Harnack- Hopf type Lemma **does not hold** anymore. Instead we use a parabolic version of Serrin's boundary corner point Lemma.

Theorem (A.S. 2014)

Let $B_+ := B(e_N)$, $I := [0,1]$, β_0, $k > 0$ *and* $v \in C^{2,1}(\overline{B_+} \times I)$ *be a nonegative function satisfying* $\|v\|_{L^\infty(B_+ \times (\frac{1}{7}, \frac{4}{7}))} \geq k$, $\|v\|_{C^{2,1}} + \|c\|_{L^\infty} \leq \beta_0$ *and* $v_t - \Delta v - cv \geq 0$ *in* $B_+ \times I$ *with Dirichlet boundary conditions.*

Theorem (Continuation...)

Then there are $\delta, \varepsilon > 0$ depending only on β_0, k, and B such that

$$\begin{aligned} v(x,1) &\geq x_N \varepsilon && \textit{for } x \in A_1, \\ \frac{\partial^2 v(x,1)}{\partial s^2} &> \varepsilon && \textit{for } x \in A_2 \textit{ (corner points)}, \\ \frac{\partial v}{\partial \nu}(x,1) &> \varepsilon && \textit{for } x \in A_3. \end{aligned}$$

where $s = \nu + e_N$, ν is the inwards unit normal vector field on ∂B, and

$$\begin{aligned} A_1 &:= \{x \in \overline{B_+} \ : \ \mathrm{dist}(x, \partial B) \geq \delta\}, \\ A_2 &:= \{x \in \overline{B_+} \ : \ \mathrm{dist}(x, H(e_N) \cap \partial B) \leq \delta\}, \\ A_3 &:= \overline{B_+} \backslash (A_1 \cup A_2), \\ H(e_N) &:= \{x \in \mathbb{R}^N \ : \ x \cdot e_N = 0\}. \end{aligned}$$

Some open questions:

- What can be said for competitive systems of three or more equations?
- Under which conditions can these results be extended to equations in unbounded domains?
- Is there an analogous result for predator-prey models?

Further symmetry characterization: Radial symmetry of semi-trivial limit profiles

Theorem (A.S. 2014)

Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ *be nonnegative bounded functions such that* $u = (u_1, u_2)$ *is a classical solution of*

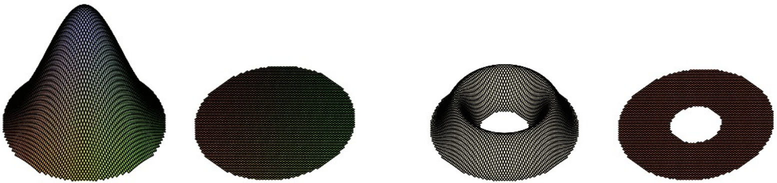
$$\begin{aligned} (u_1)_t - \Delta u_1 &= a_1 u_1 - u_1^2 - \alpha_1(x,t) u_1 u_2 && \text{in } B \times (0,\infty),\\ (u_2)_t - \Delta u_2 &= a_2 u_2 - u_2^2 + \alpha_2(x,t) u_1 u_2 && \text{in } B \times (0,\infty),\\ u_i &= 0 && \text{on } \partial B \times (0,\infty),\\ u_i(x,0) &= u_{0,i}(x) && \text{for all } x \in B,\ i = 1,2, \end{aligned}$$

where a_1, a_2 *are larger than the first Dirichlet eigenvalue of* B, $u_{0,1}, u_{0,2} \in C_0(B)$ *are not identically zero, and* α_1, α_2 *are bounded functions.*

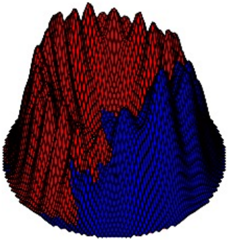
Further symmetry characterization: Radial symmetry of semi-trivial limit profiles

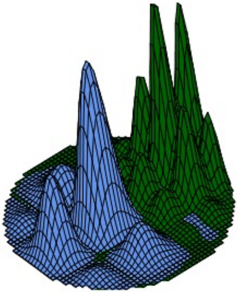
Theorem (Continuation...)

If $(z, 0) \in \omega(u)$ *then* z *is a positive radially symmetric function and it is the unique solution of the elliptic problem* $-\Delta z = a_1 z - z^2$ *in* B, *with Dirichlet boundary conditions. The analogous conclusion holds if* $(0, z) \in \omega(u)$.



Thank you!





Fully nonlinear scalar equations

Theorem

Let u be a bounded classical solution of

$$\begin{aligned} u_t(x,t) &= F(t,x,u,\nabla u, D^2 u) \quad &&\text{in } B\times(0,\infty), \\ u(x,0) &= u_0(x) &&\text{for } x\in B, \end{aligned}$$

satisfying Dirichlet boundary conditions, where F is assumed to satisfy some reflection invariance, regularity, and ellipticity assumptions; and there is $e\in\mathbb{S}^{N-1}$ such that $u_0 \geq u_0\circ\sigma_e$ in $B(e)$ and u_0 is not invariant with respect to σ_e. Then, there is $p\in\mathbb{S}^{N-1}$ such that all elements of $\omega(u)$ are foliated Schwarz symmetric with respect to p.
Here

$$\omega(u) := \{z\in C(\overline{B}) : \|u(\cdot,t_n)-z\|_{L^\infty(B)} \to 0 \text{ for some } t_n\to\infty\}$$

Theorem

Let $J := \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}$ *and let* $u = (u_1, \ldots, u_n)$ *with* $u_i \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ *be a bounded solution of*

$$\begin{aligned} (u_i)_t &= \Delta u_i + F_i(t,|x|,u) && \text{in } B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) && \text{for all } x \in B,\ i \in J, \end{aligned}$$

satisfying Neumann boundary conditions, where the following holds.

(A1) *For each* $i \in J$ *the function* $F_i : [0,\infty) \times I_B \times \mathbb{R}^n \to \mathbb{R}$; $(t,r,v) \mapsto F_i(t,r,v)$ *is locally Lipschitz in* v *uniformly with respect to* r *and* t*. Moreover,* $\max_{i \in J} \sup_{r \in I_B, t > 0} |F_i(t,r,0)| < \infty$.

(A2) *For every* $i, j \in J$, $i \neq j$, *one has that* $\partial F_i(t,r,u)/\partial u_j \geq 0$ *for all* $t \in [0,\infty)$, $r \in I_B$, *and* $u \in \mathbb{R}^n$ *such that the derivative exists.*

Theorem

Continuation...

(A3) *For each* M *there is a constant* $\sigma = \sigma(M) > 0$ *such that the following holds: for every nonempty subsets* $I_1, I_2 \subset J$ *with* $I_1 \cap I_2 = \emptyset$ *and* $I_1 \bigcup I_2 = J$, *there are* $i \in I_1$ *and* $j \in I_2$ *such that* $\partial F_i(t, r, u)/\partial u_j \geq \sigma$ *for all* $r \in I_B$, $t \in [0, \infty)$, *and* $u \in \mathbb{R}^n$ *with* $|u| \leq M$, *such that the derivative exists.*

(A4) *There is* $e \in \mathbb{S}^{N-1}$ *such that* $u_{0,i} \geq u_{0,i} \circ \sigma_e$ *and* $u_{0,i}$ *is not invariant with respect to* σ_e *for* $i \in J$.

Then there is some $p \in \mathbb{S}^{N-1}$ *such that all elements of* $\bigcup_{i=1}^{n} \omega(u_i)$ *are foliated Schwarz symmetric with respect to* p.

Harnack-Hopf type principle

Let $a, b \in \mathbb{R}$, $a < b$, $I := (a, b)$, $B_+ := \{x \in \overline{B} \ : \ x_N > 0\}$.
Suppose that $v \in C^{2,1}(\overline{B_+} \times I) \cap C(\overline{B_+ \times I})$ satisfies

$$\begin{aligned} v_t - \mu\Delta v - cv &\geq 0 && \text{in } B_+^\circ \times I, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \Sigma_2 \times I, \\ v &= 0 && \text{on } \Sigma_1 \times I, \\ v(x, a) &\geq 0 && \text{for } x \in B_+, \end{aligned}$$

where

$$\frac{1}{M} \leq \mu(x, t) \leq M \quad \text{and} \quad |c(x, t)| \leq M \qquad \text{for } (x, t) \in B_+ \times I$$

with some positive constant $M > 0$. Then $v \geq 0$ in $B_+ \times (a, b)$.
Moreover, if $v(\cdot, a) \not\equiv 0$ in B_+, then

$$v > 0 \text{ in } B_+ \times I \qquad \text{and} \qquad \frac{\partial v}{\partial e_N} > 0 \text{ on } \Sigma_1 \times I.$$

Furthermore, for every $\delta_1 > 0$, $\delta_2 \in (0, \frac{b-a}{4}]$, there exist $\kappa > 0$ and $p > 0$ depending only on δ_1, δ_2, B, and M such that

$$v(x,t) \geq x_N \, \kappa \left(\int_{Q(\delta_1,\delta_2)} v^p \; d(x,t) \right)^{\frac{1}{p}} \tag{1}$$

for all $x \in B_+$ and $t \in [a + 3\delta_2\,,\, a + 4\delta_2]$. where

$$Q(\delta_1,\delta_2) := \{(x,t) \;:\; x \in B_+,\; x_N \geq \delta_1,\; a + \delta_2 \leq t \leq a + 2\delta_2\}.$$