Derived Canonical Algebras as One-Point Extensions

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Abstract. Canonical algebras have been intensively studied, see for example [**12**], [**3**] and [**11**] among many others. We are interested in the question when a one-point extension of a finite-dimensional algebra Σ by a Σ-module M is derived canonical, i.e. derived equivalent to a canonical algebra. We give necessary conditions on the algebra Σ and the module M. If the canonical algebra associated with Σ is tame the conditions are even sufficient.
As a further result we obtain that, if Σ is derived canonical then the one-point extension of Σ by M is derived canonical again if and only if M is derived simple, i.e. M is indecomposable and belongs to the mouth of a tube in the bounded derived category of Σ-modules.

1. Prerequisites

We work over an algebraically closed base field k. By algebra we mean always a basic, finite dimensional k-algebra and by module we mean finitely generated right module. For an algebra Σ we denote by $\operatorname{mod}\Sigma$ the category of Σ-modules.

An algebra Σ which is derived equivalent to a canonical algebra will be called *derived canonical*. This terminology replaces the less suggestive term quasi-canonical used in [**10**]. For each derived canonical algebra there exists a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ such that the (bounded) derived category $\mathrm{D}^b(\Sigma)$ of $\operatorname{mod}\Sigma$ is equivalent to the (bounded) derived category of $\operatorname{coh}\mathbb{X}$, the category of coherent sheaves over $\mathbb{X}$, see [**3**]. Two derived canonical algebras are derived equivalent if and only if their associated weighted projective lines are isomorphic, see [**4**] and [**10**] for further details. Quite important information on $\mathbb{X}$, the weight type $\underline{p} = (p_1, \dots, p_t)$, can already be read from the Coxeter polynomial of Σ which has the shape

$$(T-1)^2 \cdot \prod_{i=1}^{t} \frac{T^{p_i}-1}{T-1}.$$

Note that, if $t \leq 3$, then $\mathbb{X}$ is completely determined by its weight type. The *genus* of $\mathbb{X}$, hence of Σ, defined by

$$\mathrm{g}_{\mathbb{X}} = \mathrm{g}_{\Sigma} := 1 + \frac{1}{2}\left((t-2)p - \frac{p}{p_1} - \cdots - \frac{p}{p_t}\right), \text{ where } p = \text{l.c.m.}(p_1, \dots, p_t),$$

1991 *Mathematics Subject Classification.* 16G20,16E30.

The first author gratefully acknowledges support from University of Paderborn and PADEP, UNAM.

determines the representation type of $\operatorname{coh}\mathbb{X}$ and of the canonical algebra Λ associated with $\mathbb{X}$, see [**3**]: for $g_{\mathbb{X}} \leq 1$ both $\operatorname{coh}\mathbb{X}$ and Λ are tame, whereas $\operatorname{coh}\mathbb{X}$ and Λ are wild if $g_{\mathbb{X}} > 1$. If $g_{\mathbb{X}} = 1$ the algebra Λ is *tubular*, see [**12, 10**]. This happens if and only if the weight type is — up to permutation — one of $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. Note, in this context, that passing from the canonical algebra Λ to a derived equivalent algebra Σ may simplify the representation type.

For a hereditary category $\mathcal{C}$ like $\operatorname{coh}\mathbb{X}$ or $\operatorname{mod} H$, H hereditary, the derived category $\mathrm{D}^b(\mathcal{C}) = \bigvee_{n\in\mathbb{Z}} \mathcal{C}[n]$, the additive closure of the union of all $\mathcal{C}[n]$'s, is known as well as $\mathcal{C}$. Here, each $\mathcal{C}[n]$ is a copy of $\mathcal{C}$ with objects written $X[n]$, $X \in \mathcal{C}$, and morphisms are given by

$$\mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}(X[m], Y[n]) = \mathrm{Ext}^{n-m}_{\mathcal{C}}(X, Y).$$

Note that we have natural identifications for the Grothendieck-groups

$$\mathrm{K}_0(\mathbb{X}) = \mathrm{K}_0(\Sigma) = \mathrm{K}_0(\mathrm{D}^b(\Sigma)),$$

and that the identifications preserve the *Euler forms* given on classes of Σ-modules (coherent sheaves, objects from the derived category, respectively) by the formula

$$\langle [X], [Y]\rangle = \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \mathrm{Hom}(X, Y[i]).$$

Let A be an algebra of finite global dimension. We recall from [**5**] that the bounded derived category $\mathrm{D}^b(A)$ of finite dimensional A-modules has Auslander-Reiten triangles. We say that an indecomposable A-module M is *derived peripheral* if the "middle term" E of the Auslander-Reiten triangle $\tau M \to E \to M \to \tau M[1]$ is indecomposable. An indecomposable A-module M is further called *derived simple* or also *derived simple regular*, if M is derived peripheral and τ-periodic for the Auslander-Reiten translation τ of $\mathrm{D}^b(A)$. Moreover, still assuming that M is indecomposable, we call M *derived quasi-simple* if M is derived peripheral and lies in a component of the form $\mathbb{Z}\mathbb{A}_\infty$ in $\mathrm{D}^b(A)$.

Assuming that Σ is derived canonical, it follows from [**3, 11**] and [**10**] that M is derived simple if and only if there exists a self-equivalence φ of $\mathrm{D}^b(\Sigma) = \mathrm{D}^b(\operatorname{coh}\mathbb{X})$ and a simple sheaf S on $\mathbb{X}$ such that $M = \varphi(S)$. For $g_\Sigma \neq 1$ the situation simplifies, and we may choose φ to be a translation functor $X \mapsto X[n]$.

Assuming that the algebra Σ is derived equivalent to a wild hereditary algebra H, a Σ-module M is derived quasi-simple if and only if there exists an integer n such that $M[n]$ is a regular H-module of quasi-length one.

For a representation-finite connected hereditary algebra H we fix an identification of the Auslander-Reiten quiver of $\mathrm{D}^b(H)$ with the translation quiver $\mathbb{Z}\Delta$ of the Dynkin diagram Δ attached to H. Relative to such an identification we define the *derived type* of an indecomposable Σ-module, with Σ derived equivalent to H, as the vertex v of Δ such that M belongs to the τ-orbit of v in $\mathbb{Z}\Delta$.

2. Derived canonical one-point extensions

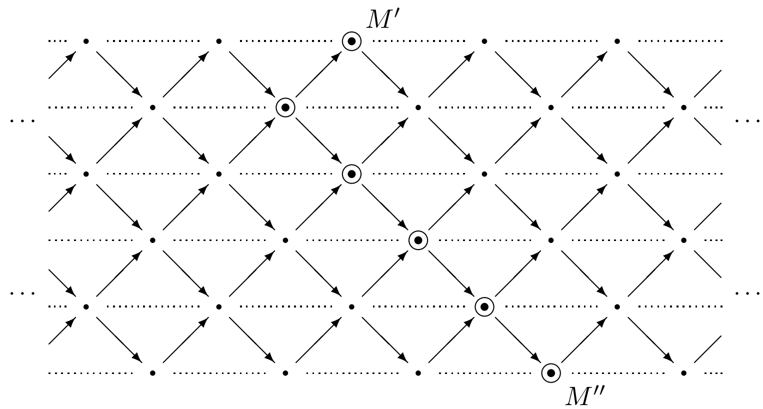
Given an algebra Σ and a Σ-module M, we will consider the *one-point sink extension* or just *sink extension* $[M]\Sigma$ which is given as

$$[M]\Sigma = \begin{bmatrix} \Sigma & 0 \\ M & k \end{bmatrix}$$

with the corresponding matrix operations. Dually we define the *one-point source extension* or just *source extension* $\Sigma[M]$ (they are also called *one-point coextension* and *one-point extension* in the literature).

We are now going to investigate when a one-point extension is derived canonical. Since an algebra is derived canonical if and only if its opposite algebra is it suffices to investigate under which conditions a sink extension $[M]\Sigma$ is derived canonical.

We need to introduce some notation for the special case where Σ is derived equivalent to $k[\mathbb{A}_n]$, i.e. Σ is a branch with n points, see [**8, 1**]. In this case, the indecomposable objects in the derived category $\mathrm{D}^b(\Sigma)$ form a single Auslander-Reiten component of type $\mathbb{Z}\mathbb{A}_n$. A slice S in $\mathrm{D}^b(\Sigma)$ is called a *(p,q)-slice* if in the quiver of S, there are p arrows pointing upwards and the remaining $n-p-1=q$ arrows point downwards.



In the above figure, we have marked a $(1,4)$-slice, and the two peripheral objects belonging to that slice.

Back in the general situation, where we consider the sink extension $[M]\Sigma$, we denote by $\overline{M}$ the indecomposable projective module corresponding to the sink vertex of $\overline{\Sigma}=[M]\Sigma$, thus the radical of $\overline{M}$ is just M. We view $\operatorname{mod}\Sigma$ as the full exact subcategory of $\operatorname{mod}\overline{\Sigma}$ consisting of all $\overline{\Sigma}$-modules X with $\operatorname{Hom}_{\overline{\Sigma}}(\overline{M},X)=0$. The inclusion $\operatorname{mod}\Sigma\subset\operatorname{mod}\overline{\Sigma}$ induces an inclusion $\mathrm{D}^b(\Sigma)\subset\mathrm{D}^b(\overline{\Sigma})$. In particular, identifying modules with stalk complexes in the corresponding derived category, $\operatorname{mod}\Sigma$ and $\operatorname{mod}\overline{\Sigma}$ become full subcategories of $\mathrm{D}^b(\overline{\Sigma})$.

THEOREM 1. *Let Σ be an algebra and M a Σ-module. Assume that $\overline{\Sigma}=[M]\Sigma$ is derived canonical. Then M is the middle term of the Auslander-Reiten triangle $\tau\overline{M}\to M\to\overline{M}\to\tau\overline{M}[1]$ in $\mathrm{D}^b(\overline{\Sigma})$ associated with the projective module $\overline{M}$ attached to the sink vertex of $\overline{\Sigma}$.*

If further $\Sigma=\Sigma_1\times\cdots\times\Sigma_s$ is the decomposition of Σ into connected algebras, and $M=M_1\times\cdots\times M_s$ is the corresponding decomposition of M into Σ_i-modules M_i, then $s\le 4$ and exactly one of the following cases happens:

1. a. *Σ is derived equivalent to $k[\mathbb{A}_\ell]$ and $M=M'\oplus M''$ is the direct sum of two indecomposable modules M' and M'' forming the periphery of a $(p-1,q-1)$-slice, $\ell=p+q-1$, of the component $\mathbb{Z}\mathbb{A}_\ell$ of $\mathrm{D}^b(\Sigma)$. Conversely, for each such choice for Σ and M the sink extension $[M]\Sigma$ is derived canonical of weight type (p,q).*

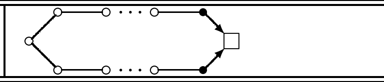
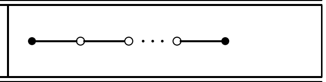
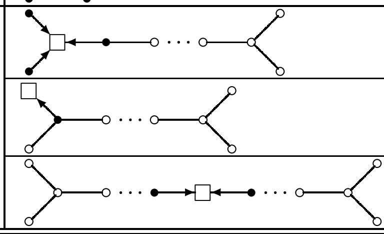
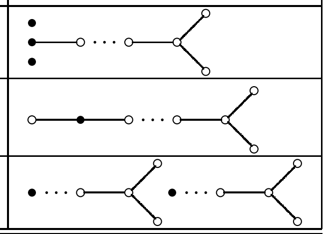
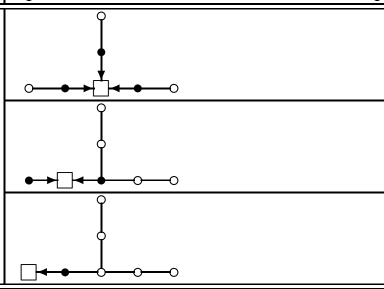
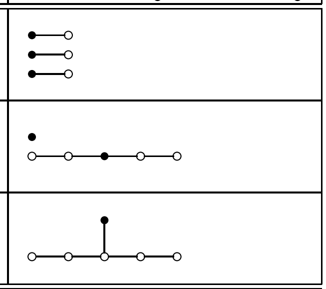
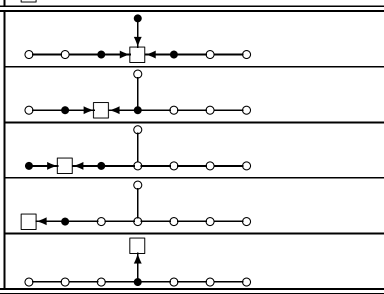
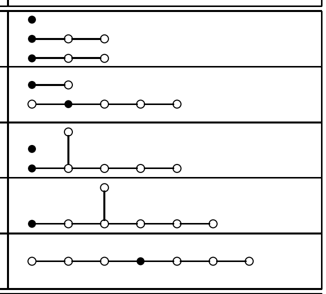
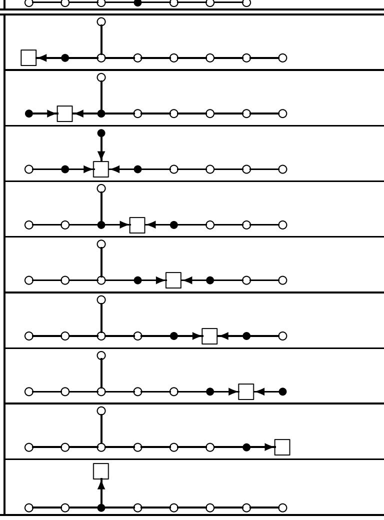
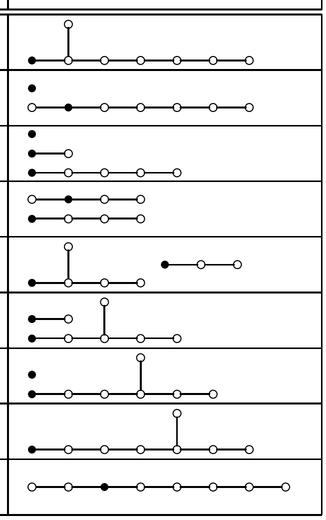
weight type($[M]\Sigma$)	hereditary type of $[M]\Sigma$	derived type of (Σ, M)
(p, q)		
$(2, 2, 2)$		
$(2, 2, n)$		
$(2, 3, 3)$		
$(2, 3, 4)$		
$(2, 3, 5)$		

Table 1: Choices for (Σ_i, M_i)

b. *Each Σ_i is derived equivalent to a representation-finite connected hereditary algebra H_i, and each M_i is an indecomposable Σ_i-module, where the Dynkin types for the Σ_i's and the derived types for the M_i's are listed in the table below.*
Conversely, for each choice of (Σ, M) conforming to the table, the sink extension $[M]\Sigma$ is derived canonical with the type given by the first column of the table.

2. *$\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_1 \neq 0$ is derived canonical and Σ_2 is derived equivalent to a hereditary algebra of type $\mathbb{A}_\ell$, $0 \leq \ell$. Moreover, M_1 is derived simple over Σ_1, and — if Σ_2 is non-zero — the Σ_2-module M_2 is derived peripheral.*
Conversely, for each such choice of (Σ, M), the sink extension $[M]\Sigma$ is derived canonical of weight type $(p_1, \dots, p_{t-1}, p_t + \ell)$, if Σ_1 has weight type $(p_1, \dots, p_t)$ and M_1 has τ-period p_t in the derived category.
3. *$\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_1 \neq 0$ is derived equivalent to a connected wild hereditary algebra H, and Σ_2 is derived equivalent to a hereditary algebra of type $\mathbb{A}_\ell$, $0 \leq \ell \leq 5$. Moreover, the Σ_1-module M_1 is derived quasi-simple, and — if Σ_2 is non-zero — the Σ_2-module M_2 is derived peripheral.*

Concerning statement 3 it is an interesting open question when the sink extension $[M]\Sigma$ of a wild hereditary algebra Σ by a regular quasi-simple module M is derived canonical. The problem is related to the question when sink and source extension algebras are quasi-tilted [**6**].

Proof. The first assertion is proved in [**2**]. Next we show that the conditions listed in 1, 2 and 3 are necessary, and exhaust all possible cases. Since $\overline{\Sigma}$ is derived canonical, there is a weighted projective line $\overline{\mathbb{X}}$ such that $\overline{\Sigma}$ can be realized as a *tilting complex* in the derived category $\mathrm{D}^b(\overline{\mathcal{C}})$, where $\overline{\mathcal{C}} = \operatorname{coh} \overline{\mathbb{X}}$, i.e. $\overline{\Sigma}$ is a full subcategory of $\mathrm{D}^b(\overline{\mathcal{C}})$ consisting of indecomposable objects $\overline{\Sigma}_j$, $j = 1, \dots, n+1$, which satisfy the condition

$$\operatorname{Hom}_{\mathrm{D}^b(\overline{\mathcal{C}})}(\overline{\Sigma}_i, \overline{\Sigma}_j[m]) = 0 \text{ for all } m \in \mathbb{Z} \setminus \{0\} \text{ and all } i, j = 1, \dots, n+1$$

and generate $\mathrm{D}^b(\overline{\mathcal{C}})$ as a triangulated category. In the present setting this latter condition is satisfied if and only if $n+1$ equals the rank of $\mathrm{K}_0(\overline{\mathcal{C}})$.

We denote by $\overline{M}$ the indecomposable object $\overline{\Sigma}_{n+1}$ corresponding to the coextension vertex, and by Σ the full subcategory consisting of $\overline{\Sigma}_1, \dots, \overline{\Sigma}_n$. By translation we may moreover assume that $\overline{M}$ lies in $\overline{\mathcal{C}}$. Notice that $\overline{M}$ is an *exceptional object* of $\overline{\mathcal{C}}$, i.e. has trivial endomorphism ring and no self-extensions. Let $\mathcal{H} = \overline{M}^{\perp}_{\overline{\mathcal{C}}}$ denote the perpendicular category

$$\mathcal{H} = \left\{ X \in \overline{\mathcal{C}} \mid \operatorname{Hom}_{\overline{\mathcal{C}}}(\overline{M}, X) = 0 = \operatorname{Ext}^1_{\overline{\mathcal{C}}}(\overline{M}, X) \right\}$$

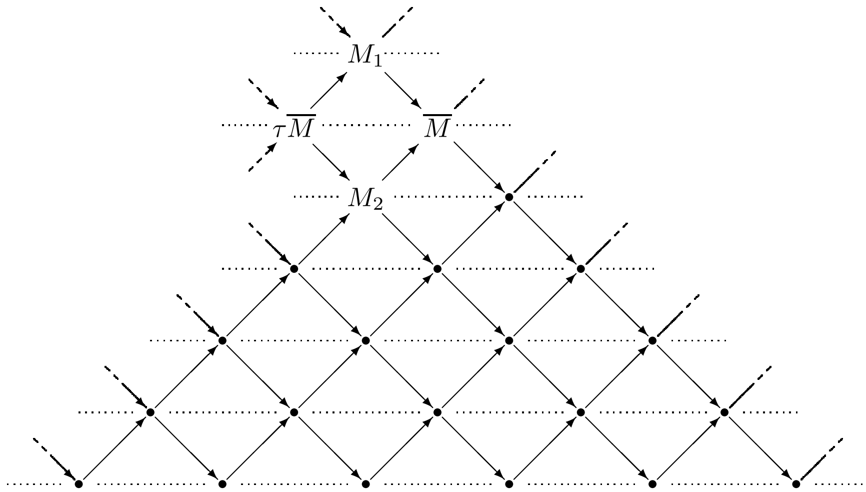
of $\overline{M}$ formed in $\overline{\mathcal{C}}$, and let $\mathcal{D} = \overline{M}^{\perp}_{\mathrm{D}^b(\overline{\mathcal{C}})}$ denote the perpendicular category

$$\mathcal{D} = \left\{ X \in \mathrm{D}^b(\overline{\mathcal{C}}) \mid \operatorname{Hom}_{\mathrm{D}^b(\overline{\mathcal{C}})}(\overline{M}, X[m]) = 0 \text{ for all } m \in \mathbb{Z} \right\}$$

of $\overline{M}$ formed in the derived category $\mathrm{D}^b(\overline{\mathcal{C}})$. It is easily checked that $\mathcal{H}$ is an abelian hereditary category, and that $\mathcal{D} = \bigvee_{n \in \mathbb{Z}} \mathcal{H}[n]$. Moreover, Σ is a tilting complex in $\mathcal{D}$.

The structure of $\mathcal{H}$ is given as follows:

Case (i): If $\overline{M}$ has finite length n, then $\overline{M}$ lies in an exceptional tube and $\mathcal{H} \cong \mathcal{C} \times \operatorname{mod} H$, where $\mathcal{C} = \operatorname{coh} \mathbb{X}$ for a weighted projective line of a weight type dominated by the weight type of $\overline{\mathbb{X}}$ and where $H = k[1 \to 2 \to \cdots \to n-1]$. This follows from [**4**, Thm. 9.5] invoking an argument of Strauß [**13**]. The figure below shows the relevant part of the component of $\mathrm{D}^b(\overline{\mathcal{C}})$ containing $\overline{M}$:



Here, the indecomposable H-modules form the subwing with "top" M_2. According to the decomposition $\mathrm{D}^b(\mathcal{H}) = \mathrm{D}^b(\mathcal{C}) \times \mathrm{D}^b(\mathrm{mod}\, H)$, the algebra Σ decomposes into two connected algebras Σ_1 and Σ_2, where Σ_1 is a tilting complex in $\mathrm{D}^b(\mathcal{C})$ and Σ_2 is a tilting complex in $\mathrm{D}^b(\mathrm{mod}\, H)$. Note that M_2 is derived peripheral over H, hence over Σ_2.

Further the object M_1 at the top of the figure is a simple object in $\mathcal{C}$, and therefore M_1 becomes a derived simple Σ_1-module. This proves the first part of the statement 2.

Case (ii): $\overline{M}$ is an exceptional vector bundle in $\mathrm{coh}\,\overline{\mathbb{X}}$. Here, it follows from [**7**] that $\mathcal{H}$ is equivalent to a module category over a (not necessarily connected) hereditary algebra. For a more complete analysis, we need to distinguish the various representation types for $\overline{\mathcal{C}} = \mathrm{coh}\,\overline{\mathbb{X}}$:

1. $\overline{\mathbb{X}}$ has genus < 1, i.e. the weight type $\Delta = (p, q, r)$ is of Dynkin type. Here the vector bundles form one component of type $\mathbb{Z}\overline{\Delta}$, where $\overline{\Delta}$ is the extended Dynkin diagram corresponding to Δ. To calculate the perpendicular category of $\overline{M}$ in $\mathrm{D}^b(\overline{\Sigma})$ we choose a slice $\overline{H}$ of $\mathbb{Z}\overline{\Delta}$ such that $\overline{M}$ becomes a sink in $\overline{H}$, which is a tilting object of $\overline{\mathcal{C}}$ whose endomorphism ring, here identified with $\overline{H}$, is a tame hereditary algebra. Since $\mathrm{D}^b(\overline{\mathcal{C}}) = \mathrm{D}^b(\mathrm{mod}\,\overline{H})$, the perpendicular category of $\overline{M}$ in $\mathrm{D}^b(\overline{\mathcal{C}})$ equals the derived category of $\overline{M}^{\perp}_{\mathrm{mod}\,\overline{H}}$. This category $\overline{M}^{\perp}_{\mathrm{mod}\,\overline{H}}$ is equivalent to the module category of a not necessarily connected hereditary algebra H, whose indecomposable objects consist of the objects of the slice $\overline{H}$ different from $\overline{M}$. The arising cases for H are listed in the table. Moreover, the almost-split sequence $0 \to \tau\overline{M} \to M \to \overline{M} \to 0$ in $\overline{\mathcal{C}}$, obtained from the first assertion, yields the

derived types of the M_i, $M = \bigoplus_{i=1}^{s} M_i$, as marked in the table. Since

$$M^{\perp}_{\mathcal{D}} = \mathrm{D}^b\,(\overline{M}^{\perp}_{\mathrm{mod}\,\overline{H}}) = \mathrm{D}^b\,(\mathrm{mod}\,H) = \prod_{i=1}^{s} \mathrm{D}^b\,(\mathrm{mod}\,H_i)$$

the tilting complex Σ decomposes into s connected pieces Σ_i, where each Σ_i as a tilting complex in $\mathrm{D}^b\,(\mathrm{mod}\,H_i)$ is derived-equivalent to H_i. This shows the first part of statement 1.

2. $\overline{\mathbb{X}}$ has genus one. By an automorphism of the derived category we can in this case achieve that $\overline{M}$ has finite length, see [**10**]. The assertion thus reduces to case (i).
3. $\overline{\mathbb{X}}$ has genus > 1. In this case, $\overline{M}$ belongs to a component of $\overline{\mathcal{C}}$ having type $\mathbb{Z}\mathbb{A}_\infty$ [**11**], and it is known that the quasi-length ℓ of $\overline{M}$ is at most 5 [loc. cit.]. Invoking arguments of [**13**], it further follows from [**7**] that $\overline{M}^{\perp}_{\overline{\mathcal{C}}}$ is equivalent to the product of $\mathrm{mod}\,A_{\ell-1}$, where $A_{\ell-1} = k[1 \to \cdots \to \ell-1]$, with the module category $\mathrm{mod}\,H$ over a connected wild hereditary algebra H. Accordingly Σ decomposes into a product $\Sigma_1 \times \Sigma_2$, where Σ_1 is connected and derived wild hereditary, and where Σ_2 is derived equivalent to $A_{\ell-1}$, i.e. a branch in the sense of [**8, 1**]. Following arguments of [**13**] and [**9**] it follows moreover that M_1 is derived quasi-simple and M_2 is derived peripheral. This proves the first part of the statement 3.

Now we show the second part of the statements 1 and 2. So, first let Σ be derived equivalent to $k[\mathbb{A}_\ell]$ and $M = M' \oplus M''$ a Σ-module such that M' and M'' are indecomposable and form the periphery of a $(p-1, q-1)$-slice S of the component $\mathbb{Z}\mathbb{A}_\ell$ of $\mathrm{D}^b\,(\Sigma)$, where p and q are such that $\ell = p+q-1$. By the first statement of the theorem, we have that $S \oplus \overline{M}$ is a tilting complex in $\mathrm{D}^b\,([M]\Sigma)$ with endomorphism algebra isomorphic to $k[\tilde{\mathbb{A}}_\ell]$. Thus $[M]\Sigma$ is derived canonical.

With the same argument we show that $[M]\Sigma$ is derived canonical, when $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$ and $M = M_1 \times \cdots \times M_s$ where Σ_j is derived hereditary and M_j is an indecomposable Σ_j-module $(j = 1, \ldots, s)$ such that the pair (Σ, M) is listed in Table 1.

Let now $\Sigma = \Sigma_1 \times \Sigma_2$, where Σ_1 is derived canonical and Σ_2 is derived equivalent to $k[\mathbb{A}_\ell]$ for some $\ell \geq 0$. Further let $M = M_1 \times M_2$, where M_1 is derived simple and, if $l > 0$, then let M_2 be deriphed peripheral. Let $\mathbb{X}(\underline{p}, \underline{\lambda})$ be the weighted projective line associated to Σ_1, where $\underline{p} = (p_1, \ldots, p_t)$ is its weight type. Let $\overline{\mathbb{X}}$ be the weighted projective line with weight type $(p_1, \ldots, p_{t-1}, p_t + \ell)$ and with the same parameter sequence $\underline{\lambda}$ as $\mathbb{X}$. We fix an indecomposable sheaf E of length $\ell+1$ concentrated at λ_t, and form the perpendicular category $\mathcal{H} = E^{\perp}$ of E in $\overline{\mathcal{C}} = \mathrm{coh}\,\mathbb{X}$. Then $\mathcal{H} = \mathcal{C} \times \mathrm{mod}\,H$, where $\mathcal{C} = \mathrm{coh}\,\mathbb{X}$ and $H = k[1 \to \cdots \to \ell]$. Moreover, the middle term of the almost-split sequence $0 \to \tau E \to S \oplus M_2 \to E \to 0$ decomposes into a simple sheaf S in $\mathcal{C}$, concentrated at λ_t and in the indecomposable projective-injective H-module M_2. Next, we realize Σ_1 as a tilting complex in $\mathrm{D}^b\,(\mathcal{C})$ so that, by means of the identification $\mathrm{D}^b\,(\Sigma_1) = \mathrm{D}^b\,(\mathcal{C})$ the module M_1 corresponds to the simple sheaf S. Further, we realize the branch Σ_2 as a tilting complex in $\mathrm{D}^b\,(H)$ such that, in the identification $\mathrm{D}^b\,(\Sigma_2) = \mathrm{D}^b\,(H)$, the H-module M_2 becomes a (derived peripheral) module over Σ_2. Following [**2**], it is easily checked that E together with Σ_1 and Σ_2 forms a tilting complex in $\mathrm{D}^b\,(\mathrm{coh}\,\mathbb{X})$ with endomorphism algebra $\overline{\Sigma} = [M_1 \times M_2]\Sigma$. Hence $\overline{\Sigma}$ is derived canonical of type $\overline{\mathbb{X}}$.

This completes the proof of the Theorem. □

In Theorem 1 we have seen that the request for an algebra Σ to admit a derived canonical source or sink extension is very restrictive for Σ and for the "extension module" M. The information is even more specific if we start the extension procedure with a derived canonical algebra Σ.

Corollary 1. *Let Σ be derived canonical, and let M be a finite dimensional not necessarily indecomposable Σ-module.*

Then the sink extension $[M]\Sigma$ is derived canonical if and only if M is derived simple, in particular indecomposable.

Proof. By Theorem 1, we only need to show that the condition is necessary. So let us assume that $[M]\Sigma$ is derived canonical. As a derived canonical algebra Σ is connected, hence Theorem 1 implies that $\overline{M}$ is derived simple. This uses that a wild hereditary or a representation-finite hereditary algebra is never derived equivalent to a canonical algebra since wild hereditary algebras always have spectral radius greater one and representation-finite hereditary algebras do not admit 1 as a root of their Coxeter polynomial, while canonical algebras have spectral radius one, and 1 is a root of their Coxeter polynomial. Thus cases 1 and 3 of Theorem 1 will not occur. □

We now consider the case where a branch $\mathcal{B}$ is attached to the extension vertex of $[M]\Sigma$, see [**1**] for definitions. We denote by $[\mathcal{B}, M]\Sigma$ the resulting algebra.

Corollary 2. *Let Σ be a derived canonical algebra, M a Σ-module and $\mathcal{B}$ a branch.*

Then $[\mathcal{B}, M]\Sigma$ is derived canonical if and only if M is derived simple.

Proof. First let $\overline{\Sigma} = [\mathcal{B}, M]\Sigma$ be derived canonical. Clearly, $\overline{\Sigma}$ is derived equivalent to $\overline{\Sigma}' = [\mathcal{B}', M]\Sigma$, where $\mathcal{B}'$ denotes the linearly ordered branch with its sink α as root point. Thus we may identify $\overline{\Sigma}'$ with the algebra $[M \times P_\alpha](\Sigma \times \mathcal{B}')$, where P_α denotes the projective indecomposable associated to the point α. Therefore, we may apply Corollary 1. Once again, only the case 2 remains possible, and we infer that M is a derived simple Σ-module.

The converse is covered by Theorem 1, part 2. □

2.1. Criteria for derived canonical algebras. Theorem 1 provides us with a necessary condition for an algebra Σ to be derived canonical. So we might use this in order to prove that certain algebras are not derived canonical. We shall exhibit this in an example. Let A_n be the algebra given by the linear bound quiver with $n \geq 8$ vertices

$$\bullet \xrightarrow{x} \bullet \xrightarrow{x} \bullet \text{---} \cdots \longrightarrow \bullet \xrightarrow{x} \bullet$$

which satisfies the $n-6$ relations $x^7 = 0$. It is not difficult to check that A_9, $A_1$0 and A_{11} are derived canonical of weight type $(2, 3, 5)$, $(2, 3, 6)$ and $(2, 3, 7)$, respectively. Also the Coxeter polynomial of A_{22} has canonical type $(2, 7, 14)$. However, the algebra A_{22} is not derived canonical.

Using Theorem 1, this can be seen as follows: First, we write A_{22} as a sink extension of the algebra A_{21} by a module M over A_{21}, thus $A_{22} = [M]A_{21}$. By Theorem 1, it suffices to show that A_{21} is neither derived equivalent to a representation-finite hereditary algebra, nor derived canonical, nor derived equivalent to a wild hereditary algebra:

Introducing the polynomials

$$V_n = \frac{T^n - 1}{T - 1},$$

the Coxeter polynomial $\mathbf{C}(A_{21})$ of A_{21} is seen to be $(T-1)^2 V_2 V_7 V_8 V_9 / V_4$. In particular, $\mathbf{C}(A_{21})$ is not of canonical type and hence A_{21} is not derived canonical. Since all roots of $\mathbf{C}(A_{21})$ lie on the unit circle in the complex plane and further 1 is a root of $\mathbf{C}(A_{21})$, the algebra A_{21} cannot be derived equivalent to a hereditary algebra which is wild or representation-finite.

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