DERIVED-TAME TREE ALGEBRAS OF TYPE $\mathbb{E}$

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Let A be a finite dimensional algebra over an algebraically closed field k. The derived category of the category $\operatorname{mod} A$ of (left-)modules over A is denoted by $D^b(A)$. For few algebras A, the description of $D^b(A)$ is known. For example, for $A = kQ$ an hereditary algebra of finite or tame type, the description of $D^b(A)$ is well-known [12], in case A is a tubular algebra, the indecomposable objects of $D^b(A)$ were described in [13].

A useful tool for describing the derived category $D^b(A)$ is the *repetitive category* $\hat{A}$ (see [12]). If A has finite global dimension, then $D^b(A)$ is triangle equivalent to $\underline{\operatorname{mod}}\, \hat{A}$, the quotient of $\operatorname{mod} \hat{A}$ by the maps factorizing through projective $\hat{A}$-modules. Following [10], we say that A is *derived-tame* if $\operatorname{gldim} A < \infty$ and the category $\hat{A}$ is tame. We recall that a k-category B is said to be tame if each factor by a cofinite set of objects is tame. We give examples in section 1.

If $\operatorname{gldim} A < \infty$, the *homological bilinear form* is given for the classes $[X]$ and $[Y]$ of the modules X and Y in the Grothendieck group $K_0(A)$ of A, by $\langle [X], [Y] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}^i_A(X,Y)$. The associated quadratic form χ_A is called the *Euler form* of A.

A basic algebra A of the form $A = kQ/I$ is a *tree algebra* if the underlying graph of Q is a tree. A tree algebra A always has finite global dimension. It has been *conjectured* in [10], that a tree algebra A is derived-tame if and only if χ_A is non-negative. The aim of this note is to show the following partial result.

Theorem. *Let A be a tree algebra containing a convex subcategory which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p$, $\tilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or to a tubular algebra. Then A is derived-tame if and only if χ_A is non-negative. Moreover, in this case, the algebra A itself is derived equivalent to some hereditary algebra of type $\mathbb{E}_q$, $\tilde{\mathbb{E}}_q$ $(q = 6, 7, 8)$ or a tubular algebra.*

We present the proof of the theorem in section 2. In section 1 we recall some concepts and give examples. In section 3 we present the list of all tree algebras which are derived equivalent to $\mathbb{E}_6$.

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1. Derived-tame algebras.

1.1. Let A be a basic algebra of the form $A = kQ/I$, where Q is a connected finite quiver and I an admissible ideal of the path algebra kQ. We consider A as a k-category where objects are the vertices Q_0 of Q and in which the space of maps $A(x, y)$ from x to y is $e_y A e_x$, where e_x denotes the primitive idempotent associated to the vertex x.

The *repetitive category* $\hat{A}$ is the k-category with objects $Q_0 \times \mathbb{Z}$ (denoted by $s[i]$, for $s \in Q_0$ and $i \in \mathbb{Z}$) and the only possible non-zero morphism spaces are $\hat{A}(r[i], s[i]) = A(r, s) \times \{i\}$ and $\hat{A}(r[i], s[i+1]) = DA(s, r) \times \{i\}$, where $D = \mathrm{Hom}_k(-, k)$ denotes the usual duality.

1.2. Let $F\colon D^b(A) \to D^b(B)$ be an equivalence of triangulated categories. Then there is an induced isomorphism $f\colon K_0(A) \to K_0(B)$ such that $f([X^\cdot]) = [FX^\cdot]$ for any object $X^\cdot \in D^b(A)$, where $[X^\cdot] = \sum_{i\in\mathbb{Z}} (-1)^i [X^i] \in K_0(A)$. Moreover, f is an isometry. In particular, χ_A is non-negative if and only if χ_B is, and, in this case, $\mathrm{corank}\, \chi_A = \mathrm{corank}\, \chi_B$.

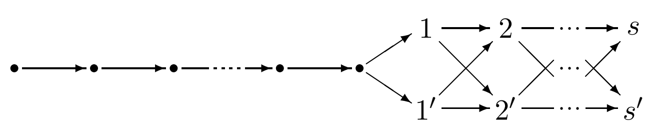
In [10], it was shown that B is derived-tame if so A is. Examples of derived-tame algebras are the following:

(a) [2] $A = kQ$ is a representation-finite hereditary algebra (hence Q is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\mathbb{E}_p$ for $p = 6, 7, 8$). In this case, χ_A is positive definite.

(b) [2, 4] $A = kQ$ is a tame hereditary algebra (hence Q is of type $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$, for $p = 6, 7, 8$). In this case χ_A is non-negative with $\mathrm{corank}\, \chi_A = 1$.

(c) [4] A is a tubular algebra in the sense of Ringel [14]. In this case, χ_A is non-negative with corank $\chi_A = 2$.

(d) [10] Let $P(n, s)$ be the algebra associated to the poset



with n vertices. In this case, χ_A is non-negative with $\mathrm{corank}\, \chi_A = s - 1$.

The above list is complete for algebras with small corank as shown in the following:

Theorem [3] *Let A be a tree algebra such that χ_A is non-negative with* $\mathrm{corank}\, \chi_A \leq 2$. *Then A is derived equivalent to one of the examples (a), (b), (c) or (d). In particular, A is derived-tame.*

1.3. Given the k-category $A = kQ/I$, we say that B is a convex subcategory of A if $B = kQ'/I'$ where Q' is a path-closed subquiver of Q and $I' = I \cap kQ'$. The following technical result will be useful. We denote $\chi_A(v, w) = \langle v, w\rangle_A + \langle w, v\rangle_A$.

Proposition. [10] *Let $A = kQ/I$ be a tree algebra. Suppose that A contains a convex subcategory C which is derived equivalent to a tame hereditary algebra $k\Delta$. Let $0 \neq v \in K_0(C)$ with $\chi_C(v) = 0$ be such that $\chi_A(v, e_s) \neq 0$ for some vertex s of Q. Then A is not derived-tame.*

1.4. Let $A = kQ/I$ be as above. We recall that the *one-point extension* $A[M]$ of A by the module M is the category with objects $Q_0 \cup \{s\}$ and morphism spaces $A[M](s,x) = M(x)$ for $x \neq s$, $A[M](s,s) = k$ and such that A is a convex subcategory.

If $F\colon D^b(A) \to D^b(B)$ is a derived equivalence such that $F(M[0]) = N[0]$ for modules $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} B$, then by [7], we get a derived equivalence $\hat{F}\colon D^b(A[M]) \to D^b(B[N])$ which is an extension of F.

We shall recall that for a derived tubular algebra A, any one-point extension $B = A[M]$ with an indecomposable A-module M has Euler form χ_B indefinite, see [3].

1.5. We recall from [6] that any non-negative connected unit form q (for example, $q = \chi_A$ as above) has an associated Dynkin graph $\operatorname{Dyn}(q)$. If B is a convex subcategory of the connected category A, and χ_A is non-negative, then χ_B is non-negative and $\operatorname{Dyn}(\chi_B) \leq \operatorname{Dyn}(\chi_A)$, where

$$\begin{aligned} \mathbb{A}_m \leq \mathbb{A}_n \leq \mathbb{D}_n \leq \mathbb{D}_p \quad &\text{for } m \leq n \leq p; \\ \mathbb{D}_p \leq \mathbb{E}_p \leq \mathbb{E}_q \quad &\text{for } 6 \leq p \leq q \leq 8. \end{aligned}$$

Remarks. [5] (a) If an algebra A is derived equivalent to $\mathbb{E}_q$ or $\tilde{\mathbb{E}}_q$ ($q = 6, 7, 8$), then $\operatorname{Dyn}(\chi_A) = \mathbb{E}_q$. If A is derived tubular with more than 6 vertices then $\operatorname{Dyn}(\chi_A) = \mathbb{E}_p$ for some $p = 6, 7, 8$.
(b) If an algebra A is derived tubular with 6 vertices, then A is not a tree algebra.
(c) If $A = P(n,s)$ then $\operatorname{Dyn}(\chi_A) = \mathbb{D}_{n-s}$.

1.6. The following Lemmas will be useful in the forthcoming.

Lemma 1 *Let A be a connected and directed algebra with non-negative Euler form χ_A. Then there exists a full subcategory B of A with positive Euler form χ_B such that* $\operatorname{Dyn}(\chi_A) = \operatorname{Dyn}(\chi_B)$.

Proof: Let C_A be the Cartan matrix of A. Recall that χ_A is the quadratic form associated to C_A^{-1}. Denote by χ_A^* the quadratic form associated to C_A. Note that χ_A^* and χ_A are equivalent forms. By [6], there exists a restriction q of χ_A^* which is positive and such that $\operatorname{Dyn}(\chi_A^*) = \operatorname{Dyn}(q)$. Clearly, we have $q = \chi_B^*$, if B denotes the full subcategory of A given by the vertices of q. Hence the result. $\square$

Lemma 2 *Let q be a positive unit form of Dynkin-type $\mathbb{E}_p$ for some $p = 6$, 7 or 8. Then q admits a restriction of Dynkin type $\mathbb{E}_6$.*

Proof: The proof is obtained by a case by case checking (using a computer program) of the whole list of positive unit forms of Dynkin type $\mathbb{E}_7$ and $\mathbb{E}_8$. □

Corollary. *Let A be an algebra derived equivalent to a hereditary algebra of type $\mathbb{E}_p$ for some $p = 6$, 7 or 8. Then A contains a full subcategory B which is derived equivalent to a hereditary algebra of type $\mathbb{E}_6$.*

Proof: By Lemma 2, there exists a full subcategory B for which the Euler form has Dynkin-type $\mathbb{E}_6$. Since $\operatorname{mod} A$ is cycle-finite, hence so is $\operatorname{mod} B$. Thus by [1], B is derived equivalent to a hereditary algebra of Dynkin type, of extended Dynkin type or to a tubular algebra. By the properties of χ_B, the algebra B must thus be derived equivalent to a hereditary algebra of type $\mathbb{E}_6$.

□

2. The theorem and consequences.

2.1. Proposition. *Let A be a tree algebra with non-negative Euler form. Then the following assertions are equivalent.*

(i) A is derived equivalent to some $\mathbb{E}_q$ or $\tilde{\mathbb{E}}_q$ $(q = 6, 7, 8)$ or to a tubular algebra.

(ii) A contains a convex subcategory E which is derived equivalent to some $\mathbb{E}_p$ or $\tilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or to a tubular algebra.

(iii) A contains a full subcategory which is derived equivalent to $\mathbb{E}_6$ and $\operatorname{corank} \chi_A \leq 2$.

Proof: Clearly (i) implies (ii). So assume now (ii). By (1.5), $\operatorname{Dyn}(\chi_A) \geq \mathbb{E}_p$, which implies $\operatorname{Dyn}(\chi_A) = \mathbb{E}_q$ for some $q = 6$, 7 or 8. Now, if $\operatorname{corank} \chi_A \leq 2$, then (1.2) applies and A is derived equivalent to an algebra of the desired type or to $P(n, s)$ for some $2 \leq s \leq 3$ and n. But $A = P(n, s)$ is impossible, since $\operatorname{Dyn}(\chi_{P(n,s)}) = \mathbb{D}_{n-s}$.

Assume $\operatorname{corank} \chi_A > 2$ and choose B a maximal connected convex subcategory of A with $\operatorname{corank} \chi_B = 2$ with E contained in B. As above B is a derived tubular algebra and since B is properly contained in A, there is a one-point extension $B[M]$ with an indecomposable B-module M contained convexely in A. By (1.4), the Euler form $\chi_{B[M]}$ is not non-negative, a contradiction. This shows (i).

Now, assume (iii). By (1.5), $\operatorname{Dyn}(\chi_A) \geq \mathbb{E}_6$, hence $\operatorname{Dyn}(\chi_A) = \mathbb{E}_p$ for some $p = 6, 7$ or 8. Thus by (1.2), assertion (i) holds. It remains to show that (i) implies (iii). Clearly we have $\operatorname{corank} \chi_A \leq 2$. By Lemma 1, there exists a full subcategory B such that χ_B is positive and $\operatorname{Dyn}(\chi_B) = \mathbb{E}_p$. By [1], the derived categrory $\mathrm{D}^{\mathrm{b}}(A)$ is cycle-finite and hence so is $\mathrm{D}^{\mathrm{b}}(B)$. Again by [1], B is derived equivalent to a hereditary algebra of Dynkin type or extended Dynkin type or to a tubular algebra. Since χ_B is positive of Dynkin type $\mathbb{E}_p$, B has to be derived equivalent to $\mathbb{E}_p$. Thus (iii) follows by Corollary 1.6. □

2.2. Proof of the theorem. Let E be a convex subcategory of A which is derived equivalent to some $\mathbb{E}_p$ or $\tilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or to a tubular algebra. If χ_A is non-negative, by (2.1) and (1.2), A is a derived-tame (of the desired form). Hence we only need to show that in case A is derived-tame, the form χ_A is non-negative.

Assume A is derived-tame and χ_A is not non-negative. Let B be a maximal connected convex subcategory of A containing E such that χ_B is non-negative. Then by (2.1), $\operatorname{corank}\chi_B \leq 2$ and B is derived equivalent to some $\mathbb{E}_q$ or $\tilde{\mathbb{E}}_q$ $(q = 6, 7, 8)$ or to a tubular algebra. Moreover, there is a one-point extension $B[M]$ which is convex in A. Let $M = \operatorname{rad} P_a$ for the new source a of the quiver of $B[M]$. We distinguish several cases.

If B is derived tubular, by [4] we get a vector $0 \neq v \in K_0(B)$ with $\chi_B(v) = 0$ and $0 \neq \langle v, [M]\rangle_A = \chi_A(v, e_a)$. Hence (1.3) implies that A is not derived-tame.

Assume that B is derived equivalent to $\tilde{\mathbb{E}}_q$ $(q = 6, 7, 8)$, say $F\colon D^b(B) \to D^b(H)$ is an equivalence of triangulated categories, where H is a hereditary algebra of type $\tilde{\mathbb{E}}_q$. Since the indecomposable modules of $D^b(H)$ are shifts $X[i]$ $(i \in \mathbb{Z})$ of H-modules X, we may assume that $F(M[0]) = N[0]$ for an indecomposable H-module N. Then (1.4) yields an equivalence $\hat{F}\colon D^b(B[M]) \to D^b(H[N])$. The maximality assumption for B implies that $\chi_{B[M]}$ is not non-negative and therefore $\chi_{H[N]}$ is not non-negative. Therefore by [9], either $H[N]$ is wild or N is preinjective (and $[N]H$ is wild). In any case, $H[N]$ is not derived-tame and hence $B[M]$ (and A) is not derived-tame.

Finally, assume that B is derived equivalent to H a hereditary algebra of type $\mathbb{E}_q$ $(q = 6, 7, 8)$. As above $B[M]$ is derived equivalent to some $H[N]$ with N an indecomposable H-module. Since H is a hereditary representation-finite algebra, then clearly $H[N]$ is a tilted algebra of type Δ (since the Auslander-Reiten quiver of $H[N]$ has a slice). Again $\chi_{H[N]}$ is not non-negative, which means that Δ is of wild type. Since $H[N]$ and $k\Delta$ are derived-equivalent, then $H[N]$ is not derived-tame. Again this implies that A is not derived-tame. □

2.3. As a consequence we obtain a result shown with a more involved proof in [11]. We recall that the *Tits form* $q_A\colon K_0(A) \to \mathbb{Z}$ is given by

$$q_A(v) = \sum_{i=0}^{2}(-1)^i \left[\sum_{x,y\in Q_0} v(x)v(y)\dim_k \operatorname{Ext}_A^i(S_x, S_y)\right],$$

where S_x denotes the simple module associated to the vertex $x \in Q_0$.

Theorem [11]. *Let A be a tree algebra satisfying the hypothesis:*

(a) *q_A is non-negative with a sincere positive isotropic vector;*

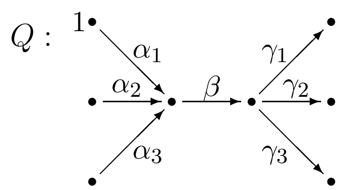
(b) *A contains a convex subcategory tilted of type $\mathbb{E}_6$.*

Then A is tame concealed or tubular.

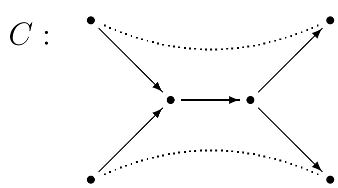
Proof: As observed in [11], hypothesis (a) implies that $\operatorname{gldim} A \leq 2$ and therefore $q_A = \chi_A$ is non-negative. The main theorem implies that A is derived equivalent to a hereditary algebra of type $\tilde{\mathbb{E}}_p$ ($p = 6, 7, 8$) or to a tubular algebra. In [4] it was shown that the existence of a sincere isotropic root yields the result. □

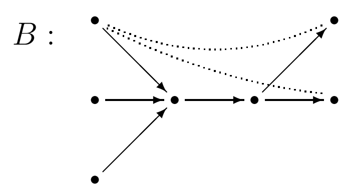
2.4. In the paper [11] it was conjectured that any sincere tree algebra A with q_A weakly non-negative and containing a convex subcategory tilted of type $\mathbb{E}_6$ should be tame of polynomial growth. The conjecture is false as shown by the following *example.*

Let $A = kQ/I$ be the algebra given by the quiver



with I generated by $\gamma_1\beta\alpha_1$, $\gamma_3\beta\alpha_3$, $\gamma_2\beta\alpha_1$, $\gamma_3\beta\alpha_2$. Consider the convex subcategories C and B of A given by the quivers (with relations indicated)





The algebra C is tilted of type $\mathbb{E}_6$, while B is pg-critical and hence A cannot be tame of polynomial growth. Since B is sincere and A is the one-point coextension $[P_1]B$, then A is sincere. Finally, A is tame since it is a full subcategory of the tame category $\widehat{B}$ associated to the derived-tame algebra B, in particular showing that q_A is weakly non-negative [9].

3. The list of tree algebras which are derived equivalent to $\mathbb{E}_6$.

Each picture represents a class of algebras which is obtained in the following way: edges without orientation may be oriented in either way and one may change the orientation of all arrows simultaneously. In this way, the list shows 208 non-isomorphic algebras.

1)

2)

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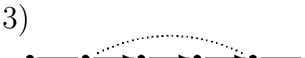
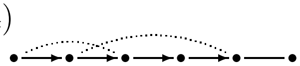
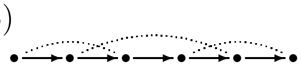
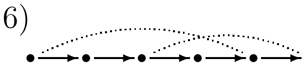
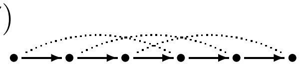
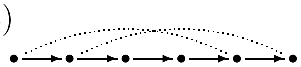


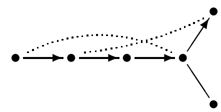
4) 

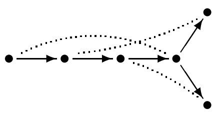
5) 

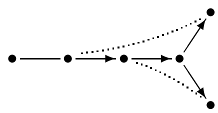
6) 

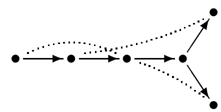
7) 

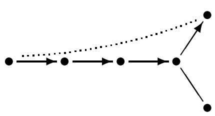
8) 

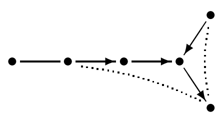
9) 

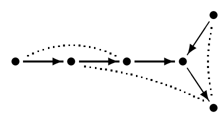
10) 

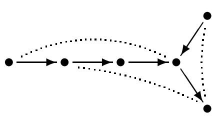
11) 

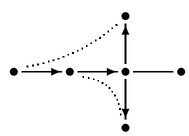
12) 

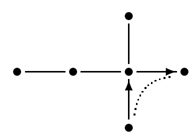
13) 

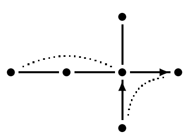
14) 

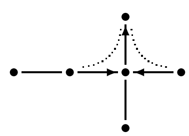
15) 

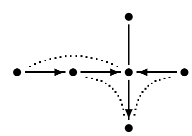
16) 

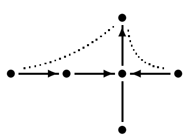
17) 

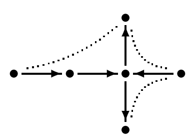
18) 

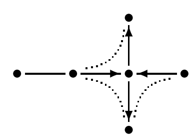
19) 

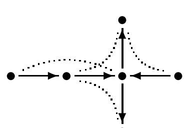
20) 

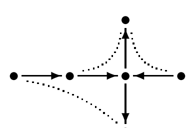
21) 

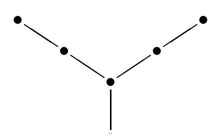
22) 

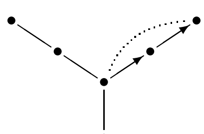
23) 

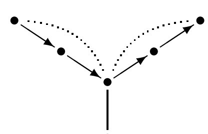
24) 

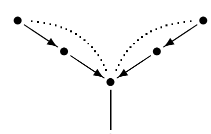
25) 

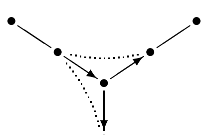
26) 

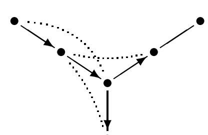
27) 

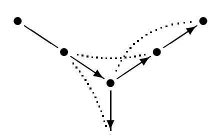
28) 

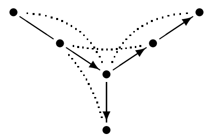
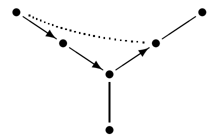
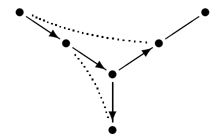
29) 

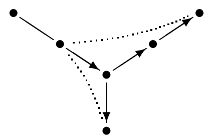
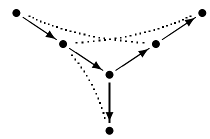
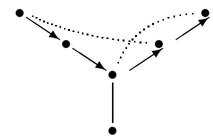
30) 

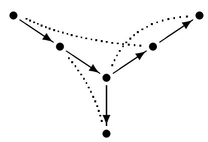
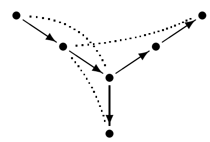
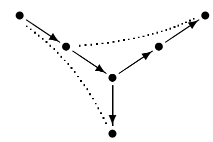
31) 

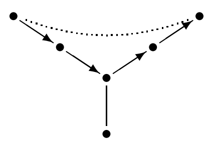
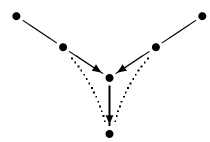
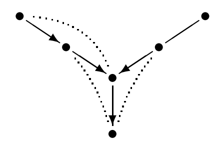
32) 

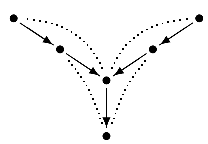
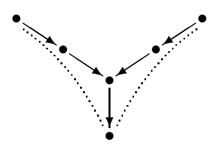
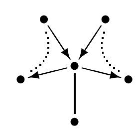
33) 

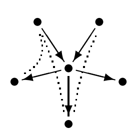
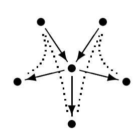
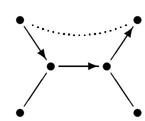
34)  35)  36) 

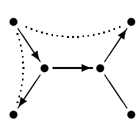
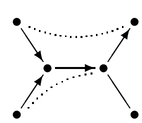
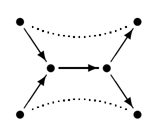
37)  38)  39) 

40)  41)  42) 

43)  44)  45) 

46)  47)  48) 

49)  50)  51) 

52)  53)  54) 

Remark. The above list and the list of positive unit forms of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ calculated using a C++-program may be obtained writing to the first named author. This list may also be calculated using the CREP program, see [8].

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