DERIVED TUBULARITY: A COMPUTATIONAL APPROACH

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Let A be a finite dimensional algebra over an algebraically closed field. We denote by mod_A the category of finite dimensional left A-modules and by $D^b(A)$ the derived category of mod_A (see for example [12] for definitions). We say that two algebras A and B are *derived equivalent* if their derived categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories.

An important problem in the Representation Theory of Algebras has been to characterize those algebras A which are derived equivalent to well understood classes of algebras, for instance to representation-finite hereditary algebras [11,12], to tame hereditary algebras [11] or to tubular algebras [13,1]. The aim of this work is to discuss the above mentioned characterizations from a computational point of view. In particular, we present an *algorithm* to decide whether or not certain classes of algebras are derived equivalent to a tubular algebra. This algorithm has been implemented in a $C{+}{+}$ program which will be available as part of the CREP package, see [8].

Let A be an algebra as above with finite global dimension. Let $K_0(A) \xrightarrow{\sim} \mathbb{Z}^n$ be its Grothendieck group which we will consider equipped with a (non-symmetric) bilinear form $\langle -,-\rangle_A$ such that for two modules $X, Y \in \mathrm{mod}_A$ we have,

$$\langle [X],[Y]\rangle_A = \sum_{i=0}^{\infty}(-1)^i \dim_k \mathrm{Ext}^i_A(X,Y).$$

The associated quadratic form $\chi_A(v) = \langle v,v\rangle_A$ is called the *Euler form* of A. For two derived equivalent algebras A and B, the Euler forms χ_A and χ_B are equivalent.

If χ_A is non-negative, then $\mathrm{rad}\,\chi_A = \{v \in \chi_A : \chi_A(v) = 0\}$ is a subgroup of $\mathbb{Z}^n$ and there is an induced positive definite form $\bar{\chi}_A : \mathbb{Z}^n/\mathrm{rad}\,\chi_A \to \mathbb{Z}$ (see (2.10)). The form $\bar{\chi}_A$ accepts only finitely many *positive roots* (a root $v \in \mathbb{Z}^n/\mathrm{rad}\,\chi_A \cong \mathbb{Z}^{n-s}$ satisfies $\bar{\chi}_A(v) = 1$), see [26]. The main result behind our algorithm is the following (for the definition of strong simple connectedness see (1.2)).

Main Theorem. *Let $A = kQ/I$ be a strongly simply connected algebra and assume that its Grothendieck group is $K_0(A) \cong \mathbb{Z}^n$ with $n > 6$. Then A is derived equivalent to a tubular algebra if and only if the following conditions are satisfied:*

(a) χ_A *is non-negative of* $\mathrm{corank}\,\chi_A = 2$*;*

(b) *there exists a source or a sink a of Q such that $Q \setminus \{a\}$ is connected and the convex subcategory B of A with vertices $Q \setminus \{a\}$ has a non-negative Euler form of corank one.*

(c) $\overline{\chi}_{A}$ *has* 36*,* 63 *or* 120 *positive roots.*

We briefly describe the contents of the paper. In section 1 we recall the relevant characterizations of derived equivalences mentioned above. In section 2 we survey some results related to algorithmic procedures for integral quadratic forms. Moreover we show in (2.11) (see also [19]):

Theorem. *Each connected non-negative unit form q determines uniquely a Dynkin graph Δ such that $\overline{q}$ is $\mathbb{Z}$-eqivalent to the form p_Δ asociated with Δ.*

Using this theorem we prove in section 3 the main result. Finally, in section 4 we present the list of all algebras whose underlying quiver is a simple oriented line and which are derived equivalent to a tubular algebra. This list was calculated using our computer program.

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1. Derived equivalence for algebras.

1.1. Let A be a basic, finite-dimensional k-algebra over a fixed algebraically closed field k. It is well-known that $A = kQ/I$ where Q is a finite quiver and I is an admissible ideal of the path algebra kQ. We shall consider A as a *spectroid*, that is, as a k-category whose objects are the vertices Q_0 of Q and the space of maps from x to y is $A(x, y) = e_y A e_x$, where e_x denotes the primitive idempotent associated to the vertex x, see [9].

We assume that $Q_0 = \{1, \dots, n\}$ and that Q is connected and has no oriented cycle. In particular, the global dimension of A is finite.

1.2. We recall that a vertex $a \in Q_0$ is *separated* in A if any two different direct summands of the radical rad P_a of the indecomposable projective module P_a (= projective cover of the simple S_a) have their supports in different connected components of the quiver $Q^{(x)} = Q \setminus \{y \in Q_0$: there is a path from y to $x\}$. The algebra A is *separated* if every vertex $a \in Q_0$ is separated.

Finally, A is *strongly simply connected* if every full and convex (= path closed) subcategory B of A is separated. See [27] for equivalent properties.

If A is representation-finite and separated, then A is strongly simply connected [7]. In any case, 'separation' is a condition which may be easily checked.

1.3. A basic, finite-dimensional and hereditary algebra H is a path algebra $k\Delta$ for Δ a finite connected quiver without oriented cycle. Recall that H is representation-finite (resp. tame) if and only if Δ is of Dynkin (resp. extended-Dynkin) type. In [11] it was shown that A is derived equivalent to H if and only if A is *tilting-cotilting equivalent* to H, that is, there is a sequence of algebras $A = A_0, A_1, \dots, A_m = H$ and a sequence of modules ${}_{A_i}T^{(i)}$ $(0 \leq i < m)$ such that $A_{i+1} = \mathrm{End}_{A_i} T^{(i)}$ and $T^{(i)}$ is either a tilting or a cotilting module. Then we may reformulate a result in [2,(5.1)] in the following way.

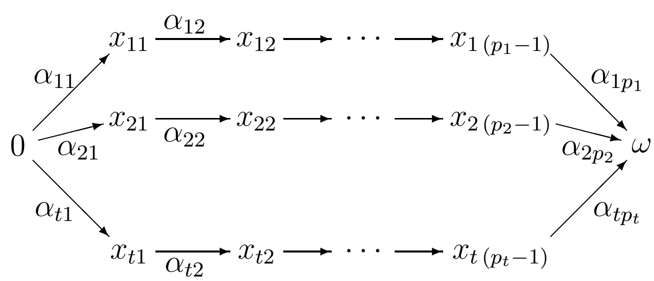
Theorem. [2] *Let $A = kQ/I$ be a separated algebra. Then A is derived equivalent to $k\Delta$ with Δ a quiver of Dynkin type if and only if χ_A is positive definite.*

1.4. That the positivity of the Euler form is preserved by derived equivalence is a consequence of the following general argument: if $F\colon D^b(A) \to D^b(B)$ is an equivalence of triangulated categories, then there is an induced isomorphism $f\colon K_0(A) \to K_0(B)$ such that $f([X^{\cdot}]) = [FX^{\cdot}]$, for any object $X^{\cdot} \in D^b(A)$, where $[X^{\cdot}] = \sum_{i\in\mathbb{Z}} (-1)^i [X^i] \in K_0(A)$. Then f commutes with the corresponding bilinear forms (we say that f is an *isometry*). In particular, χ_A is non-negative if and only if so is χ_B, and in this case, $\mathrm{corank}\,\chi_A = \mathrm{corank}\,\chi_B$.
The next natural step after (1.3) is the following:

Theorem. [2,4] *Let A be a strongly simply connected algebra. Then A is derived equivalent to $k\Delta$ with Δ of extended-Dynkin type if and only if χ_A is non-negative and* $\mathrm{corank}\,\chi_A = 1$.

The result is shown in [2] in case A is representation-finite and extended in [4] to the representation-infinite situation.

1.5. Let $t \geq 3$ and $p = (p_1, \dots, p_t)$ be a sequence of numbers $2 \leq p_i$. Consider the following quiver $Q(p)$:



Let $\lambda = (\lambda_3, \dots, \lambda_t)$ be a sequence of pairwise different elements of $k \setminus \{0\}$. Then the *canonical algebra* $C(p, \lambda)$ is defined as the quotient of $kQ(p)$ by the ideal generated by the relations: $\alpha^{(i)} = \alpha^{(1)} + \lambda_i \alpha^{(2)}$, $i = 3, \ldots, t$, where $\alpha^{(i)} = \alpha_{ip_i} \ldots \alpha_{i1}$. Canonical algebras have been extensively studied, see for instance [26,10]. Of particular interest are *canonical tubular algebras* obtained when $p = (2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. These algebras are *tame* and the module category is described in [26]. The class of *tubular algebras* is obtained in [26] as certain tilted algebras of canonical tubular algebras.

We say that A is *derived tubular* if it is derived equivalent to a canonical tubular algebra. In this case χ_A is non-negative and $\operatorname{corank} \chi_A = 2$.

1.6. The *Tits form* $q_A \colon \mathbb{Z}^n \to \mathbb{Z}$ of A is obtained as a 'truncation' of χ_A in the following way: $q_A(v) = \sum_{i=0}^{2} (-1)^i \sum_{a,b \in Q_0} v(a)v(b) \dim_k \mathrm{Ext}_A^i(S_a, S_b)$. We recall that for a representation-finite algebra A, the Tits form is *weakly positive* (that is, $q_A(v) > 0$ for $0 \neq v \in \mathbb{N}^n$). If A is separated, the converse holds [5]. Meanwhile, for a *tame* algebra, the Tits form is *weakly non-negative* (that is, $q_A(v) \geq 0$ for $v \in \mathbb{N}^n$) [21]. If A is strongly simply connected, it is conjectured that the converse holds [22].

Theorem. [3] *The algebra A is representation-finite and derived tubular if and only if the following conditions are satisfied:*

(0) *A has more than six vertices ($n > 6$);*

(1) *χ_A is non-negative and* $\operatorname{corank} \chi_A = 2$*;*

(2) *$\chi_A^{-1}(1) \cap (\operatorname{rad} \chi_A)^{\perp} = \emptyset$;*

(3) *q_A is weakly positive;*

(4) *A is separated.*

Where $V^{\perp} = \{w \in \mathrm{K}_{\circ}(A) : \langle w, v \rangle = 0 \text{ for every } v \in V\}$. Conditions (0), (1) and (4) are easy to check (also by a computer). For condition (3) there are efficient algorithms as we shall recall in section 2. Condition (2) is the hardest to check. Our work in section 3 is devoted to substitute this condition for another easier to check (by computer).

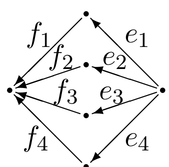
1.7. We shall consider also the following extension of (1.6) to representation-infinite algebras.

Theorem. [4] *Let $A = kQ/I$ be a strongly simply connected algebra with $n = |Q_0| > 6$. Then A is derived tubular if and only if conditions (1) and (2) above hold.*

Let us observe that it is impossible to have a criterion for derived tubularity in the case $n = 6$, which uses only properties of the Euler form. Indeed, for $p = (2, 2, 2, 2)$ and $\lambda = (1, \rho)$ with $\rho \neq 0, 1$, the algebra $C = C(p, \lambda)$ is canonical tubular, while the algebra $C' = kQ(p)/\langle \alpha^{(3)} - \alpha^{(2)} - \alpha^{(1)}, \alpha^{(4)} - \alpha^{(2)} - \alpha^{(1)} \rangle$ is not derived tubular, but $\chi_C = \chi_{C'}$. See [4].

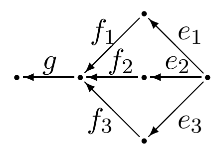
In the case $n = 6$, a complete list of all derived tubular algebras (strongly simply connected or not) was obtained in [4]. We reproduce in Fig. 1 this list.

$\mathcal{A}_\rho$: $(\rho \neq 0, 1)$



$f_1e_1 + f_2e_2 + f_3e_3 = 0$
$f_1e_1 + f_2e_2 + \rho f_4e_4 = 0$

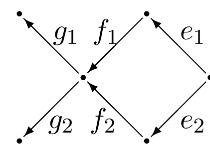
$\mathcal{B}_\rho$: $(\rho \neq 0, 1)$



$f_1e_1 + f_2e_2 + f_3e_3 = 0$
$g(f_1e_1 + \rho f_2e_2) = 0$

$\mathcal{B}_\rho^{\mathrm{op}}$

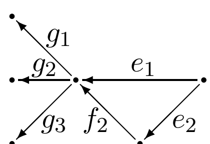
$\mathcal{C}_\rho$: $(\rho \neq 0, 1)$



$g_1(f_1e_1 + f_2e_2) = 0$
$g_2(f_1e_1 + \rho f_2e_2) = 0$

$\mathcal{C}_\rho^{\mathrm{op}}$

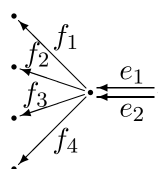
$\mathcal{D}_\rho$: $(\rho \neq 0, 1)$



$g_1e_1 = 0$
$g_2(e_1 + f_2e_2) = 0$
$g_3(e_1 + \rho f_2e_2) = 0$

$\mathcal{D}_\rho^{\mathrm{op}}$

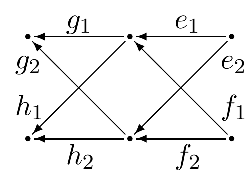
$\mathcal{E}_\rho$: $(\rho \neq 0, 1)$



$f_1e_1 = 0$
$f_2e_2 = 0$
$f_3(e_1 + e_2) = 0$
$f_4(e_1 + \rho e_2) = 0$

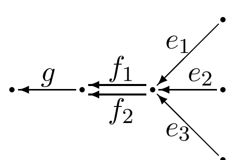
$\mathcal{E}_\rho^{\mathrm{op}}$

$\mathcal{F}_\rho$: $(\rho \neq 0, 1)$



$g_1e_1 + g_2e_2 = 0$
$h_1e_1 + h_2e_2 = 0$
$g_1f_1 + g_2f_2 = 0$
$h_1f_1 + \rho h_2f_2 = 0$

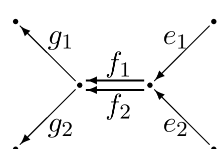
$\mathcal{G}_\rho$: $(\rho \neq 0, 1)$



$gf_1 = 0$
$f_2e_1 = 0$
$(f_1 + f_2)e_2 = 0$
$(f_1 + \rho f_2)e_3 = 0$

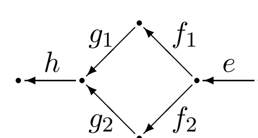
$\mathcal{G}_\rho^{\mathrm{op}}$

$\mathcal{H}_\rho$: $(\rho \neq 0, 1)$



$f_1e_1 = 0$
$f_2e_2 = 0$
$g_1(f_1 + f_2) = 0$
$g_2(f_1 + \rho f_2) = 0$

$\mathcal{I}_\rho$: $(\rho \neq 0, 1)$



$h(g_1f_1 + g_2f_2) = 0$
$(g_1f_1 + \rho g_2f_2)e = 0$

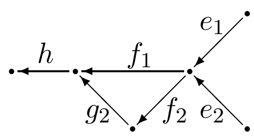
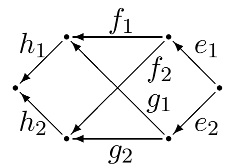
$\mathcal{J}_\rho$: $(\rho \neq 0, 1)$		$\mathcal{K}_\rho$: $(\rho \neq 0, 1)$
	$\mathcal{J}_\rho^{\mathrm{op}}$	
$f_1 e_1 = 0$ $(f_1 + g_2 f_2) e_2 = 0$ $h(f_1 + \rho g_2 f_2) = 0$		$h_1 f_1 + h_2 f_2 = 0$ $h_1 g_1 + h_2 g_2 = 0$ $f_1 e_1 + g_1 e_2 = 0$ $f_2 e_1 + \rho g_2 e_2 = 0$

Fig. 1

1.8. We need to recall some central steps of the *proof of Theorems* (1.6) and (1.7) with the purpose of sketching how condition (2) is used. We consider only the sufficiency of the conditions in Theorem (1.7).

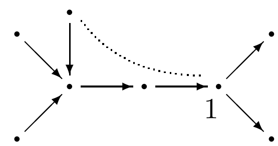
We assume that $A = kQ/I$ is strongly simply connected and such that Q has $n > 6$ vertices. Assume moreover that conditions (1) and (2) are satisfied. Choose a a source or sink in Q such that $Q \setminus \{a\}$ is connected, say a is a source. Then $A = B[M]$ is a one-point extension for $B = A/(a)$ and $M = \operatorname{rad} P_a$. Since A is strongly simply connected, M is indecomposable and B is again strongly simply connected. Clearly, χ_B is non-negative and $\operatorname{corank} \chi_B \geq 1$ (indeed, if v and w are generators of $\operatorname{rad} \chi_A$, we may always choose w satisfying $w(a) = 0$). Consider $p = [P_a] \in K_0(A)$ which is a root of χ_A. Since $\langle p, v \rangle_A = v(a)$ for any vector $v \in K_0(A)$ and by assumption (2) we have $\langle p, \operatorname{rad} \chi_A \rangle_A \neq 0$, thus $\operatorname{corank} \chi_B = 1$. By (1.4), there is a hereditary algebra $H = k\Delta$ of extended-Dynkin type which is derived equivalent to B. The argument [3,(5.4)] shows that the non-negativity of χ_A implies the existence of a derived equivalence $F\colon D^b(B) \to D^b(H)$ such that $FM \in \operatorname{mod}_H$ and F extends to an equivalence $\hat{F}\colon D^b(B[M]) \to D^b(H[FM])$. In particular $\chi_{H[FM]}$ is non-negative. By [21] and condition (1), only the following situations are possible:

(a) FM is a simple regular module and $H[FM]$ is a tubular algebra;

(b) Δ is of type $\tilde{\mathbb{D}}_{n-2}$, FM is a regular module of regular length 2 in the tube of rank $n-4$ (≥ 3) in the Auslander-Reiten quiver Γ_H. In this case $H[FM]$ is said to be 2-tubular.

Finally, condition (2) is not satisfied by 2-tubular algebras (see below) and the proof is completed.

As illustration of the last step above, we consider a 'typical' 2-tubular algebra A:



It is easy to see that χ_A is non-negative of corank $\chi_A = 2$ but $e_1 \in \chi_A^{-1} \cap (\mathrm{rad}\, \chi_A)^{\perp}$.

2. Reduction of integral quadratic forms.

2.1. An *integral quadratic form* q is a map $q\colon \mathbb{Z}^n \to \mathbb{Z}$ of the shape

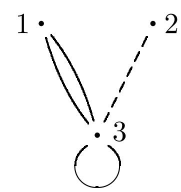
$$q(x_1,\dots,x_n) = \sum_{i=1}^{n} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$$

where $q_i \in \mathbb{Z}$ and $q_{ij} \in \mathbb{Z}$ for all $i, j \in \{1,\dots,n\}$. We say that q is a *semi-unit form* if $q_i \in \{0,1\}$ for all $i \in \{1,\dots,n\}$. Sometimes we also will consider the special case where $q_i = 1$ for all $i \in \{1,\dots,n\}$ and then say that q is a *unit form.*

With a semi-unit form q we associate a bigraph G_q (see for example [16,19]). We illustrate this in the following example:

$$q(x_1,x_2,x_3) = x_1^2 + x_2^2 - 2x_1x_3 + x_2x_3$$

G_q :



In particular, we say that q is connected if the graph G_q is connected. Conversely, any bigraph G defines a quadratic form $p_G : \mathbb{Z}^{G_0} \to \mathbb{Z}$, where G_0 denotes the set of vertices of G.

In the following observations we will deal with semi-unit forms which are weakly non-negative (or even weakly positive). Observe that this implies $q_{ij} \in \{-1,-2\}$ whenever $q_{ij} < 0$.

2.2. In [16], a reduction procedure for semi-unit forms was introduced which allows to verify the weak positivity or weak non-negativity of forms.

Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form and $i \neq j$ be indices such that $q_{ij} < 0$. Define the form $q'\colon \mathbb{Z}^{n+1} \to \mathbb{Z}$ by

$$q'(y) = q\rho(y) + y_i y_j$$

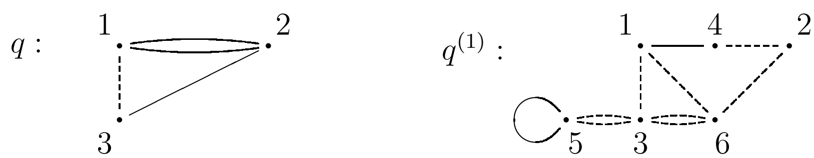
where $\rho\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ denotes the linear map sending $e_s \mapsto e_s$ if $1 \le s \le n$ and $e_{n+1} \mapsto e_i + e_j$. Then q' is said to be obtained from q by *edge reduction* with respect to i and j.

Proposition. [16,18] *Let* $q: \mathbb{Z}^n \to \mathbb{Z}$ *be a semi-unit form and* q' *be obtained from* q *by edge reduction. Then*

(a) *q is weakly positive if and only if q' is weakly positive. In this case, ρ induces a bijection between the set of positive roots $\sum^1(q) = q^{-1}(1) \cap \mathbb{N}^n$ and $\sum^1(q')$.*

(b) *q is weakly non-negative if and only if q' is weakly non-negative. In this case, ρ induces a bijection between the set of positive isotropic vectors $\sum^0(q) = q^{-1}(0) \cap \mathbb{N}^n$ and $\sum^0(q')$.*

2.3. We say that an iterated edge reduction of a semi-unit form $q: \mathbb{Z}^n \to \mathbb{Z}$ is *exhaustive* if every reduction step only involves reductions with respect to indices $\leq n$ and the resulting semi-unit form q' satisfies $q'_{ij} \geq 0$ for all $1 \leq i < j \leq n$. For instance the unit form q in the example below admits an exhaustive reduction with respect to the sequence (of couples of vertices) $\{2,3\}$, $\{1,2\}$, $\{1,2\}$, the result is indicated as $q^{(1)}$:



Theorem. [18] *Let* $q: \mathbb{Z}^n \to \mathbb{Z}$ *be a semi-unit form and* $q^{(k)}: \mathbb{Z}^{n_k} \to \mathbb{Z}$, $k = 0, 1, 2 \ldots$ *be a sequence of semi-unit forms obtained by iterated exhaustive reductions of* q.

(a) *q is weakly positive if and only if there is some $k \leq 30$ such that $q_{ij}^{(k)} \geq 0$ for all $1 \leq i < j \leq n_k$*

(b) *q is weakly non-negative if and only if $q_i^{(k)} \geq 0$ for all $k \leq 31$ and $1 \leq i \leq n_k$.*

2.4. We recall that a semi-unit form $q: \mathbb{Z}^n \to \mathbb{Z}$ is called *critical* if either $n = 1$ and $q(x_1) = 0$, $n = 2$ and $q(x_1, x_2) = (x_1 - x_2)^2$ or $n \geq 3$ and q is not weakly positive but every restriction $q^{(i)} = q(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ is weakly positive. In the latter case, q is non-negative and there is a vector $z_0 \in \mathbb{N}^n$ such that $\operatorname{rad} q = \mathbb{Z} z_0$ (for $n = 1$, $z_0 = (1)$; for $n = 2$, $z_0 = (1,1)$). See [26].
Similarly, the semi-unit form $q: \mathbb{Z}^n \to \mathbb{Z}$ with $n \geq 3$ is called *hypercritical* if q is not weakly non-negative but every restriction $q^{(i)}$ is weakly non-negative. Clearly a semi-unit form q is weakly positive (resp. weakly non-negative) if and only if there is no restriction $q|_I$ of q which is critical (resp. hypercritical). Observe that a critical semi-unit form is already a unit form. Critical and hypercritical unit forms have been classified [15,17].
A strongly simply connected algebra A is said to be a *critical algebra* (resp. *hypercritical algebra*) if its Tits form q_A is critical (resp. hypercritical). These algebras have also been classified [14,28]. Their importance is due to the following:

Theorem. *Let A be a strongly simply connected algebra. Then*

(a) [6] *q_A is weakly positive (equivalently, A is representation-finite) if and only if A does not contain a convex critical subcategory.*

(b) [22] *q_A is weakly non-negative if and only if A does not contain a convex hypercritical subcategory.*

For other results related to these problems the reader may see [24]. Criteria as the above Theorem are normally well-adapted for eye-checking but are very slow when run in a computer. The implementations of (2.3) run much faster.

2.5. Clearly, if $\operatorname{gldim} A \leq 2$, then the quadratic forms q_A and χ_A coincide. Most of the applications of these forms reduces to this case when the homological information contained in χ_A may be compared with the combinatorial information related to q_A. In the general case there are still some relations.

Lemma. [25] *Let A be a strongly simply connected algebra. If χ_A is weakly positive (resp. weakly non-negative), then q_A is also weakly positive (resp. weakly non-negative).*

Proof: Assume q_A is not weakly positive (resp. not weakly non-negative). Since A is strongly simply connected, then by (2.4), A contains a convex subcategory B which is critical (resp. hypercritical). Since $\operatorname{gldim} B \leq 2$, then $q_B = \chi_B$ is not weakly positive (resp. not weakly non-negative) which implies that χ_A is not weakly positive (resp. not weakly non-negative). □

2.6. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. A vector $v \in \mathbb{N}^n$ is called *critical* for q if the restriction q^v of q to the support $\operatorname{supp} v = \{i\colon v(i) \neq 0\}$ is a critical form and $\operatorname{rad} q^v = \mathbb{Z}v$. The set of critical vectors for q is denoted by C_q.

Lemma. [23] *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form.*

(a) *If q is weakly non-negative, then every vector $w \in \sum^0(q)$ may be written as $w = \sum_{v \in C_q} \lambda_v v$ for some numbers $\lambda_v \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the non-negative rational numbers.*

(b) *If q is non-negative, then the dimension of the space generated by $\sum^0(q)$ is the maximal number of linearly independent vectors in C_q (this number is called the* positive corank *of q and it is denoted by* $\operatorname{corank}^+ q$*).*

(c) *If q is non-negative and there is a sincere vector $v \in \sum^0(q)$, then* $\operatorname{corank} q = \operatorname{corank}^+ q$.

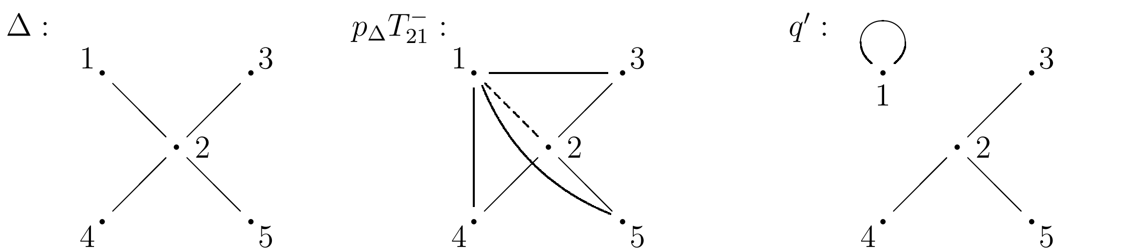
2.7. We introduce some operations on quadratic forms, the so called deflations and inflations which have been successfully applied for various problems [20,15,19].
Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form with $q_{ij} \neq 0$ and set $\varepsilon = -$ (resp. $\varepsilon = +$) if $q_{ij} < 0$ (resp. $q_{ij} > 0$). Then we define the invertible $\mathbb{Z}$-matrix $T^{\varepsilon}_{ij}\colon \mathbb{Z}^n \to \mathbb{Z}$ by the rule

$$T^{\varepsilon}_{ij}(e_s) = \begin{cases} e_s & \text{if } s \neq i \\ e_i - \varepsilon e_j & \text{if } s = i \end{cases}$$

We say that T^{-}_{ij} is a *deflation* (resp. T^{+}_{ij} is an *inflation*) for q.
In comparison to the edge reductions which preserve weak positivity and weak non-negativity, deflations and inflations preserve positivity and non-negativity. The following result shown in [19] is a generalization of $\operatorname{corank} q = 1$, considered in [20].

Theorem. [19] *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative semi-unit quadratic form. Then* $\operatorname{corank} q = \operatorname{corank}^{+} q = s$ *if and only if there is an iteration of deflations with composition T such that the form $q' = qT$ is the orthogonal sum $q^0 \oplus q^1$ of two forms such that q^0 is the zero form in s variables and q^1 is a positive definite form in $n - s$ variables.*

Example: We consider the quadratic form p_Δ associated to the following (extended Dynkin) diagram



Applying to p_Δ the transformation T^{-}_{21} we get the quadratic form associated to the bigraph in the middle. The sequence $(T^{-}_{21}, T^{-}_{31}, T^{-}_{41}, T^{-}_{51}, T^{-}_{21})$ applied to p_Δ yields the form $q' = qT^{-}_{21}T^{-}_{31}T^{-}_{41}T^{-}_{51}T^{-}_{21}$ in the variables x_1, x_2, x_3, x_4, x_5 but since x_1 does not appear explicitely, we have $q' = q^0 \oplus q^1$ where $q^1(x_2, x_3, x_4, x_5) = q'(0, x_2, x_3, x_4, x_5)$ and $q^0 = 0$. The form q^1 is positive definite and $\operatorname{corank} q = \operatorname{corank}^{+} q = 1$.

2.8. We denote by ζ the zero form in one variable.

Lemma. *Let $q \neq 0$ be a connected non-negative semi-unit form, T be a deflation or an inflation and $q' = qT$. If q' decomposes properly, say $q' = q^0 \oplus q^1$ with $q^1 \neq 0$ then $q^0 = \zeta$ and q^1 is a connected non-negative semi-unit form.*

Proof: Let $T = T_{ij}^{\varepsilon}$ be deflation or an inflation. Let $G_{q'}$ be the bigraph associated to q' and assume it is not connected. Since G_q is connected, then $G_{q'}$ has exactly two components, C_0 containing j and C_1 containing i. In particular, $q'_{ij} = 0$.
An obvious calculation yields:

$$0 = q'_{ij} = q_{ij} + 2\varepsilon \quad \text{and} \quad q'_j = q_i\varepsilon^2 + q_{ij}\varepsilon + q_j = q_i + q_j - 2$$

As observed in [19], q' is still a semi-unit form and hence $q_i = 1 = q_j$ and $q'_j = 0$. For any $s \neq j$ we should have $q'_{sj} = 0$, otherwise $q'(2e_j - q'_{sj}e_s) < 0$ which contradicts the non-negativity of q'. Therefore, C_0 contains only the vertex j and $q' = q^0 \oplus q^1$, where q^i is the form associated to C_i, $i = 0, 1$. Clearly, q^0 is the zero form in one variable. □

2.9. We present now a generalization of (2.7) which will be central in the proof of our main result (compare also with [29]).

Theorem. *Let q be a connected, non-negative semi-unit form of corank s. Then there exists an iteration of deflations and inflations with composition T such that qT is the orthogonal sum $\zeta^s \oplus p_\Delta$, where ζ^s is the zero form in s variables and p_Δ is the quadratic form associated to a (connected) Dynkin diagram Δ (hence p_Δ is positive definite). Moreover, Δ is uniquely determined by q.*

Proof: We proceed by induction on $s = \operatorname{corank} q$. If $s = 0$ the form q is already positive definite. Hence by [20], there exists a sequence of inflations with composition T such that $qT = p_\Delta$ for some Dynkin diagram Δ. By (2.8), Δ is connected.
For the induction step we first show in (a) that there is a sequence of inflations with composition T_a such that the radical of the form $q^a = qT_a$ contains a positive vector v_a and then in (b) we prove that there exists a sequence of deflations with composition T_b such that q^aT_b is the orthogonal sum of the zero form in one variable with a non-negative connected semi-unit form of corank $s - 1$.
For any $v \in \mathbb{Z}^n$ we define $|v| = \sum_{i=1}^n |v(i)|$, $\operatorname{supp} v = \{i : v(i) \neq 0\}$ and $v^+, v^- \in \mathbb{Z}^n$ by

$$v^\varepsilon(i) = \begin{cases} \varepsilon v(i) & \text{if } \varepsilon v(i) > 0 \\ 0 & \text{else} \end{cases} \quad (\varepsilon = \pm)$$

(a) Choose $v \in \operatorname{rad} q$, $v \neq 0$. If there are $i \in \operatorname{supp} v^+$ and $j \in \operatorname{supp} v^-$ such that $q_{ij} > 0$ then we apply T_{ji}^+ to q obtaining $\tilde{q} = qT_{ji}^+$ and $\tilde{v} = (T_{ji}^+)^{-1}v$. Observe that $\tilde{v} = v + v(j)e_i$, thus $|\tilde{v}^-| < |v^-|$. We repeat this procedure until this is no longer possible and obtain a quadratic form q^a and a radical vector v'_a. Now we have

$$0 = q^a(v'_a) = q^a({v'_a}^+) + q^a({v'_a}^-) + \sum_{(i,j)} (q^a)_{ij} v'_a(i) v'_a(j)$$

where the sum runs over $\operatorname{supp} {v'_a}^+ \times \operatorname{supp} {v'_a}^-$. Any summand on the right hand side is at least zero hence equals zero. So we get step (a) by setting $v_a = {v'_a}^+ + {v'_a}^-$.
(b) If there exist $i, j \in \operatorname{supp} v_a$ with $(q^a)_{ij} < 0$ we apply $T = T^-_{ij}$ if $v_a(i) \geq v_a(j)$ (or $T = T^-_{ji}$ if $v_a(i) < v_a(j)$). Observe that for $\tilde{q}_a = q^a T$ and $\tilde{v}_a = T^{-1} v_a$ we have that $\tilde{v}_a$ is positive again with $|\tilde{v}_a| < |v_a|$. We repeat this procedure as long as possible and end up with a quadratic form q^b and a positive radical vector v_b. Then we have

$$0 = q^b(v_b) = \sum_{i=1}^{n} (q^b)_i v_b(i)^2 + \sum_{i,j \in \operatorname{supp} v_b} (q^b)_{ij} v_b(i) v_b(j)$$

where any summand on the right hand side is non-negative, hence zero. In particular $(q^b)_i = 0$ whenever $v_b(i) \neq 0$. This proves step (b). □

2.10. Lemma *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative semi-unit form. Let $S, T\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be $\mathbb{Z}$-invertible such that $qS = \zeta^s \oplus p_\Sigma$ and $qT = \zeta^t \oplus p_\Delta$ where $\Sigma = \bigcup_{i=1}^{a} \Sigma_i$ and $\Delta = \bigcup_{j=1}^{b} \Delta_j$ are disjoint unions of Dynkin graphs, then we have $s = t$, $a = b$ and $\Sigma_i = \Delta_{\pi(i)}$ for $i = 1, \ldots, a$ where π is a permutation.*

Proof: Since p_Σ is positive definite, s is the corank of qS hence $s = t$.
Let $C_\Sigma = q_\Sigma^{-1}(1) \subset \mathbb{Z}^{n-s}$ and $G(C_\Sigma)$ be the graph having as points the elements of C_Σ and an edge x —— y if $x \pm y \in C_\Sigma \cup \{0\}$ for $x, y \in C_\Sigma$. The connected components of $G(C_\Sigma)$ are exactly the graphs $G(C_{\Sigma_i})$ for $i = 1, \ldots, a$. The map $T \circ S^{-1}$ induces a bijection $C_\Sigma \xrightarrow{\sim} C_\Delta$ which respects the associated graph structure, i.e. $T \circ S^{-1}$ induces an isomorphism $G(C_\Sigma) \xrightarrow{\sim} G(C_\Delta)$, hence $a = b$ and there exists a permutation π such that $G(C_{\Sigma_i}) \xrightarrow{\sim} G(C_{\Delta_{\pi(i)}})$. Therefore $\Sigma_i \xrightarrow{\sim} \Delta_{\pi(i)}$ holds for every i. □

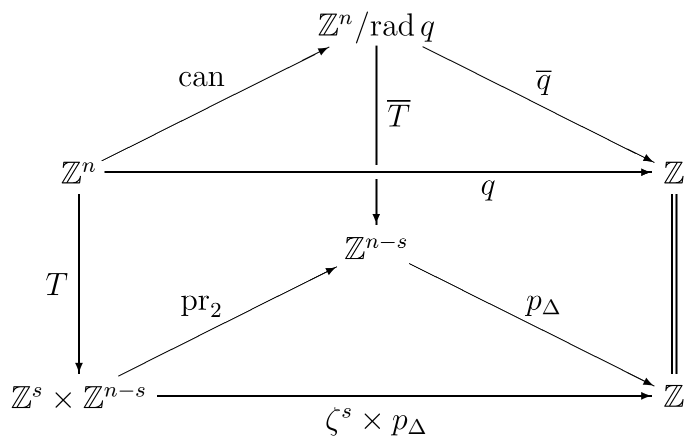
2.11. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative semi-unit form. Then $\operatorname{rad} q$ is a pure subgroup of $\mathbb{Z}^n$ (that is, if $n \in \mathbb{Z} \setminus \{0\}$ and $nv \in \operatorname{rad} q$, then $v \in \operatorname{rad} q$). Hence $\mathbb{Z}^n / \operatorname{rad} q \xrightarrow{\sim} \mathbb{Z}^{n-s}$ as $\mathbb{Z}$-modules where $s = \operatorname{corank} q$. Thus we may consider the induced map

$$\bar{q}\colon \mathbb{Z}^n / \operatorname{rad} q \to \mathbb{Z}, v + \operatorname{rad} q \mapsto q(v)$$

which is, by choosing a $\mathbb{Z}$-base of $\mathbb{Z}^n / \operatorname{rad} q$, a positive definite form.
The following result, which (surprisingly) seems to be new, answers the question whether there is a basis of $\mathbb{Z}^n / \operatorname{rad} q$, which makes $\bar{q}$ a unit form, affirmatively.

Theorem. *Let q be a non-negative semi-unit form. Then there exists a disjoint union Δ of Dynkin graphs Δ_i $(i = 1, \ldots, n)$ such that $\bar{q}$ is $\mathbb{Z}$-equivalent to p_Δ. Moreover n is the number of connected components of q and Δ is (up to the order of the Δ_i) uniquely determined by q.*

Proof: By (2.9), there is an invertible $\mathbb{Z}$-matrix $T\colon \mathbb{Z}^n \to \mathbb{Z}^n$ such that $qT^{-1} = \zeta^s \oplus p_\Delta\colon \mathbb{Z}^s \times \mathbb{Z}^{n-s} \to \mathbb{Z}$ where Δ is the disjoint union of Dynkin diagrams Δ_i $(i = 1, \dots, n)$ and n is the number of connected components of q. Since $\mathbb{Z}^s = T(\operatorname{rad} q) = \operatorname{rad}(qT^{-1})$ we get an induced $\mathbb{Z}$-isomorphism $\bar{T}\colon \mathbb{Z}^n/\operatorname{rad} q \to \mathbb{Z}^s$ making the following diagram commutative



hence $\bar{q} = p_\Delta \bar{T}$ which is the desired equivalence. The assertion about the uniqueness follows by (2.10). □

Corollary. *Each connected non-negative unit form q determines uniquely a Dynkin graph Δ such that $\overline{q}$ and p_Δ are $\mathbb{Z}$-eqivalent.*

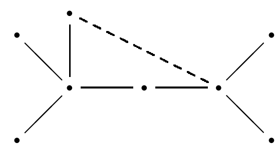
3. Derived tubularity: an algorithm.

3.1. Lemma

(a) *Let C be a canonical tubular algebra of type* $(3,3,3)$, $(2,4,4)$ *or* $(2,3,6)$. *The Dynkin diagram $\Delta(\chi_C)$ associated to the Euler form χ_C as in* (2.9) *is of the form $\mathbb{E}_p, p = 6, 7, 8$, respectively.*

(b) *Let C_0 be a representation-infinite tame concealed algebra of type $\tilde{\mathbb{D}}_n$ and let M be an indecomposable C_0-module regular of regular length* 2 *lying in a tube of rank $n-2$. Consider the one-point extension $B = C_0[M]$. Then the Dynkin diagram $\Delta(\chi_B)$ is of the form $\mathbb{D}_n$.*

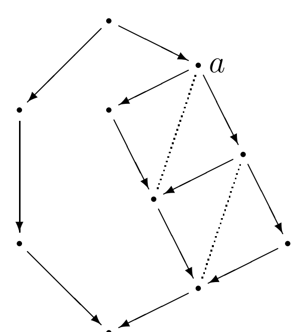
Proof: It is enough to apply deflations and inflations in the way indicated by (2.9). (a) is left as an easy exercise.
(b): Any algebra of the type $B = C_0[M]$ as indicated is tilted (and therefore derived equivalent) to an algebra E whose Euler form χ_E has the following bigraph (we draw the case $n = 6$)



Again, it is an easy exercise to transform this graph to a Dynkin diagram $\mathbb{D}_5$ by means of deflations. $\square$

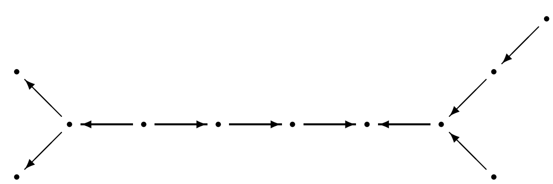
3.2. *Proof of the main Theorem:* Let A be a strongly simply connected algebra of the form $A = kQ/I$ with $Q_0 = \{1, \dots, n\}$ and $n > 6$.
Assume first that A is derived tubular. Then we know that χ_A is non-negative of corank $\chi_A = 2$ and $A = B[M]$ for some representation-infinite tilted connected algebra B such that corank $\chi_B = 1$, see for example [4]. Moreover we may assume that A is derived equivalent to a canonical tubular algebra C of type $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. Then by (2.9) and (3.1), the Dynkin graph $\Delta(\chi_A)$ is equal to $\Delta(\chi_C)$ of type $\mathbb{E}_p$, $p = 6,7,8$, respectively. The number of positive roots of $\bar{\chi}_A \simeq p_{\Delta(\chi_A)}$ is $36, 63, 120$, respectively.
Conversely, assume that the conditions (a), (b) and (c) hold. Choose a sink or source a in Q such that $Q \setminus \{a\}$ is connected and χ_B is non-negative with corank one, where $B = A/(a)$. We assume that a is a source (the case where a is sink, is handled dually), so the B-module $M = \operatorname{rad} P_a$ is indecomposable.
By (1.4), we get that B is derived equivalent to a hereditary algebra $H = k\Delta$, where Δ is an extended-Dynkin diagram of type $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ $(p = 6,7,8)$. As in (1.8), we get a derived equivalence between $A = B[M]$ and $H[N]$ for some regular H-module N. There are two possibilities: either $H[N]$ is a tubular algebra or Δ is of type $\tilde{\mathbb{D}}_m$ and N lies on a tube of rank $n-4$ (≥ 3) in Γ_H and has regular length 2. In the latter case, (3.1) implies that $\Delta(\chi_{H[N]}) = \Delta(\chi_A)$ is a Dynkin diagram of type $\mathbb{D}_n$. But then the number of positive roots of $\bar{\chi}_A = p_{\Delta(\chi_A)}$ is $n(n-1)$ which cannot be 36, 63 or 120. A contradiction showing that $H[N]$ is a tubular algebra and A is derived tubular. $\square$

3.3. We remark that the 'strong simple connectedness' hypothesis is necessary in the main theorem. Namely, consider the algebra $A = kQ/I$ given by the following quiver with commutative relations



Then we have the following facts:

- χ_A is non-negative of corank $\chi_A = 2$;
- the algebra $B = A/(a)$ is tame concealed of type $\tilde{\mathbb{E}}_8$;
- $\Delta(\chi_A)$ is of type $\mathbb{E}_8$ and hence $\bar{\chi}_A$ has 120 positive roots;
- the algebra A is wild since the universal covering $\tilde{A}$ of A contains a convex subcategory of the following form



therefore, A is not derived tubular (as observed in [25], every derived tubular algebra is tame).

3.4. In a preliminary stage of our research we suggested the following *algorithm* to check whether or not $\chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp} = \emptyset$ for a non-negative form χ_A with corank $\chi_A = 2$.

First construct a $\mathbb{Z}$-bases (v_1, v_2) of rad χ_A.

A subset $\mathcal{S}$ of $\chi_A^{-1}(1)$ is said to be *reduced* if $s_1, s_2 \in \mathcal{S}$ and $s_1 - s_2 \in \chi_A^{-1}(0)$ or $s_1 + s_2 \in \chi_A^{-1}(0)$ imply that $s_1 = s_2$. Any reduced set is finite.

Construct a maximal reduced subset $\mathcal{S}$ of $\chi_A^{-1}(0)$. Then the following hold:

- If $\langle v_1, v_2\rangle_A = 0$, then $\chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp} = \emptyset$ if and only if $\langle v_1, s\rangle_A \neq 0$ or $\langle v_2, s\rangle_A \neq 0$ for every $s \in \mathcal{S}$
- If $\langle v_1, v_2\rangle_A \neq 0$, then $\chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp} = \emptyset$ if and only if $\varepsilon_1 \frac{\langle v_2, s\rangle}{\langle v_2, v_1\rangle} v_1 + \varepsilon_2 \frac{\langle v_1, s\rangle}{\langle v_1, v_2\rangle} v_2$ is not in $\mathbb{Z}^n$ for any $s \in \mathcal{S}$ and $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$.

Clearly, the cardinality of $\mathcal{S}$ is the number of positive roots of $\bar{\chi}_A$. It was the implementation of the above algorithm which suggested the result presented in the main theorem.

4. Examples.

We append the list of all derived tubular algebras whose underlying quiver is linearly ordered. The list was calculated by our C++ program.

Each figure represents a class of algebras which is obtained from the given algebra in the picture by a sequence of the following two operations: add a relation $\beta\alpha$, if $\beta\alpha \neq 0$, and change the orientation of all arrows simultaneously. In this way, picture (6) defines 16 non-isomorphic algebras.

For each algebra we give at the right hand side generators of the radical of the Euler form. In order to obtain the generators of the radical of the Euler form of an algebra which is obtained by adding a relation of length two one calculates starting with the given vectors $[v_1\ \ v_2\ \ \cdots\ \ v_n]$ the new ones in the following way:

Add a relation from 2 to 4: $[(v_1-v_2)\ \ -v_2\ \ (v_3-v_2)\ \ v_4\ \ \cdots\ \ v_n]$.

Add a relation from 1 to 3: $[-v_1\ \ (v_2-v_1)\ \ v_3\ \ v_4\ \ \cdots\ \ v_n]$.

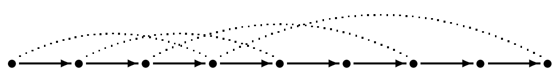
Add a relation from 2 to 4 and a relation from 1 to 3:

$$[(-v_1+v_2)\ \ -v_1\ \ (v_3-v_2)\ \ v_4\ \ \cdots\ \ v_n].$$

Similarily one calculates on the side of the sink of the algebra.

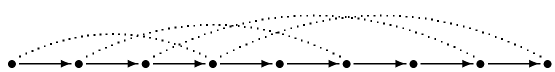
Algebras with 9 points (8 algebras)

(1)



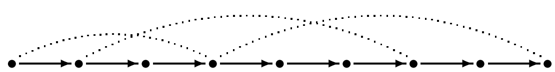
-1	0	1	0	-1	-1	-1	0	1
1	1	-1	-1	0	1	2	1	0

(2)



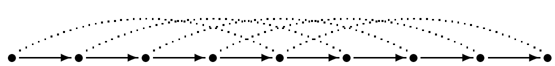
-1	1	1	0	-1	-2	-1	0	1
1	0	-1	-1	0	1	1	1	0

(3)



0	1	0	-1	-1	-1	-1	0	1
1	-1	-1	-1	0	1	2	1	0

(4)



-1	0	1	1	0	-1	-1	0	1
1	1	0	-1	-1	0	1	1	0

(5)

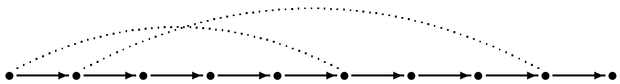
0	1	1	0	-1	-1	-1	0	1
1	0	-1	-1	-1	0	1	1	0

Algebras with 10 points (123 algebras)

(6)

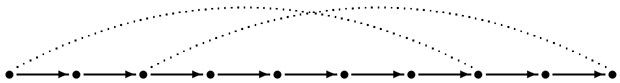
-1	-2	-1	0	1	2	1	0	0	0
0	0	-2	-1	0	1	2	3	2	1

(7)



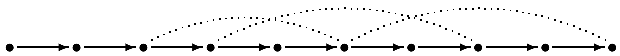
−1 −2 −1 0 1 2 2 2 2 1
−2 −1 0 1 2 3 2 1 0 0

(8)



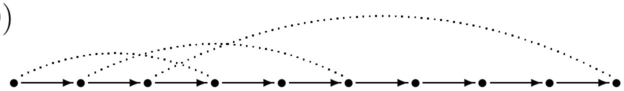
−1 0 1 1 1 1 1 1 0 −1
−2 −2 −2 −1 0 1 2 3 2 1

(9)



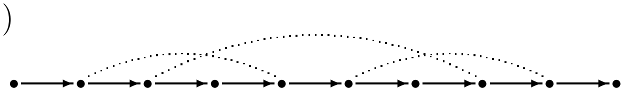
0 0 0 1 1 0 −1 −1 0 1
1 2 3 0 −1 −2 0 2 1 0

(10)



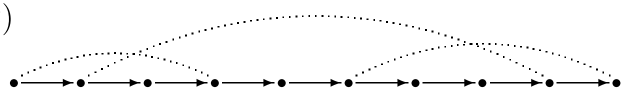
0 1 0 −2 −1 0 2 3 2 1
1 0 −1 −1 0 1 1 1 1 1

(11)



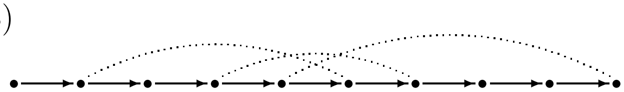
0 0 1 0 −1 −2 −1 0 2 1
1 2 0 −1 −2 −1 0 1 0 0

(12)



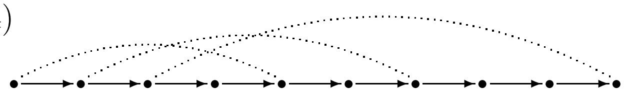
0 −1 0 1 2 3 2 1 0 −2
−1 0 1 2 2 2 1 0 −1 −1

(13)



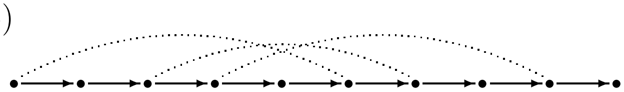
0 0 1 2 0 −1 −2 −1 0 1
1 2 0 −2 −1 0 3 2 1 0

(14)



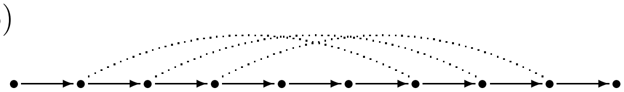
1 0 −1 −1 −1 0 1 1 1 1
0 −2 −1 0 1 2 3 2 1 0

(15)



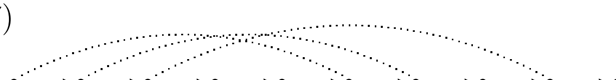
1 0 −1 −2 −1 0 2 2 2 1
0 −1 −2 −1 0 1 2 1 0 0

(16)



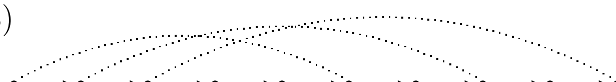
0 0 −1 −2 −1 0 1 2 2 1
1 2 2 1 0 −1 −2 −1 0 0

(17)



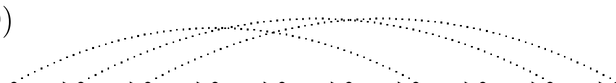
0 −1 −2 −1 0 1 2 2 2 1
−1 −2 −1 0 1 2 2 1 0 0

(18)



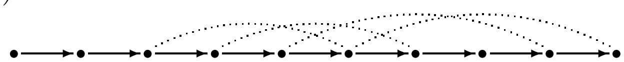
1 0 −1 −1 −1 −1 0 1 1 1
−1 −2 −1 0 1 2 2 2 1 0

(19)



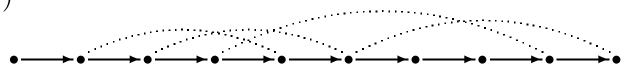
−1 0 1 1 1 1 1 0 −1 −1
−2 −2 −1 0 1 2 3 2 1 0

(20)



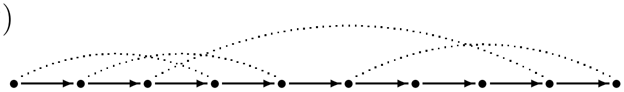
1	2	3	0	-1	0	2	1	0	-2
0	0	0	1	0	-1	-1	0	1	1

(21)



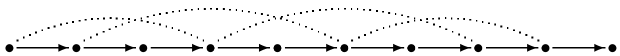
1	2	0	-1	0	3	2	1	0	-2
0	0	1	0	-1	-2	-1	0	1	1

(22)



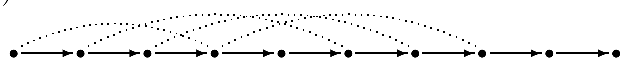
-1	0	1	0	-2	-2	-1	0	2	1
0	1	0	-1	-2	-1	0	1	1	0

(23)



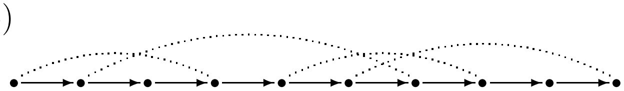
-1	0	1	2	0	-2	-1	0	2	1
0	-1	0	1	1	1	0	-1	0	0

(24)



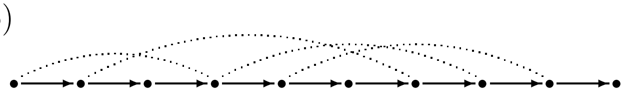
0	1	0	-2	-1	0	2	3	2	1
1	0	-1	-1	0	1	1	0	0	0

(25)



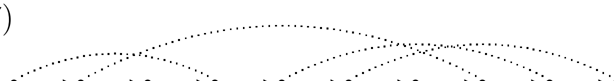
-1	0	1	2	2	0	-1	-1	0	1
0	1	0	-1	-2	-1	0	2	1	0

(26)



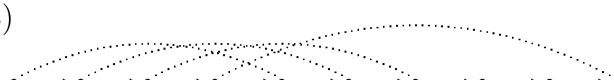
0	1	0	-1	-2	-1	0	2	2	1
1	0	-1	-2	-1	0	1	1	0	0

(27)



0	-1	0	1	2	2	1	0	-2	-1
-1	0	1	2	2	1	0	-1	-1	0

(28)



0	-1	-2	0	1	2	2	1	0	-1
1	0	-2	-1	0	2	3	2	1	0

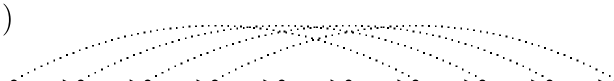
(29)

0	1	2	2	0	-1	-2	-1	0	1
1	0	-1	-2	-1	0	2	2	1	0

(30)

1	1	0	-1	-1	-1	0	1	1	1
0	-1	-2	-1	0	1	2	2	1	0

(31)

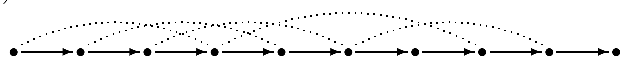


0	1	2	2	1	0	-1	-2	-2	-1
-1	-2	-2	-1	0	1	2	2	1	0

(32)

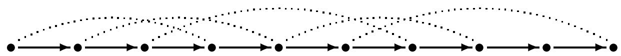
0	0	1	0	-1	0	1	1	0	-1
-1	-2	0	1	0	-1	0	2	1	0

(33)



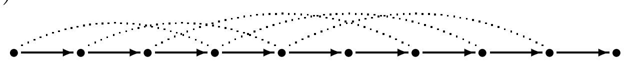
$$\begin{array}{cccccccccc} 0 & -1 & 0 & 1 & & -2 & -1 & 0 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \end{array}$$

(34)



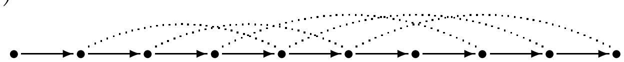
$$\begin{array}{cccccccccc} 0 & -1 & 0 & 1 & 2 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -2 & -1 & 0 & 2 & 1 & 0 \end{array}$$

(35)



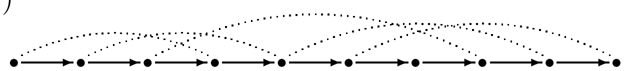
$$\begin{array}{cccccccccc} -1 & -2 & 1 & 2 & 0 & -1 & -2 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

(36)



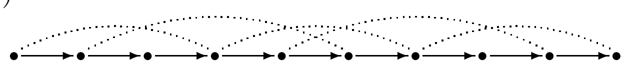
$$\begin{array}{cccccccccc} 1 & 2 & 0 & -1 & 0 & 2 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \end{array}$$

(37)



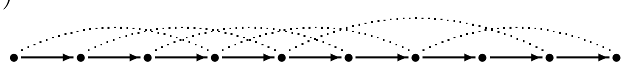
$$\begin{array}{cccccccccc} -1 & 0 & 1 & 0 & -2 & -2 & -1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & -2 & -1 & 0 & 1 & 1 & 0 \end{array}$$

(38)



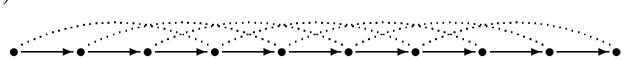
$$\begin{array}{cccccccccc} 0 & 1 & 0 & -1 & -1 & 0 & 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 2 & 0 & -1 & -1 & 0 & 1 & 0 \end{array}$$

(39)



$$\begin{array}{cccccccccc} 1 & 0 & -1 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}$$

(40)



$$\begin{array}{cccccccccc} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{array}$$

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