THE DYNKIN TYPE OF A NON-NEGATIVE UNIT FORM

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Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form of the shape

$$q(x) = \sum_{i=1}^{n} q_i x(i)^2 + \sum_{i<j} q_{ij} x(i)x(j).$$

Following [5], we say that q is a *unit* (resp. *semi-unit*) form if $q_i = 1$ (resp. $q_i \in \{0,1\}$) for every $1 \leq i \leq n$.

Unit forms appear in different fields of mathematics. Our motivation comes from unit forms associated to algebraic structures such as Lie algebras, finite dimensional algebras and others. The study of the roots ($x \in \mathbb{Z}^n\colon q(x) = 1$) and isotropic vectors ($x \in \mathbb{Z}^n\colon q(x) = 0$) of these unit forms provide important information on the algebraic structures. Moreover, the positivity or non-negativity of the forms is important. For example, the semi-simple Lie algebras are those whose Killing form is positive and are therefore associated to Dynkin diagrams; in the representation theory of finite dimensional algebras, a path algebra $k\Delta$ is of finite (resp. tame) representation type if and only if the corresponding Tits form is positive (resp. non-negative) if and only if Δ is of Dynkin type (resp. extended-Dynkin type), see [4].

In this work we are concerned with non-negative semi-unit forms. Although, it is simple to verify if a quadratic form is non-negative (for example, applying Lagrange's method), we find convinient the following criterion for semi-unit forms. We shall consider the *polar form* $q(-,-)$ of q as a bilinear map $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by $q(x,y) = q(x+y) - q(x) - q(y)$.

Theorem (Non-negativity criterion). *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. Then q is non-negative if and only if the following conditions hold:*

(C1) $-2 \leq q_{ij} \leq 2$, *for $i < j$.*

(C2) q *is* balanced, *that is, for every $v \in \mathbb{Z}^n$ with $q(v) = 0$, we have $q(v,-) = 0$.*

Associated to a semi-unit form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ we have a *bigraph* G_q with vertices $1, \ldots, n$; two vertices $i \neq j$ are joined by $|q_{ij}|$ full edges if $q_{ij} < 0$ and by q_{ij} dotted edges if $q_{ij} > 0$; for every vertex i, there are $1 - q_i$ full loops at i. We say that q is connected if so is G_q. For a given graph Δ (of full edges) with at most one loop at each vertex, we consider the semi-unit form q_Δ with associated bigraph Δ.

If q is non-negative, then $q^{-1}(0) = \{x \in \mathbb{Z}^n \colon q(x,-) = 0\}$ is a direct summand of $\mathbb{Z}^n$. For Dynkin diagrams we consider the order relation given by:

$$\mathbb{A}_m \leq \mathbb{A}_n \leq \mathbb{D}_n \leq \mathbb{D}_p, \quad \text{for} \ \ 0 \leq m \leq n \leq p \ \ \text{and}$$

$$\mathbb{D}_p \leq \mathbb{E}_p \leq \mathbb{E}_q, \quad \text{if} \ \ 6 \leq p \leq q \leq 8.$$

We consider the diagram without points as Dynkin diagram and denote it by $\mathbb{A}_0$. Observe that $q_{\mathbb{A}_0} : \mathbb{Z}^0 = \{0\} \to \mathbb{Z}$ is a positive form by definition.

The main result of the work is the following.

Theorem (Dynkin-types). *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form.*

(a) *There exists a $\mathbb{Z}$-invertible transformation $T \colon \mathbb{Z}^n \to \mathbb{Z}^n$ such that*

$$qT(x_1, \ldots, x_n) = q_\Delta(x_1, \ldots, x_{n-c}),$$

where $c = \operatorname{corank} q$ and Δ is a Dynkin diagram uniquely determined by q. Write $\operatorname{Dyn}(q) = \Delta$ and call it the Dynkin-type of q.

(b) *If q' is a connected restriction of q, then $\operatorname{Dyn}(q') \leq \operatorname{Dyn}(q)$.*

(c) *There exists a connected restriction q' of q such that q' is a positive unit form and $\operatorname{Dyn}(q') = \operatorname{Dyn}(q)$.*

Part of this result was shown in [1]. The proof of the above results is based on the use of *inflations* and *deflations*, a technique introduced in [9] and recently used in [6, 7]. Note that the theorem implies immediatly the following result (two forms q, q' are called *equivalent* if there is a liner $\mathbb{Z}$-invertible transformation T such that $q' = qT$, see also 1.2).

Corollary *Two connected non-negative semi-unit forms are equivalent if and only if they have the same corank and the same Dynkin-type.*

The paper is organized as follows. In section 1 we recall some basic facts, in particular, the definitions of inflations and deflations. In section 2 we give the proofs of our Theorems. Section 3 is devoted to some considerations on Coxeter matrices associated to non-negative unit forms. We show that the eigenvalues of those matrices have modulus one.

Applications of our results to finite dimensional algebras may be found in the recent papers [2, 3, 7, 10].

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1. Basic facts.

1.1. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$, $x \mapsto \sum\limits_{i=1}^{n} q_i x(i)^2 + \sum\limits_{i<j} q_{ij} x(i)x(j)$ be an integral quadratic form.

The *radical* of q is $\operatorname{rad} q = \{x \in \mathbb{Z}^n \colon q(x,-) = 0\}$. Since $\operatorname{rad} q$ is a pure subgroup of $\mathbb{Z}^n$, it is a direct summand of $\mathbb{Z}^n$. We define $\operatorname{corank} q = \operatorname{rank}(\operatorname{rad} q)$. Observe that $\operatorname{rad} q \subset q^{-1}(0)$.

If q is non-negative, then $\operatorname{rad} q = q^{-1}(0)$. In this case, the induced form $\bar{q} = q/\operatorname{rad} q \colon \mathbb{Z}^{n-c} \to \mathbb{Z}$, where $c = \operatorname{corank} q$, is well-defined.

For the sake of simplicity we set $q_{ji} = q_{ij}$ for $i < j$.

1.2. We say that two integral forms $q, q' \colon \mathbb{Z}^n \to \mathbb{Z}$ are $\mathbb{Z}$-*equivalent* if there exists a $\mathbb{Z}$-invertible linear transformation $T\colon \mathbb{Z}^n \to \mathbb{Z}^n$ with $q' = qT$. Observe that then q is non-negative if and only if q' is non-negative, and in that case $\operatorname{corank} q = \operatorname{corank} q'$ (in particular, q is positive if and only if q' is positive). For $\varepsilon \in \{+,-\}$ and $i, j \in \{1, \ldots, n\}$, $i \neq j$ we define $T_{ij}^{\varepsilon}\colon \mathbb{Z}^n \to \mathbb{Z}^n$ as the linear transformation given by

$$T_{ij}^{\varepsilon}(e_s) = \begin{cases} e_s, & \text{if } s \neq i \\ e_i - \varepsilon e_j, & \text{if } s = i. \end{cases}$$

If $q_{ij} > 0$ we call T_{ij}^{+} an *inflation* for q, whereas if $q_{ij} < 0$ we call T_{ij}^{-} a *deflation* for q. Some simple facts are given in the following lemma.

Lemma. *Let $T = T_{ij}^{-}$ be a deflation for $q\colon \mathbb{Z}^n \to \mathbb{Z}$ and set $q' = qT$. Then we have:*

(a) *q and q' are $\mathbb{Z}$-equivalent.*

(b) *If q is a unit form and $q_{ij} = -1$, then q' is a unit form.*

(c) *If q is a semi-unit form and $0 < -q_{ij} \leq q_i + q_j$, then q' is a semi-unit form.*

(d) *If q is a non-negative semi-unit form, then so is q'.*

Proof: Straightforward, observing that $q'_i = q_i + q_j + q_{ij}$. □

1.3. For $q : \mathbb{Z}^n \to \mathbb{Z}$ an integral quadratic form and $i \in \{1, \ldots, n\}$ define $q^{(i)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ by $q^{(i)}(x_1, \ldots, x_{n-1}) = q(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$.

Now, let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. Then we have the following facts.

(a) If q is a weakly positive (i.e. $q(v) > 0$ for all $v \in \mathbb{N}^n \setminus \{0\}$), then q is a unit form and $-1 \leq q_{ij} \leq 1$.

(b) If q is weakly non-negative (i.e. $q(v) \geq 0$ for all $v \in \mathbb{N}^n$), then $-2 \leq q_{ij} \leq 2$.

(c) The form is said to be *critical* if q is not weakly positive but every restriction $q^{(i)}$ is weakly positive. By [9], a critical form q is non-negative of corank one and there exists a sincere vector $z \in \mathbb{N}^n$ such that $\operatorname{rad} q = \mathbb{Z}z$ (we recall that a vector $v \in \mathbb{Z}^n$ is *sincere* if $v(i) \neq 0$ for $1 \leq i \leq n$). Note that a critical form is connected.

The following result was shown in [9] (see also [5, 7]).

Proposition. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form.*

(a) *If q is positive, then there is a sequence of inflations with composition T such that the bigraph of qT is a disjoint union of Dynkin diagrams.*

(b) *If q is critical, then there is a sequence of inflations with composition T such that the bigraph of qT is an extended-Dynkin diagram.*

This result was used in [5] to classify the critical unit forms.

2. Proof of the Theorems.

2.1. A quadratic form q is *balanced* if $q^{-1}(0) \subset \operatorname{rad} q$. In that case, $q^{-1}(0) = \operatorname{rad} q$. Clearly, if q is non-negative, then q is balanced. The converse is not true as shown by the unit form $q(x, y) = x^2 + y^2 - 3xy$.

Proof of the non-negativity criterion: The 'only if' follows from the remarks above and (1.3 b). For the 'if' part, assume that q satisfies conditions (C1) and (C2).

(i) *We show that, if T is a deflation or an inflation for q, then qT again satisfies conditions* (C1) *and* (C2)

Let $T = T_{ij}^{-}$ be a deflation for q and $q' = qT$. Since $q'^{-1}(0) = T^{-1}(q^{-1}(0)) = T^{-1}(\operatorname{rad} q) = \operatorname{rad} q'$, we only need to verify that q' satisfies (C1).

Assume that $q'_{st} \leq -3$ for some $s < t$.

Since $q'_{st} = q'(e_s + e_t) - q'(e_s) - q'(e_t)$, we shall have that either $s = i$ or $t = i$. Let $s = i$, then $q'_{st} = q_{it} + q_{jt} \leq -3$. Since q satisfies (C1), we get that either ($q_{it} = -2$ and $q_{jt} \leq -1$) or ($q_{it} \leq -1$ and $q_{jt} = -2$). We consider the first possibility. For $v = e_i + e_t$, we have then $q(v) = q_i + q_t + q_{it} \leq 0$.

This implies that $q_i = 1$ (otherwise, $q(e_i) = 0$ and $q(e_i, e_t) = q_{it} < 0$ contradicting that q is balanced). Similarly $q_t = 1$ and hence $q(v) = 0$. From this we get $q(v, e_j) = q_{ij} + q_{jt} < q_{jt} \leq -1$, again a contradiction. This shows that $-2 \leq q'_{st}$. By dual arguments we get $q'_{st} \leq 2$, that is q' satisfies condition (C1).

Similarly, qT_{ij}^{+} satisfies (C1) and (C2) for any inflation T_{ij}^{+} for q.

(ii) *We shall show that there is a composition of inflations and deflations T such that $qT = \zeta \oplus p$, where ζ is the trivial quadratic form in $c = \operatorname{corank} q$ variables and p is a semi-unit form in $n-c$ variables satisfying* (C1), (C2) *and*

(C3) $p^{-1}(0) = \{0\}$.

Since q is balanced, we have $q^{-1}(0) \cong \mathbb{Z}^c$. We proceed by induction on c.

If $c = 0$, there is nothing to prove. Assume $c > 0$ and choose $0 \neq v \in \operatorname{rad} q$. We define $|v| = \sum_{i=1}^{n} |v(i)|$, $\operatorname{supp} v = \{i \colon v(i) \neq 0\}$ and $v^+, v^- \in \mathbb{Z}^n$ by $v^\varepsilon(i) = \max(0, \varepsilon v(i))$.

If there are $x \in \operatorname{supp} v^+$ and $y \in \operatorname{supp} v^-$ such that $q_{xy} > 0$, then we apply T^+_{xy} to q to obtain $\tilde{q} = qT^+_{xy}$ and $\tilde{v} = (T^+_{xy})^{-1} v$. Observe that $\tilde{v} = v + v(x)e_y$, thus $|\tilde{v}^-| < |v^-|$. We repeat this procedure until this is no longer possible and obtain a composition of inflations T' such that the radical of the form $q' = qT'$ contains a positive vector v', see [1, (2.9)] for details.

Now we claim that there exists a sequence of deflations with composition T'' such that $qT'T'' = \zeta' \oplus p'$, where ζ' is the trivial form in one variable and p' is a semi-unit form with corank $c-1$ satisfying (C1) and (C2). We proceed as in [1, (2.9)]: if there exist $x, y \in \operatorname{supp} v'$ with $q'_{xy} < 0$, we apply $T = T^-_{xy}$ if $v'(y) \geq v'(x)$ (or $T = T^-_{xy}$ if $v'(y) < v'(x)$). Observe that for $\tilde{q} = q'T$ and $\tilde{v} = T^{-1}v'$ we have that $\tilde{v}$ is positive again with $|\tilde{v}| < |v'|$. We repeat the procedure as long as possible to get a composition of deflations T'', a semi-unit form $q'' = q'T''$ and a positive vector $v'' \in \operatorname{rad} q''$ such that $q''_{xy} \geq 0$ for all $x, y \in \operatorname{supp} v''$. Then $q''_x = 0$ whenever $x \in \operatorname{supp} v''$. The claim follows. We are now in position to apply the induction hypothesis on p'.

To complete the proof it is enough to show that a semi-unit form satisfying conditions (C1), (C2) and (C3) is positive. This is the content of the next result. □

2.2. Theorem (positivity-criterion). *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. Then q is positive if and only if q satisfies:*

(C1) $-2 \leq q_{ij} \leq 2$ *for $i < j$ and*

(C3) $q^{-1}(0) = \{0\}$.

Proof. Assume that q is not positive.

(i) *First, we verify that we may restrict to the case where q is not weakly positive.*

Let $v \in \mathbb{Z}^n \setminus \{0\}$ be a vector with $q(v) \leq 0$. If there exist $j \in \operatorname{supp} v^+$ and $i \in \operatorname{supp} v^-$ with $q_{ij} > 0$, then as in the proof (2.1) we apply the inflation T^+_{ij} to q to obtain $q' = qT^+_{ij}$ and $v' = (T^+_{ij})^{-1} v$ satisfying $q'(v') = q(v) \leq 0$ and $|v'^-| < |v^-|$. Repeating this procedure produces a semi-unit form $\tilde{q} = qT$ where T is a composition of inflations and a vector $\tilde{v} \neq 0$ with $\tilde{q}(\tilde{v}) \leq 0$ and such that $\tilde{q}_{ij} \leq 0$ for every $i \in \operatorname{supp} \tilde{v}^-$, $j \in \operatorname{supp} \tilde{v}^+$. Hence $0 \geq \tilde{q}(\tilde{v}) = \tilde{q}(\tilde{v}^+) + \tilde{q}(\tilde{v}^-) + \sum_{i<j} \tilde{q}_{ij} \tilde{v}^+(i) \tilde{v}^-(j) \geq \tilde{q}(\tilde{v}^+) + \tilde{q}(\tilde{v}^-)$ and either $\tilde{q}(\tilde{v}^+) \leq 0$ or $\tilde{q}(\tilde{v}^-) \leq 0$. By the proof of (2.1), we know that qT satisfies (C1) and (C3). Moreover, if qT is positive, then q is positive. Therefore, we may assume that $0 \neq v \in \mathbb{N}^n$ with $q(v) \leq 0$, that is, q is not weakly positive.

(ii) *We show, that this leads to a contradiction.*

We shall proceed by induction on n. First, observe that $q_i = 1$ and $-1 \leq q_{ij} \leq 1$. Indeed, $q_i = 0$ implies that $q(e_i) = 0$ in contradiction to (C3); $q_{ij} = \pm 2$ implies that $q(e_i \pm e_j) = 0$, again a contradiction.

If $n = 1$, then $q(x_1) = x_1^2$; if $n = 2$, then $q(x_1, x_2) = x_1^2 + x_2^2 \pm x_1 x_2$ which are positive forms. We may assume that $n \geq 3$ and that every restriction $q^{(i)} \colon \mathbb{Z}^{n-1} \to \mathbb{Z}$ is positive (since $q^{(i)}$ satisfies (C1) and (C3)). Hence q is critical by (1.3). In particular, q is non-negative and $\operatorname{corank} q = 1$, in contradiction to (C3). □

2.3. The next result, proved in [1], shows part (a) of the Dynkin-type Theorem.

Theorem. *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form. Then there exist a composition T of deflations and inflations such that $qT(x_1, \ldots, x_n) = q_\Delta(x_1, \ldots, x_{n-c})$ where $c = \operatorname{corank} q$ and Δ is a Dynkin diagram uniquely determined by q.*

Sketch of proof: First, we need to know that, if $T = T_{ij}^{\varepsilon}$ is an inflation or a deflation for q and $q' = qT$ is not connected, then we have

$$qT(x_1, \ldots, x_n) = \tilde{q}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

for some $1 \leq j \leq n$, where $\tilde{q}$ is a connected non-negative semi-unit form. Indeed, if q' is not connected, then $q'_{ij} = 0$. Using the non-negativity of q, we get $q'_j = 0$ and $q'_{js} = 0$ for any $s \neq j$. Since for $s \neq i, j$ we have $q_{js} = q'_{js}$, we infer that $\tilde{q}$ is connected.

As in (2.1), we get a composition T_1 of deflations and inflations such that $qT_1 = \zeta \oplus p$, where ζ is the trivial quadratic form in c variables and p is positive. By the first remark, p is connected. Then (1.3 a) yields a composition T_2 of inflations such that $pT_2 = q_\Delta$, where Δ is a Dynkin diagram. □

Corollary. *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form. Then there exists a Dynkin diagram Δ such that the induced form $\bar{q}$ is $\mathbb{Z}$-equivalent to q_Δ.* □

2.4. *Proof of part* (b) *of the Dynkin-type Theorem:* Let $q: \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form and $i \in \{1, \dots, n\}$ be such that $q^{(i)}$ is connected. We shall prove that $\mathrm{Dyn}\,(q^{(i)}) \leq \mathrm{Dyn}\,(q)$.

(i) *First we suppose that* $\mathrm{corank}\, q^{(i)} = \mathrm{corank}\, q = c$.

We take $i = n$. Let $T: \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}$ be a $\mathbb{Z}$-invertible linear transformation such that

$$q^{(n)} T(x_1, \dots, x_{n-1}) = q_\Delta(x_1, \dots, x_{n-1-c})$$

for some Dynkin diagram Δ. Let $\tilde{T} : \mathbb{Z}^n \to \mathbb{Z}^n$ be such that $\tilde{T}(e_n) = e_n$ and $\tilde{T}(e_i) = [\, T(e_i) \;\; 0\,]^{\mathrm{T}}$ for all $i = 1, \dots, n-1$. If we apply $\tilde{T}$ to q we obtain

$$q\tilde{T}(x_1, \dots, x_{n-1}, x_n) = \tilde{q}(x_1, \dots, x_{n-1-c}, x_n),$$

where $\tilde{q}$ is a connected positive unit form. By (1.3), there is a composition S of inflations such that

$$\tilde{q}S(x_1, \dots, x_{n-1-c}, x_n) = q_{\Delta'}(x_1, \dots, x_{n-1-c}, x_n)$$

for a Dynkin diagrams Δ'. We shall prove that $\Delta \leq \Delta'$ in the order given in the introduction.

Let $v \in q_\Delta^{-1}(1)$ be a maximal root, that is, for all $w \in q_\Delta^{-1}(1)$, the vector $v - w$ is positive or zero. Let $\bar{v} = (v_1, \dots, v_{n-c-1}, 0) \in {q'}^{-1}(1)$. Observe that for all $j \in \{1, \dots, n-1-c, n\}$, we have $\bar{v}(j) \leq (S^{-1}\bar{v})(j)$. Let $j_0 \in \{1, \dots, n-1-c, n\}$ be such that $\bar{v}(j_0) \geq \bar{v}(t)$ for all t. If Δ is of type $\mathbb{D}_n$ (resp. $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$) we have $\bar{v}(j_0) = 2$ (resp. $3, 4, 5$). Therefore $q_{\Delta'}$ contains a root $S^{-1}\bar{v}$ with $(S^{-1}\bar{v})(j_0) \geq 2$ (resp. $3, 4, 5$). Hence $\Delta \leq \Delta'$.

(ii) *Suppose that* $\mathrm{corank}\, q^{(i)} < \mathrm{corank}\, q$.

Hence $\mathrm{corank}\, q^{(i)} = \mathrm{corank}\, q - 1$. Denote $p = q^{(i)}$. Observe that extending by zero gives an inclusion $\overline{p}^{-1}(1) \hookrightarrow \overline{q}^{-1}(1)$ and that the graphs $\mathrm{Dyn}\,(p)$ and $\mathrm{Dyn}\,(q)$ have the same number of vertices.

For a Dynkin graph Δ, the number of roots $|q_\Delta^{-1}(1)|$ is given as follows:

Δ	$\mathbb{A}_m$	$\mathbb{D}_m$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert q_\Delta^{-1}(1)\rvert$	$m(m+1)$	$2m(m-1)$	72	126	240

Clearly, if Δ and Δ' are two Dynkin diagrams with the same number of vertices and $|q_{\Delta'}^{-1}(1)| \leq |q_\Delta^{-1}(1)|$, then $\Delta' \leq \Delta$. This shows $\mathrm{Dyn}\,(q^{(i)}) = \mathrm{Dyn}\,(p) \leq \mathrm{Dyn}\,(q)$ $\square$

2.5. The following result proves the induction step for part (c) of the Dynkin-type Theorem. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form. We say that $i \in \{1, \dots, n\}$ is *omissible* (or an *omissible point*) for q if there exists a vector $v \in \operatorname{rad} q$ such that $v(i) = 1$.

Proposition. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative semi-unit form, $q \neq 0$. Then for any omissible point i, the restriction $q^{(i)}$ is again connected and* $\operatorname{Dyn}(q^{(i)}) = \operatorname{Dyn}(q)$.

Moreover, if $\operatorname{corank} q > 0$ *then q admits an omissible point.*

Proof. If $\operatorname{corank} q = 0$ then there is no omissible point and we have nothing to show.

Let i be any omissible point of q, and $v \in \operatorname{rad} q$ such that $v(i) = 1$. Let $\underline{\overline{x}} = \{\overline{x}_1, \dots, \overline{x}_\ell\}$ be a $\mathbb{Z}$-base of $\mathbb{Z}^n/\operatorname{rad} q$ and let $x_j \in \mathbb{Z}^n$ be a representative of $\overline{x}_j$ with $x_j(i) = 0$. Thus, if $\iota : \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ denotes the canonical inclusion with $q^{(i)} = q\,\iota$, then we have that x_j belongs to the image of ι for all j, say $\iota(x'_j) = x_j$. Since $\iota(\operatorname{rad} q^{(i)}) \subset \operatorname{rad} q$, we obtain that $\underline{\overline{x}}' = \{\pi(x'_1), \dots, \pi(x'_\ell)\}$ are linearily independent and therefore build a $\mathbb{Z}$-base in $\mathbb{Z}^{n-1}/\operatorname{rad} q^{(i)}$, where $\pi : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}/\operatorname{rad} q^{(i)}$ is the canonical projection. If T denotes the transition matrix of the base $\underline{\overline{x}}$ in the base $\underline{\overline{x}}'$ we have $\overline{q} = \overline{q}^{(i)} T$.

This implies first that, since $\overline{q}$ is connected, $\overline{q}^{(i)}$ and therefore also $q^{(i)}$ is connected, and secondly, that $\operatorname{Dyn}(q^{(i)}) = \operatorname{Dyn}(q)$.

Now, we assume that $\operatorname{corank} q > 0$ and have to show that there exists at least one omissible point for q. We first study the case where $\operatorname{corank} q = 1$. Let v be the generator of the radical of q, which has at least one positive entry. As shown in the proof of the Non-neagtivity Criterion (2.1), there exists a sequence of inflations with composition T such that $\tilde{v} = T^{-1} v$ is a positive vector. Observe that we have $\tilde{v}(i) = v(i)$ for all $i \in \operatorname{supp} v^+$ and that the restriction of qT to the support of $\tilde{v}$ is a critical form. Thus the result follows from [5], where it is shown, that a critical form always admits an omissible point.

If the corank of q is bigger than one, we take q' to be a restriction of q with corank one and apply the above argument on q' in order to see that there always exists an admissible point. □

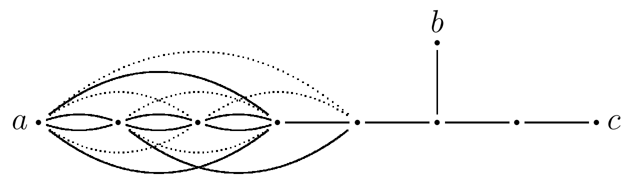
As direct consequence we obtain the following result.

Corollary 1 *Let q be a connected, non-negative semi-unit form. Then for any c with $0 \leq c \leq$* $\operatorname{corank} q$ *there exists a restriction q' of q with* $\operatorname{corank} q' = c$ *and* $\operatorname{Dyn}(q') = \operatorname{Dyn}(q)$.

Proof: By induction on $\operatorname{corank} q - c$. □

Part (c) in the Dynkin-type Theorem is just the special case $c = 0$ of the Corollary.

2.6. *Example:* Let q be the unit form associated to the following bigraph



Then q is non-negative and $\operatorname{corank} q = 3$. Moreover, $\mathrm{Dyn}\,(q) = \mathbb{E}_6$, $\mathrm{Dyn}\,(q^{(a)}) = \mathbb{E}_6$, $\mathrm{Dyn}\,(q^{(b)}) = \mathbb{A}_5$ and $\mathrm{Dyn}\,(q^{(c)}) = \mathbb{D}_5$.

3. Coxeter matrices.

3.1. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form and consider the upper triangular matrix $C = (c_{ij})$ given by $c_{ii} = q_i = 1$, $c_{ij} = q_{ij}$ if $i < j$. Then $q(x, y) = x^t(C + C^t)y$ and C is a $\mathbb{Z}$-invertible matrix. The matrix $\phi = -C^{-1}C^t$ is called the *Coxeter matrix* associated to q.

Part (a) of the following remark is proved in [8].

Proposition. *Let q and ϕ be as above.*

(a) *If q is positive, then all eigenvalues of ϕ have modulus one and* 1 *is not an eigenvalue of ϕ.*

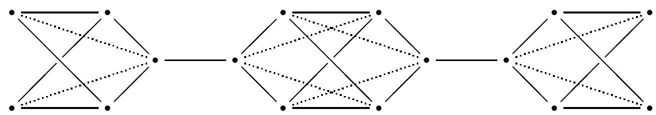
(b) *If q is non-negative, then all eigenvalues of ϕ have modulus one. Moreover, if q is not positive, then* 1 *is an eigenvalue of ϕ.*

Proof. The first statement of (a) is [8, (3.1)]. If $\phi v = v$ for some vector $v \neq 0$, then $q(v, -) = 0$ and q is not positive.

(b): Assume q is non-negative. Let $\varepsilon > 0$ and consider the triangular matrix $N_\varepsilon = C + \varepsilon I_n$. The quadratic form $q^\varepsilon(x) = x^t(N_\varepsilon + N_\varepsilon^t)x = q(x) + 2\varepsilon\|x\|^2$ is positive. Hence by (a), the eigenvalues of $\phi^\varepsilon = -N^{-1}N^t$ have modulus one. Moreover, ϕ^ε depends continuously on the parameter ε. Hence $\operatorname{Spec}\phi \subset \mathbb{S}^1$.

If q is not positive, let $0 \neq v \in \mathbb{Z}^n$ be such that $q(v) = 0$. Hence $0 = (C + C^t)v$ and $\phi v = v$, showing that $1 \in \operatorname{Spec}\phi$. □

3.2. *Example:* Let q be the quadratic form associated to the following bigraph



is non-negative with $\operatorname{corank} q = 5$. We have $\operatorname{Dyn}(q) = \mathbb{D}_{11}$ and the characteristic polynomial of the Coxeter matrix ϕ is $(T+1)^6(T-1)^6(T^4+T^3+T^2+T+1)$.

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