REPRESENTATION-FINITE DERIVED TUBULAR ALGEBRAS

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Abstract. The use of quadratic forms as a tool for characterizing classes of algebras is well known and widely accepted. However, untill now a characterization of the interesting class of derived tubular algebras, by properties of the (non-symmetric) Euler form only, failed because of the existence of so-called pg-critical algebras, whose Euler form manifest similar properties as those of tubular algebras. The present article fills this gap and provides a distinction between derived tubular algebras and derived pg-critical algebras. Furthermore, an explicit characterization of representation-finite derived tubular algebras in terms of their Euler forms will be given.

1. Known properties of tubular algebras, the main result

For definitions and more details we refer, unless otherwise stated, in this section always to [13]. By an algebra we mean a finite-dimensional, basic algebra over an algebraically closed field k, and by a module we mean a finite dimensional left module.

Tubular algebras were introduced by Ringel in [13], they are branch-enlargements of tame concealed algebras with *tubular type* $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. Two tubular algebras, A and B, are *derived equivalent*, that is their derived categories $\mathrm{D}^{\mathrm{b}}(A)$ and $\mathrm{D}^{\mathrm{b}}(B)$ are triangular equivalent [7], if and only if they have the same tubular type [8]. Therefore we may speak of the tubular type of a *derived tubular* algebra, that is an algebra which is derived equivalent to a tubular algebra.

In [2], it was shown, that for a derived tubular algebra A there exists a tubular algebra T such that A and T are even *reflection-equivalent*, that is A may be obtained from T by a series of source- and sink-reflections, or equivalently, their repetitive categories are equivalent [9].

Tubular algebras are *directed*, that is, their ordinary quiver Q_A contains no cycle, their global dimension is always 2. In the forthcoming we shall identify the Grothendieck group $\mathrm{K}_\circ(A)$ with $\mathbb{Z}^n$, where n is number of points of Q_A, and the class of an A-module X, denoted by $\underline{\dim}\, X \in \mathrm{K}_\circ(A)$ with the dimension vector of the corresponding representation of Q_A.

We denote by $\langle -,- \rangle_A : \mathrm{K}_\circ(A) \times \mathrm{K}_\circ(A) \to \mathbb{Z}$ the homological bilinear form of an algebra of finite global dimension and by $\chi_A : \mathrm{K}_\circ(A) \to \mathbb{Z}$ the associated quadratic form, called *Euler-Poincaré characteristic*, or shortly *Euler form*, whereas the *Tits form* will be denoted by q_A.

If χ_A is non-negative, that is $\chi_A(v) \geq 0$ for all $v \in \mathrm{K}_\circ(A)$, then $\mathrm{rad}\,\chi_A$ equals $\chi_A(0)^{-1}$ and is a direct summand of $\mathrm{K}_\circ(A)$. In this case the *corank* of χ_A is

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defined as $\operatorname{corank}\chi_A = \operatorname{rank}(\operatorname{rad}\chi_A)$. Further, we define the *right orthogonal space* of a subset $U \subset \mathrm{K}_\circ(A)$ to be $U^\perp = \{v \in \mathrm{K}_\circ(A) \mid \forall u \in U, \langle u, v\rangle_A = 0\}$.

In the following we consider an algebra A as a *spectroid*, that is a k-category where distinct objects are non-isomorphic, morphism spaces are finite dimensional and endomorphism algebras are local, see [10]. We say that a full subcategory B of A is *convex* in A if the quiver Q_B is path closed in Q_A and by abusing the language we say then that B is a *full subalgebra* of A. An algebra A is called *strongly simply connected* if for every convex algebra B in A, the first Hochschild cohomology vanishes [14].

The main result of this paper is the following.

Main Theorem. *Let A be a finite-dimensional, connected algebra over an algebraically closed field. Then A is of derived tubular and representation-finite if and only if A satisfies the following five conditions:*

(i) Q_A *has more than six points,*
(ii) χ_A *is non-negative and has corank two,*
(iii) $\chi_A^{-1}(1) \cap (\operatorname{rad}\chi_A)^\perp = \phi$
(iv) *the Tits form q_A does not admit a positive isotropic root, and*
(v) *A is strongly simply connected.*

The article is organized as follows. Section 2 and 3 deal with the conditions (i) and (iii) respectively. The proof of the Main Theorem is given in section 4.

This work is part of my Ph.D. thesis done at UNAM under the supervision of J. A. de la Peña. Results of this work have been used in further developpments [4].

2. Derived tubular algebras of type $(2, 2, 2, 2)$

2.1. The following result is a special case of a result in [11]. The proof we present is more elemetary.

Proposition. *A derived tubular algebra of type* $(2, 2, 2, 2)$ *is representation-infinite.*

2.2. We shall need the following lemma.

Lemma. *Let A be an algebra such that there exist two points, x and y, in A with $\dim_k A(x, y) \geq 2$. Then all algebras which are reflection-equivalent to A are of infinite representation type.*

Proof. We denote the repetitive category of an algebra A by $\widehat{A}$ and their points by $a[i]$, for $a \in A$ and $i \in \mathbb{Z}$. Let B be a algebra, which is reflection-equivalent to A. Then by [9], B can be considered as a full, complete slice of $\widehat{A}$, hence there exist integers i_x and i_y such that $x[i_x]$ and $y[i_y]$ belong to B. By the convexity of B in $\widehat{A}$ follows, that either $i_x = i_y$ or $i_x = i_y - 1$. In both cases there exist two points, x' and y', in B such that $\dim_k B(x', y') \geq 2$. The full subalgebra of B given by this two points is therefore of representation-infinite, and hence so is B itself. □

2.3. *Proof of Proposition 2.1.* Let A be a derived tubular algebra of tubular type $(2, 2, 2, 2)$. By [2], there exists a tubular algebra T, which is reflection-equivalent to A. Let C be a tame concealed algebra such that T is a branch-source-extension of C. Clearly C has one of the following tubular types: $()$, (2), $(2, 2)$ or $(2, 2, 2)$. In the first two cases there exist two points, x and y, in C such that $\dim_k C(x,y) \geq 2$, so the assertion follows by (2.2). In the last case, C is hereditary with underlying graph $\mathbb{D}_4$, $T = C[M]$ (one-point source extension), where M is an indecomposable C-module lying in a homogeneous tube. Since for all possible orientations we have $\dim_k M(c) = 2$, if c denotes the "center-point" of $\mathcal{C}$, we obtain $\dim_k T(\alpha, c) = 2$, where α denotes the extension vertex of $T[M]$, and the assertion follows again by (2.2). In the remaining case C is hereditary with one of the two quivers in Figure 1,

Figure 1

from which the left hand case can be handled as before. So it remains to consider the case where C is hereditary and has as quiver the one in Figure 1 on the right hand side. In order to obtain T from C, the latter must be extended by two non-isomorphic modules, M_λ and M_μ, both lying in homogeneous tubes. Hence $\mathcal{T}$ is isomorphic to $k\mathrm{Q}/\mathrm{I}_\rho$ where the quiver Q looks as depicted in Figure 2 on the left hand side,

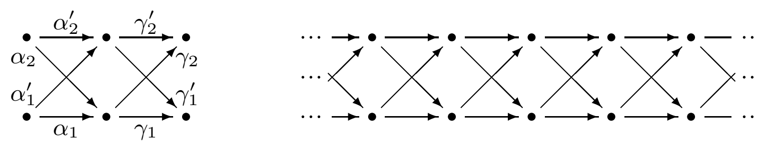

Figure 2

and the ideal I_ρ is generated by $\alpha_1\gamma_1 - \alpha_1'\gamma_1'$, $\alpha_1\gamma_2 - \alpha_1'\gamma_2'$, $\alpha_2\gamma_1 - \alpha_2'\gamma_1'$ and $\alpha_2\gamma_2 - \rho\cdot\alpha_2'\gamma_2'$ for some $\rho \in k$ with $\rho \neq 0, 1$. The quiver of its repetitive algebra is depicted on the right hand side in Figure 2. By [9], A is a full complete slice of $\widehat{\mathcal{T}}$ and, by the convexity, it follows that $\mathcal{R}$ contains a full complete subalgebra, which is isomorphic to $\mathcal{C}$. Therefore B is of infinite representation type. □

3. Distinction Lemma

3.1. In order to charaterize the derived tubular algebras of finite representation-type, we may restrict to algebras with more than six points, see (2.1). A 2-*tubular* algebra A is a one-point source extension $A = \mathcal{D}[M]$, where $\mathcal{D}$ is domestic tubular of tubular type $(2, 2, t)$ for some $t \geq 2$, and M an indecomposable, regular $\mathcal{D}$-module of colength two (that is M lies on the second position in its coray), which belongs to the tube $\mathcal{T}_\lambda$ of rank t.

Distinction Lemma. *Let A be a finite algebra with more than six points.*

(a) *If A is derived tubular, then* $\chi_A^{-1}(1) \cap (\mathrm{rad}\,\chi_A)^\perp = \emptyset$.

(b) *If A is derived 2-tubular, then* $\chi_A^{-1}(1) \cap (\mathrm{rad}\,\chi_A)^\perp \neq \emptyset$.

For algebras with six points the statement (b) of the Distinction Lemma fails [4]. The proof of the Distinction Lemma is given in 3.7.

3.2. Denote by N the indecomposable, regular $\mathcal{D}$-module, which lies on the same coray than M and has colength one.

Let $\alpha = \alpha_M$, the additional point of the source-extension $A = \mathcal{D}[M]$. Furthermore denote τ_A the Auslander-Reiten-translation in $\operatorname{ind} A$, and $\Phi_A : \mathrm{K}_\circ(A) \longrightarrow \mathrm{K}_\circ(A)$ the Coxeter transformation.

Lemma. *With the above notations, the following holds:*

$$\Phi_A \underline{\dim}\, N + \underline{\dim}\, N = \underline{\dim}\, \mathrm{P}_\alpha - \underline{\dim}\, \mathrm{S}_\alpha.$$

Proof. If N is not projective, then clearly $\underline{\dim}\, \tau_{\mathcal{D}} N + \underline{\dim}\, N = \underline{\dim}\, M = \underline{\dim}\, \mathrm{P}_\alpha - \underline{\dim}\, \mathrm{S}_\alpha$. Calculating $\tau_A N$ by considering the almost-split sequence in $\operatorname{ind} A$, which ends at N, one verifies, that $\tau_A N = \tau_{\mathcal{D}} N$. Using, that the Coxeter transformation of a source-extension $A = \mathcal{D}[M]$ can be expressed by the Coxeter transformation of $\mathcal{D}$ and the bilinear form $\langle ?', ? \rangle_{\mathcal{D}}$, one obtains $\Phi_A \underline{\dim}\, N = \Phi_{\mathcal{D}} \underline{\dim}\, N - \langle \underline{\dim}\, N, \underline{\dim}\, M \rangle_{\mathcal{D}} \cdot \underline{\dim}\, \mathrm{S}_\alpha$. Therefore the assertion follows, since $t > 2$ implies $\langle \underline{\dim}\, N, \underline{\dim}\, M \rangle_{\mathcal{D}} = 0$.

If N is projective, $N = \mathrm{P}_\beta$, then $\Phi_A \underline{\dim}\, N = -\underline{\dim}\, \mathrm{I}_\beta = -\underline{\dim}\, \mathrm{S}_\beta$, where the last equation is due to the position of N in the tube $\mathcal{T}_\lambda$. Therefore $\Phi_A \underline{\dim}\, N + \underline{\dim}\, N = -\underline{\dim}\, \mathrm{S}_\beta + \underline{\dim}\, N = \underline{\dim}\, M$. □

3.3. **Lemma.** *With the notations of (3.1), the following holds:*

$$\text{for all } v \in \operatorname{rad} \chi_A, \quad \left\langle v, \Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha \right\rangle_A = 0$$

Proof. First apply the general rules $\langle x, y \rangle_A \overset{(\natural)}{=} -\langle y, \Phi_A x \rangle_A$ and $\Phi \underline{\dim}\, \mathrm{P}_s \overset{(\flat)}{=} -\underline{\dim}\, \mathrm{I}_s$ in order to obtain $\left\langle v, \Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha \right\rangle_A = \langle \Phi_A \underline{\dim}\, N, v \rangle_A - v(\alpha)$.

Now by (3.2), $\langle \Phi_A \underline{\dim}\, N, v \rangle_A = -\langle \underline{\dim}\, N, v \rangle_A + v(\alpha) - \langle \underline{\dim}\, \mathrm{S}_\alpha, v \rangle_A$. Applying to the first and the last term on the right hand side twice ($\natural$) and the fact, that for all $v \in \operatorname{rad} \chi_A$, $\Phi_A v = v$ holds, we get $2 \langle \underline{\dim}\, \mathrm{S}_\alpha, v \rangle_A = v(\alpha) - \langle \Phi_A \underline{\dim}\, \mathrm{S}_\alpha, v \rangle_A \overset{(\flat)}{=} v(\alpha) - \langle \underline{\dim}\, \mathrm{P}_\alpha, v \rangle_A = 2v(\alpha)$, hence the result. □

3.4. **Lemma.** *With the notations of (3.1), the following holds:*

$$\chi_A(\Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha) = 1.$$

Proof. First, if N is not projective, then $\Phi_A \underline{\dim}\, N = \underline{\dim}\, \tau_{\mathcal{D}} N$ holds, as we have already seen in the proof of Lemma (3.2). Since $t > 2$, we have $\langle \underline{\dim}\, \tau_{\mathcal{D}} N, \underline{\dim}\, M \rangle_{\mathcal{D}} = 1$ and therefore $\Phi_A \underline{\dim}\, \tau_{\mathcal{D}} N = \Phi_{\mathcal{D}} \underline{\dim}\, \tau_{\mathcal{D}} N - \langle \underline{\dim}\, \tau_{\mathcal{D}} N, \underline{\dim}\, M \rangle_{\mathcal{D}} \underline{\dim}\, \mathrm{S}_\alpha = \Phi_{\mathcal{D}} \underline{\dim}\, \tau_{\mathcal{D}} N - \underline{\dim}\, \mathrm{S}_\alpha$. Altogether we get

$$\begin{aligned} \chi_A(\Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha) &= \chi_A(\Phi_{\mathcal{D}} \underline{\dim}\, \tau_{\mathcal{D}} N) \\ &= \chi_{\mathcal{D}}(\Phi_{\mathcal{D}} \underline{\dim}\, \tau_{\mathcal{D}} N) \\ &= \chi_{\mathcal{D}}(\underline{\dim}\, \tau_{\mathcal{D}} N) \\ &= 1, \end{aligned}$$

where the second equality is due to the convexity of $\mathcal{D}$ in A and the third to (3.3 $\natural$).

Assume now $N = \mathrm{P}_\beta$. In that case the position of N in $\mathcal{T}_\lambda$, and the definition of branch-source-extensions imply, that there exists a point $\gamma \in \mathcal{D}$, such that $\operatorname{rad} \mathrm{P}_\alpha = \operatorname{rad} \mathrm{P}_\beta = \mathrm{P}_\gamma$. Therefore $\langle \underline{\dim}\, \mathrm{I}_\beta, \underline{\dim}\, \mathrm{I}_\alpha \rangle_A = 0$ and $\Phi_A \underline{\dim}\, \mathrm{I}_\beta = \Phi_A(\underline{\dim}\, \mathrm{P}_\beta - \underline{\dim}\, \mathrm{P}_\gamma) = \underline{\dim}\, \mathrm{I}_\beta - \underline{\dim}\, \mathrm{I}_\gamma$, and thus $\langle \Phi_A \underline{\dim}\, \mathrm{I}_\beta, \underline{\dim}\, \mathrm{I}_\alpha \rangle_A = - \langle \underline{\dim}\, \mathrm{I}_\gamma, \underline{\dim}\, \mathrm{I}_\alpha \rangle_A = -1$.

So we conclude again

$$\begin{aligned}\chi_A(\Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha) &= \chi_A(\Phi_A \underline{\dim}\, \mathrm{I}_\beta) + \chi_A(\underline{\dim}\, \mathrm{I}_\alpha) \\ &\quad + \langle \Phi_A \underline{\dim}\, \mathrm{I}_\beta, \underline{\dim}\, \mathrm{I}_\alpha \rangle_A + \langle \underline{\dim}\, \mathrm{I}_\alpha, \Phi_A \underline{\dim}\, \mathrm{I}_\beta \rangle_A \\ &= 1 + 1 + (-1) + 0.\end{aligned}$$

□

Remark. If the module $\tau_A^2 N$ exists, then its dimension vector equals $\Phi_A^2 \underline{\dim}\, N + \underline{\dim}\, \mathrm{S}_\alpha$.

3.5. **Proof of the Distinction Lemma.** The main work, namely part (b), was already done in (3.3) and (3.4).

So assume that A is a tubular algebra. Then A contains two full, convex and critical subalgebras, $\mathcal{C}_\circ$ and $\mathcal{C}_\infty$ [13]. Denote by $h_\circ$ and h_∞ its radical vectors (by the convexity of $\mathcal{C}_\circ$ and $\mathcal{C}_\infty$ in A they belong automatically to $\operatorname{rad} \chi_A$). The full subcategory $\mathcal{T}^{(1)}$ of $\operatorname{ind} A$ formed by all indecomposable modules M, which satisfy $\langle h_\circ, \underline{\dim}\, M \rangle_A = - \langle h_\infty, \underline{\dim}\, M \rangle_A$ build a stable tubular family of the same tubular type than A [13]. More general, let $U^{(1)} = \{u \in \mathrm{K}_\circ(A) \mid \langle h_\circ, u \rangle_A = - \langle h_\infty, u \rangle_A\}$. Since χ_A controls $\mathcal{T}^{(1)}$ [13], it follows that for any positive u ($u(s) \geq 0$ for all $s \in A$, and $u \neq 0$) in $\mathrm{K}_\circ(A)$ with $\chi_A(u) = 1$ there exists an indecomposable module M in $\mathcal{T}^{(1)}$ with $\underline{\dim}\, M = u$. From the additivity of the linear functions $\langle h_\circ, ? \rangle_A$ and $\langle h_\infty, ? \rangle_A$ it follows for a positive $u \in U^{(1)}$, that $\langle h_\circ, u \rangle_A = n \cdot \langle h_\circ, h_\infty \rangle_A$ implies $\chi_A(u) = 0$.

Therefore, if there would exist a vector $v \in \chi_A^{-1}(1) \cap (\operatorname{rad} \chi_A)^\perp$, then for a large $N \in \mathbb{N}$, the vector $u := v + N(h_\circ + h_\infty)$ would be positive, lie in $U^{(1)}$ and $\langle h_\circ, u \rangle_A$ would be a multiple of $\langle h_\circ, h_\infty \rangle_A$, and hence $\chi_A(u) = 0$, but at the same time it would hold $\chi_A(u) = \chi_A(v) + N^2 \chi_A(h_\circ + h_\infty) + N \langle h_\circ + h_\infty, v \rangle_A + N \langle v, h_\circ + h_\infty \rangle_A = \chi_A(v) + 0 + 0 + 0 = 1$ in contradiction to the calculation above. This shows part (a) of the Distinction Lemma. □

4. Proof of the Main Theorem

First assume that A is derived tubular of finite representation-type. Then by (2.1), A has more than six points and thus the Distinction Lemma (3.1) shows condition (iii), whereas (ii) is well-known [13]. Condition (iv) follows directly from the famous Tits argument, namely that a representation-finite algebra has a weakly positive Tits form [5]. Finally for the last condition we use, that $\underline{\mathrm{mod}}\, \widehat{A}$ is cycle-finite, and apply [1] in order to conclude, that A is simply connected. So, by [6], A is strongly simply connected.

Now assume that the finite algebra A satisfies the conditions (i) to (v). Choose then $x \in A$ extremal, such that $A_\circ = A \setminus \{x\}$ is connected (an easy inductive argument, left to the reader, shows that it is always possible to find such an x, whenever A is directed). If x is minimal then by condition (v) x is separating and $\mathrm{I}_x/\mathrm{soc}\,\mathrm{I}_x$ is indecomposable. So, in this case we first may reflect x, in order to obtain in any case the following situation: $A = A_\circ[M]$, where $A_\circ$ is a finite, directed algebra of finite representation type and strongly simply connected, and M is an indecomposable $A_\circ$-module. Since $A_\circ$ is convex in A, the homological quadratic form, $\chi_{A_\circ}$, is just the restriction of χ_A to $A_\circ$, and therefore non-negative again. The free abelian group $\mathrm{rad}\,\chi_A$ has rank two, and it is therefore easy to find a non-zero vector $v \in \mathrm{rad}\,\chi_A$ with $v(x) = 0$. Thus the corank of $\chi_{A_\circ}$ is at least one. But it is exactly one, since the assumption that there exist two linarly independent radical vectors of $\chi_{A_\circ}$ would imply directly $\underline{\dim}\,\mathrm{P}_x \in \chi_A^{-1}(1) \cap (\mathrm{rad}\,\chi_A)^\perp$ (where P_x is the indecomposable projective A-module corresponding to x) - in contradiction to condition (iii). Altogether $A_\circ$ is a algebra of finite representation type with a non-negative homological quadratic form of corank one. Thus, by [4], $A_\circ$ is derived equivalent to hereditary algebra $B_\circ$ of extended Dynkin type and the indecomposable $A_\circ$-module M corresponds by this equivalence to an indecomposable $B_\circ$-module N. By [3], the two alegbras $A = A_\circ[M]$ and $B = B_\circ[N]$ are derived equivalent. By [12], the algebra B is either domestic tubular or tubular or 2-tubular. Now, the first possibility is excluded by condition (ii) and the latter by conditions (i) and (iii) together with the Distinction Lemma (3.1). □

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