**ALGEBRAS WHOSE EULER FORM IS NON-NEGATIVE**

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Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. We denote by $\text{mod}_A$ the category of finite dimensional left $A$-modules and by $D^b(A)$ the derived category of $\text{mod}_A$. We say that two algebras, $A$ and $B$, are derived equivalent if their derived categories, $D^b(A)$ and $D^b(B)$, are derived equivalent as triangulated categories. See [11] for definitions and basic concepts.

In recent years a considerable effort has been devoted in the characterizations of algebras which are derived equivalent to well understood classes of algebras (tame hereditary algebras, tubular algebras, some special biserial algebras) [1, 12, 3, 9]. An important invariant entering in all these characterizations is the Euler form: if $A$ has finite global dimension, the Grothendieck group $K_0(A) \simeq \mathbb{Z}^n$ is equipped with a (non-symmetric) bilinear form $\langle -, - \rangle_A$ such that for two modules $X, Y \in \text{mod}_A$ we have

$$\langle [X], [Y] \rangle_A = \sum_{i=0}^{\infty} \dim_k \text{Ext}^i_A(X, Y),$$

where $[X]$ denotes the class of $X$ in $K_0(A)$. The associated quadratic form $\chi_A(v) = \langle v, v \rangle_A$ is the Euler form of $A$. For two derived equivalent algebras, $A$ and $B$, the Euler forms $\chi_A$ and $\chi_B$ are equivalent. In particular, $\chi_A$ is non-negative if and only if so is $\chi_B$.

Algebras $A$ whose form $\chi_A$ is non-negative are important. Examples include the algebras which are derived equivalent to tame hereditary and tubular algebras [11, 12]; certain tree algebras which are derived tame [16] and others. Recent results in [5] show that for the non-negative form $\chi_A$ of a connected algebra $A$, there exists an invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\chi_A T(x_1, \ldots, x_n) = q_\Delta(x_1, \ldots, x_{n-s})$, where $s = \text{corank} \chi_A$ and $q_\Delta$ is the quadratic form associated to a uniquely determined Dynkin graph $\Delta$. The graph $\Delta = \text{Dyn}(\chi_A)$ is called the Dynkin-type of $\chi_A$.

The main result of this work completes the description of the algebras $A$ whose Euler form $\chi_A$ is non-negative with corank $\chi_A \leq 2$ (at least for some classes of algebras).

**Theorem.** Let $A = kQ_A/I$ be a connected finite dimensional $k$-algebra such that $\chi_A$ is non-negative of corank 2. Assume that $A$ is in one of the following classes: (1) tree algebras; (2) strongly simply connected poset algebras. Then $A$ is derived equivalent to a tubular algebra or to a poset algebra $P(n)$ of the form
Moreover, if $A$ has more than 6 vertices, then $A$ is derived equivalent to a tubular algebra (resp. to $P(n)$) if and only if $\text{Dyn}(\chi_A) = E_p$ ($p = 6, 7, 8$) (resp. $\text{Dyn}(\chi_A) = D_{n-2}$).

The work is organized as follows. In Section 1, we recall some examples and properties of algebras whose Euler form is non-negative. In Section 2, we describe the Dynkin-type of algebras derived equivalent to well-known classes of algebras. In particular we show the following result.

**Proposition.** Let $A$ be a strongly simply connected algebra whose Euler form is non-negative and of Dynkin-type $A_n$. Then $A$ is derived equivalent to a hereditary algebra of type $A_n$.

In Section 3, we prove a useful lemma about the connectedness of the radical of a strongly simply connected algebra. In Section 4 and 5, we give the proofs of the above theorem for tree algebras and strongly simply connected poset algebras respectively. Finally in the last section, we treat the case where the associated Euler form is non-negative but has higher corank.

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1. Some algebras whose Euler form is non-negative

1.1 Let $A = kQ_A/I$ be a finite-dimensional algebra. We shall assume that $Q_A$ is connected and without oriented cycle (we say $A$ is connected and triangular, respectively). In particular, $A$ has finite global dimension. By $Q_0$, we denote the set of vertices of $Q_A$.

A module $X \in \text{mod}_A$ is also considered as a representation of $Q_A$. The dimension vector $\dim X$ is also identified with the class $[X]$ of $X$ in the Grothendieck group $K_0(A) \cong \mathbb{Z}^n$.

For $x \in Q_0$ we denote by $S_x$ the simple module at $x$. By $P_x$ (resp. $I_x$) we denote a projective cover (resp. injective envelope) of $S_x$. We also write $e_x$ instead of $\dim S_x$.

1.2 Given two derived equivalent algebras $A$ and $B$ with $F : \text{D}^b(A) \to \text{D}^b(B)$ a triangular equivalence, there is an induced isometry $f : K_0(A) \to K_0(B)$, satisfying $\langle x, y \rangle_A = \langle f(x), f(y) \rangle_B$.

Recall that $A[M]$ denotes the one-point extension of $A$ by a module $M$ (see [17]). The following result will be basic for our considerations.

**Theorem.** [2] Let $A$ and $B$ be two algebras and $M \in \text{mod}_A$, $N \in \text{mod}_B$ two modules. Suppose there is a triangular equivalence $F : \text{D}^b(A) \to \text{D}^b(B)$
which maps the stalk complex $M[0]$ to $N[0]$. Then there exists a triangular equivalence $\mathcal{F}: D^b(A[M]) \rightarrow D^b(B[N])$ extending $F$.

1.3 Given a subset $J$ of the vertices of $Q_A$, the algebra $\text{End}_A(\bigoplus_{x \in J} P_x)^{\text{op}} = B$ is said to be fully contained in $A$. If $J$ is path closed in $Q_A$, then $B$ is said to be convex in $A$. If $Q_o$ denotes the vertex set of $Q_A$ and $J = Q_o \setminus \{y\}$, we denote $A \setminus \{y\} = B$.

**Lemma.** [3] Let $B$ be fully contained in $A$ and assume that $\chi_A$ is non-negative. Then $\chi_B$ is non-negative and $\text{corank} \chi_B \leq \text{corank} \chi_A$.

1.4 We recall that an algebra $A$ is said to be strongly simply connected if for every algebra $B$ convex in $A$, the first Hochschild cohomology $H^1(B)$ vanishes [18]. Equivalently, $A$ is strongly simply connected if and only if every algebra $B$ convex in $A$ is separated, that is, $B = kQ_B/I'$ and for every vertex $x$ in $Q_B$ the following condition is satisfied: let $\text{rad}P_x = \bigoplus_{i=1}^m M_i$ be a decomposition into indecomposable modules of the $B$-module $\text{rad}P_x$, then for any $i \neq j$, the support of $M_i$ and $M_j$ are contained in different connected components of $Q_B \setminus \{y\}$: there is a path from $y$ to $x$. See also [6, 18].

**Examples:** (a) If $A = kQ_A/I$ is a tree algebra (that is, the underlying graph of $Q_A$ is a tree), then $A$ is strongly simply connected.

(b) Let $A = k[S]$ be a poset algebra (that is, $S$ is a poset and $A = kQ_S/I_S$ where $Q_S$ is the quiver, $kQ_S$ is the path algebra of $kQ_A$ of $S$ and $I_S$ the ideal in $kQ_S$ generated by the differences of parallel paths in $kQ_S$, see [10]). Then $A$ is strongly simply connected if and only if $A$ has no crowns, see [7]. We recall that a crown in $A$ is an algebra $C$, fully contained in $A$, of the form

![Diagram](https://via.placeholder.com/150)

and such that the convex closure $\{a_i, b_i\}$ of $\{a_i, b_i\}$ intersects $\{a_{i+1}, b_i\}$ (resp. $\{a_i, b_{i-1}\}$) in $b_i$ (resp. in $a_i$), for $i = 1, \ldots, m$ and $a_{m+1} = a_1$, $b_0 = b_m$.

The following results are central in our considerations.

**Theorem.** Let $A$ be a strongly simply connected algebra.

(i) [1, 4] $A$ is derived equivalent to a tame hereditary algebra $k\Delta$ if and only if $\chi_A$ is non-negative with $\text{corank} \chi_A = 1$. In this case, $\Delta$ is of type $D_n$ ($n \geq 4$) or $E_p$ ($p = 6, 7, 8$).

(ii) [3] If $Q_A$ has more than 6 vertices, then $A$ is derived equivalent to a tubular algebra $k\Delta$ if and only if $\chi_A$ is non-negative with $\text{corank} \chi_A = 2$ and $\chi_A^{-1}(1) \cap \chi_A^{-1}(0) = \emptyset$ (where $V = \{w \in K_0(A) : \langle v, w \rangle_A = 0$ for all $v \in V\}$).
1.5 Following [16], we say that $A$ is derived-tame if $A$ has finite global dimension and the repetitive category $\tilde{A}$ is tame. Examples of derived-tame algebras are the following:

(a) By [11], hereditary tame algebras are also derived-tame. By [12], tubular algebras are also derived-tame.

(b) If $A$ is derived tame and $D^b(A) \simeq D^b(B)$ is a triangular equivalence, then $B$ is also derived-tame, see [16].

(c) Let $C$ be a hereditary tame algebra of type $\widetilde{D}_n$ and let $M$ be an indecomposable regular $C$-module of regular length 2 lying in a tube of rank $n-2$ in the Auslander-Reiten quiver $\Gamma_C$. Then the one-point extension $C[M]$ is called a 2-tubular algebra (see [14]). In [16], it is shown that $B$ is derived tame and derived equivalent to the poset algebra $P(n+2)$ as defined in the introduction.

(d) Other examples of derived tame algebras are provided by the poset algebras associated to posets of the form

\[
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Remark. (1) All algebras in the above examples have a non-negative Euler form.

(2) Information on the structure of the module category of a derived tame algebra was recently obtained in [9].

2. THE DYNKIN-TYPE OF NON-NEGATIVE EULER FORMS

2.1 Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form of the shape $q(v) = \sum_{i=1}^{n} q_i v(i)^2 + \sum_{i<j} q_{ij} v(i)v(j)$. We say that $q$ is a unit (resp. semi-unit) form if $q_i = 1$ (resp. $q_i \in \{0, 1\}$).

Associated with a semi-unit form we define a bigraph $G_q$ with vertices $1, \ldots, n$: two vertices $i \neq j$ are joined by $|q_{ij}|$ full edges if $q_{ij} < 0$ and by $q_{ij}$ dotted edges if $q_{ij} \geq 0$; for every vertex $i$, there are $1 - q_i$ full loops at $i$. We say that $q$ is connected if $G_q$ is connected. The following are elementary facts.

(a) If $A = kQ/I$ is a connected and triangular algebra, then $\chi_A$ is a connected unit form.

(b) Given a connected graph $\Delta$ formed by full edges and at most one loop at each vertex, there is a semi-unit form $q_\Delta$ such that $G_{q_\Delta} = \Delta$. Then $q_\Delta$ is positive (resp. non-negative) if and only if $\Delta$ is a Dynkin diagram (resp. extended Dynkin diagram).
For Dynkin diagrams we consider the following partial order:

\[ A_m \leq A_n \leq D_n \leq D_p \text{ for } m \leq n \leq p \quad \text{and} \]
\[ D_p \leq E_p \leq E_q \text{ for } 6 \leq p \leq q \leq 8. \]

The following result is relevant in our discussion.

Theorem. \([5]\) Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a connected, non-negative semi-unit form. Then there exists a \( \mathbb{Z} \)-invertible linear transformation \( T : \mathbb{Z}^n \to \mathbb{Z}^n \) such that \( qT(x_1, \ldots, x_n) = q\Delta(x_1, \ldots, x_{n-c}) \), where \( c = \text{corank } q \) and \( \Delta = \text{Dyn}(q) \) is a Dynkin diagram uniquely determined by \( q \). Moreover, if \( q' \) is a connected restriction of \( q \), then \( \text{Dyn}(q') \leq \text{Dyn}(q) \).

2.2 Proposition. Let \( A \) be a strongly simply connected algebra with a non-negative Euler form \( \chi_A \) of type \( \mathbb{A}_n \). Then \( A \) is derived equivalent to a hereditary algebra of type \( \mathbb{A}_n \) and \( \text{corank } A = 0 \).

Proof. We show first that \( \text{corank } A > 0 \), that is, \( \chi_A \) is positive. Suppose that \( \text{corank } A > 0 \). Then there exists an algebra \( B \) convex in \( A \) such that \( \text{corank } B = 1 \). By (2.1), \( \text{Dyn}(\chi_B) \leq \text{Dyn}(\chi_A) = \mathbb{A}_n \), thus \( \text{Dyn}(\chi_B) = \mathbb{A}_m \) for some \( m \leq n \). By (1.4), the algebra \( B \) is derived equivalent to a hereditary algebra of type \( \mathbb{D}_{m-1} \) or \( \mathbb{E}_{m-1} \) \( (m = 7, 8, 9) \), which implies \( \text{Dyn}(\chi_B) = \mathbb{D}_{m-1} \) or \( \text{Dyn}(\chi_B) = \mathbb{E}_{m-1} \), respectively - in any case a contradiction. Hence we have \( \text{corank } A = 0 \).

By [1], \( A \) is derived equivalent to a hereditary algebra \( k\Delta \), where \( \Delta \) is a quiver of Dynkin type. Clearly, we have \( \text{Dyn}(\chi_A) = \Delta \).

2.3 Let us restate the results in [1, 3] mentioned in (1.3). Let \( A = kQ/I \) be a connected and strongly simply connected algebra. Then we have:

1. A is derived equivalent to a tame (but not representation-finite) hereditary algebra if and only if \( \chi_A \) is non-negative and \( \text{corank } \chi_A = 1 \). In this case, \( \text{Dyn}(\chi_A) = \mathbb{D}_n \) \( (n \geq 4) \) or \( \mathbb{E}_p \) \( (p = 6, 7, 8) \).

2. If \( A \) is derived equivalent to a tubular algebra (resp. to a 2-tubular algebra), then \( \chi_A \) is non-negative and \( \text{corank } \chi_A = 2 \). If \( Q_A \) has more than 6 vertices, then \( \text{Dyn}(\chi_A) = \mathbb{E}_p \) \( (p = 6, 7, 8) \) (resp. \( \text{Dyn}(\chi_A) = \mathbb{D}_n \) \( (n \geq 4) \)), whereas if \( Q_A \) has 6 vertices in both cases we have \( \text{Dyn}(\chi_A) = \mathbb{D}_4 \).

3. Assume \( A = B[M] \) such that \( \chi_A \) is non-negative, \( \text{corank } \chi_A = 2 \) and \( \text{corank } \chi_B = 1 \). Then \( A \) is derived equivalent to a tubular or a 2-tubular algebra.

We conjecture that the following hold for a strongly simply connected algebra \( A \).

4. If \( \text{corank } \chi_A = 2 \), then
   4.1 if \( \text{Dyn}(\chi_A) = \mathbb{D}_n \) and \( n \geq 5 \), then \( A \) is derived equivalent to a 2-tubular algebra,
   4.2 if \( \text{Dyn}(\chi_A) = \mathbb{E}_p \) \( (p = 6, 7, 8) \), then \( A \) is derived equivalent to a tubular.
(5) If corank $\chi_A \geq 3$ then $\text{Dyn}(\chi_A) = E_n$.

The results we show in this work are special cases of conjecture (4). In [9], special cases of conjecture (5) are considered.

2.4 We recall from [4, 5] examples showing that the above conjectures may be expected only in the strongly simply connected case.

(a) Let $A$ be the algebra given by the following quiver with commutativity relations as indicated by dotted lines.

Then $\chi_A$ is non-negative with corank $\chi_A = 2$ and $\text{Dyn}(\chi_A) = E_8$. Moreover, $A$ is wild and hence $A$ cannot be derived tame, by (1.5).

(b) Let $A$ be the algebra given by the following quiver with zero relations as indicated by dotted lines.

Then $\chi_A$ is non-negative with corank $\chi_A = 3$ and $\text{Dyn}(\chi_A) = E_6$.

3. Connectivity of the radical

In the following we prove a general result about the convex closure of the support of the radical of a strongly simply connected algebra with non-negative Euler form. Although the proof is quite technical, it will be of great use in the forthcoming considerations.

**Proposition.** Let $A = kQ/I$ be a strongly simply connected algebra with non-negative Euler form. Then the convex closure $\overline{\text{rad} \chi_A}$ of the support of $\text{rad} \chi_A$ is connected in $A$.

**Proof.** Suppose that there exists a strongly simply connected algebra $A$ such that $\overline{\text{rad} \chi_A}$ is not connected in $A$. We assume that $A$ is a minimal such example and let $\overline{\text{rad} \chi_A} = \bigcup_{i=1}^t R_i$, $(t \geq 2)$, be a decomposition into connected algebras $R_i$, which are convex in $A$.

The proof is done in several steps:

(i) We first show that corank $\chi_A \geq 2$.

Any vector $v \in \text{rad} \chi_A$ decomposes as $v = \sum_{i=1}^t v_i$ with $v_i \in \text{K}_0(R_i) \subset \text{K}_0(A)$. Hence $0 = \chi_A(v) = \sum_{i=1}^t \chi_{R_i}(v_i)$ (since for $i \neq j$, $x \in \text{supp} R_i$ and
We get simply connected algebra and convex in $A$.

(ii) We show $t = 2$, that is $\text{rad}_A = R_1 \cup R_2$ where $R_1$, $R_2$ are connected and convex in $A$.

Choose $i \neq j$ such that there is a walk $\gamma$ between $R_i$ and $R_j$ in $Q_A$ of minimal length. Then the convex closure of $R_i$, $R_j$ and $\gamma$ in $A$ is a strongly simply connected algebra $A_0$ with $\text{rad}_{A_0} = R_i \cup R_j$. By the minimality of $A$ we get $A = A_0$ and $t = 2$.

(iii) Next we verify for $i = 1, 2$ that there is a source or a sink $y_i$ such that $A \setminus \{y_i\}$ is connected and $y_i \in R_i$.

First observe that $A \setminus \{R_1 \cup R_2\}$ contains a vertex $x_0$ which is a source or a sink in $Q_A$, and that for any such point $x_0$, by the minimality, $A \setminus \{x_0\}$ is not connected.

Choose such a point $x_0 \in A \setminus \{R_1 \cup R_2\}$, say $x_0$ is a source, and set $A \setminus \{x_0\} = B_1 \cup B_2$ with $B_1 \subset R_1$ and $B_2 \subset R_2$. Now, choose a sink $x_1 \in B_1$. If $A \setminus \{x_1\}$ decomposes, say $A \setminus \{x_1\} = C_1 \cup C_2$ with $R_2 \subset C_2$, we choose a source $x_2 \in C_1$. Again, if $A \setminus \{x_2\}$ decomposes, say $A \setminus \{x_2\} = D_1 \cup D_2$ with $B_2 \subset D_2$ we choose a sink $x_3 \in D_1$. Observe that $|B_1| > |C_1| > |D_1| > \cdots$.

This process may be continued until we find a source or a sink $x_m$ not belonging to $R_2$ such that $A \setminus \{x_m\}$ is connected. By the above, we thus have $y_1 := x_m \in R_1$. Dually we find $y_2$.

(iv) Now we shall prove that $A$ is tubular or 2-tubular.

Since $y_i \in R_i$, we have $\text{corank}_{A \setminus \{y_i\}} \leq \text{corank}_A$ for $i = 1, 2$, and hence we obtain by the minimality that $\text{corank}_{R_1} = 1 = \text{corank}_{R_2}$. Therefore we have $\text{corank}_{A} = 2$.

We may assume that $y_1$ is a source. Since $A$ is strongly simply connected, $M = \text{rad}_{P_{y_1}}$ is indecomposable and $A' = A \setminus \{y_1\}$ is strongly simply connected with $\text{corank}_{A'} = 1$. By (2.3.3), $A$ either derived equivalent to a tubular or to a 2-tubular algebra.

(v) Finally, we show that this leads to a contradiction.

In both cases we have $\text{rad}_A = k v_1 \oplus k v_2$ with $\langle v_1, v_2 \rangle_A \neq 0$ (in the tubular case this follows from [17], in the 2-tubular case it may be easily verified for the poset algebra $P(n)$). But this contradicts the fact that there are vectors $w_1, w_2 \in \text{rad}_A$ with $w_i \in K_0(R_i) \subset K_0(A)$ for $i = 1, 2$, which implies that $\langle w_1, w_2 \rangle_A = 0$. This completes the proof of the proposition. \hfill $\square$

4. The tree case

4.1 The first result we state provides the inductive step dealing with tree algebras with non-negative Euler form.

**Proposition.** Let $A$ be a tree algebra with non-negative Euler form and $\text{corank}_A = c$. Then there exists an algebra $B$ satisfying the following two properties.
4.2 Lemma. Let $A = kQ_A/I$ be a tree algebra. Consider the convex closure $\text{rad}\chi_A$ of the support of $\text{rad}\chi_A$ and let $x$ be a source or a sink in $\text{rad}\chi_A$. Then $A \setminus \{x\}$ is again a tree algebra.

Proof. Suppose $A \setminus \{x\}$ is not a tree. Denote by $y_1, \ldots, y_t$ the vertices in $Q_A$ such that there exists an arrow $\alpha_i : y_i \to x$ and denote by $z_1, \ldots, z_s$ those vertices with an arrow $\beta_i : x \to z_i$. Since $\chi_A$ is non-negative, we have $t \leq 4$ and $s \leq 4$. Since $A \setminus \{x\}$ is not a tree, it fully contains an algebra $B$ of the form

\[
\begin{array}{c}
y_1 \quad y_2 \quad \ldots \quad y_r \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
z_1 \quad z_2 \quad \ldots \quad z_r
\end{array}
\]

with $r \geq 2$ and the arrows are compositions $\beta_j \alpha_i$ for some $1 \leq i \leq t$ and $1 \leq j \leq s$. Then there exists a vector $v \in \text{rad}\chi_A$ with $v(y_i) \neq 0 \neq v(z_j)$. This contradicts the fact that $x$ was chosen to be a source or a sink in $\text{rad}\chi_A$. \qed

4.3 Let $A = kQ_A/I$ be a triangular algebra and $x$ a source in $Q_A$. Let $A_0 = A \setminus \{x\}$ and write $A = A_0[M]$ as a one-point extension with $M = \text{rad}P_x$. Then $S^+_x A = [M]A_0$ is the source-reflection of $A$ at $x$. In [11] it is shown that $A$ and $S^+_x A$ are derived equivalent. We denote the extension-vertex in $Q_{S^+_x A}$ by $x^*$, that is $I_{x^*}/\text{soc}I_{x^*} = M$.

For any vertex $x \in Q_A$ we assume that

\[
x_A^x := \{a \neq x : \text{ there is a path from } a \text{ to } x\} = \{a_1, \ldots, a_t\}
\]

is enumerated in such a way that the existence of a path from $a_i$ to $a_j$ implies that $i \leq j$. Define the algebra $A^x = S^+_x \cdots S^+_x A$, which is derived equivalent to $A$. Clearly, $x$ is then a source in $A^x$. For any point $u \in x_A^x$ and $y = u^* \in Q_{A^x}$ we also write $u = y^*$.

Lemma. Let $A$ be a tree algebra such that $\chi_A$ is non-negative, and let $x$ be a source in $\text{rad}\chi_A$. Then for any arrow $\alpha : x \to y$ in $Q_{A^x}$ we have $y \not\in \text{rad}\chi_A$.

Proof. We assume that there exists $y \not\in \text{rad}\chi_A$ such that there is an arrow $\alpha : x \to y$ in $Q_{A^x}$ and proceed in several steps.

(i) First we show, that $x$ is the only vertex in $\text{rad}\chi_A$ which is the starting point of a path in $Q_{A^x}$ to $y$.

Assume there exists a vertex $x' \in \text{rad}\chi_A$, $x' \neq x$ and a path

\[
x' \to z_0 \to z_1 \to \cdots \to z_t \to y
\]
in $Q_A$. By Proposition 3, we know that $x$ and $x'$ may be connected by a walk inside $\text{rad}_x^A$, thus, since $A$ is a tree algebra, we have $y \in (x_{A}^x)^*$. If there exists $i > 0$ such that $z_{i-1}$ does not belong to $(x_{A}^x)^*$, we choose $i$ maximal with this property. Then in $A$ we have the following paths.

$$x' \to z_0 \to z_1 \to \cdots \to z_{i-1}$$

$$z_i^* \to z_{i+1}^* \to \cdots \to z_t^* \to y^* f x$$

where $f$ itself is a path. Together with a path from $z_i^*$ to $z_{i-1}$ and a walk inside $\text{rad}_x^A$ between $x'$ and $x$, we obtain a closed walk in $Q_A$, in contradiction to the assumption that $A$ is a tree algebra. The case where $z_i^* \in x_{A}^x$ for all $i = 0, \ldots, t$ is similar.

(ii) Now we show that the assumption leads to a contradiction.

Let $A' = A \setminus \{y_{A}^x \setminus x_{A}^x\}$. Clearly, $A'$ is convex in $A^x$ and $\text{rad}_x^A$ is fully contained in $A'$. It is thus sufficient to show that for $A'$ the assumption leads to a contradiction.

Consider a projective resolution in $\text{mod}_{A'}$

$$0 \to P(n) \to P(n-1) \to \cdots \to P(0) \to S_y \to 0,$$

then $\langle \text{dim} P(i), v \rangle_{A'} = 0$ for all $i = 0, \ldots, n$ and $v \in \text{rad}_x^A$.

Let $v \in \text{rad}_x^A$ be such that $v(x) \neq 0$. Then

$$\langle v, e_y \rangle_{A'} = \langle v, \text{dim} I_y \rangle_{A'} - \langle v, \text{dim} I_x \rangle_{A'} = -v(x) \neq 0$$

because $y \not \in \text{rad}_x^A$ and $x$ is the only predecessor of $y$ in $Q_{A'}$. On the other hand,

$$\langle e_y, v \rangle_{A'} = \sum_{i=0}^{n} (-1)^i \langle \text{dim} P(i), v \rangle_{A'} = 0.$$

Therefore $\chi_{A'}(2v + e_y) < 0$ contradicting the non-negativity of $\chi_{A'}$. \qed

Obviously, the dual statement may be proved similarly.

4.4 Proof of Proposition 4.1. By Proposition 3, $\text{rad}_x^A$ is connected. Choose a source or a sink $x$ in $\text{rad}_x^A \setminus \{x\}$ such that $\text{rad}_x^A \setminus \{x\}$ is still connected. Say $x$ is a source in $\text{rad}_x^A$. Consider the algebra $A_0 = A \setminus \{x\}$ which is fully contained in $A$. By (4.2), $A_0$ is again a tree algebra and clearly, $\text{corank} \chi_{A_0} = c - 1$.

As in the proof of Lemma 4.3, we have $\text{rad}_x^A = \text{rad}_x^{A_0}$ and in particular $\text{rad}_x^{A_0} = \text{rad}_x^{A_0^x}$. Observe that $x$ is a source in $A^x$ and define $B = A^x \setminus \{x\}$. Hence $B = S_{a_1}^+ \cdots S_{a_t}^+ A_0$ (where $x_{A}^x = \{a_1, \ldots, a_t\}$ is supposed to be “well-enumerated”) and $\text{corank} \chi_{B} = \text{corank} \chi_{A_0} = c - 1$.

It remains to show that the $B$-module $M = \text{rad} P_x$ is indecomposable. First, observe that $M' = \text{rad} P_x |_{\text{rad}_x^{A_0^x}}$ is indecomposable (because $\text{rad}_x^A = \text{rad}_x^{A_0^x}$). By (4.3), any arrow $x \to y$ in $Q_{A_0^x}$ belongs to $\text{rad}_x^{A_0^x}$. Therefore a decomposition of $M$ yields a decomposition of $M'$, thus $M$ is indecomposable. \qed

4.5 Proof of the Main Theorem for tree algebras. Let $A$ be a tree algebra with non-negative Euler form of corank 2. By (4.1), there exists a triangular,
connected algebra $B$ which is derived equivalent to a tree algebra $C$ and such that $\chi_B$ is non-negative of corank one and there exists an indecomposable $B$-module $M$ such that $A$ is derived equivalent to $B[M]$. In particular, $\chi_C$ is non-negative of corank one.

By (1.4), the algebra $C$ is derived equivalent to a hereditary algebra of type $\tilde{\Delta}$ and moreover $\Delta = \text{Dyn}(\chi_A) = \mathbb{D}_n$ ($n \geq 4$) or $\mathbb{E}_p$ ($p = 6, 7, 8$). By (1.1), there exists an indecomposable $H$-module $N$ such that $B[M]$ is derived equivalent to $H[N]$. The result follows from (2.3.3).

5. The poset case

A rereading of the proof of the Main Theorem in the tree case reveals, that the assumption for $A$ to be a tree algebra is only needed in the proof of Lemma 4.2 and in the step (i) of Lemma 4.3.

In the following we just give the arguments which establish the same assertions as (4.2) and (4.3) if $A$ is a strongly simply connected poset algebra.

5.1 Lemma. Let $A$ be a strongly simply connected poset algebra. Let $x$ be a source or a sink in $\text{rad} \chi_A$. Then $A \setminus \{x\}$ is again a strongly simply connected poset algebra.

Proof. The algebra $B = A \setminus \{x\}$ is clearly a poset algebra. To show that $B$ is strongly simply connected, it is enough to show that $B$ admits no crown (1.4). This is shown exactly as in the proof of Lemma 4.2.

5.2 Lemma. Let $A$ be strongly simply connected poset algebra such that $\chi_A$ is non-negative, and let $x$ be a source in $\text{rad} \chi_A$. Then for any arrow $\alpha : x \to y$ in $Q_A$, we have $y \in \text{rad} \chi_A$.

Proof. Again, we assume that there exists an arrow $\alpha : x \to y$ such that $y \not\in \text{rad} \chi_A$.

And again, we first show that then $x$ is the only start point of a path from $\text{rad} \chi_A$ to $y$ in $Q_A$. So assume that this is not so: let $x' \in \text{rad} \chi_A$ be different from $x$ such that there exists a path

$$x' \to z_0 \to z_1 \to \cdots \to z_t \to y$$

in $A$. Since $\text{rad} A^x$ is connected there exists a fully contained algebra $C$ in $\text{rad} A^x$ of the form $(\ast)$

First, suppose $y \not\in (x')^*$. Then we have $x' \not\leq x$ in $A$ because there is an arrow $x \to y$ and $A$ is a poset algebra. If there exists a $j$ such that $c_j < y$ then we choose $j$ maximal with that property. Observe that we have $j < t$. Thus $\{b_{j+1}, c_{j+1}, \ldots, b_t, c_t, x, y\}$ is a crown in $A$, in contradiction to the fact that $A$ is strongly simply connected, see (1.4 b). On the other hand, if there does not exist a $j$ with $c_j < y$ then $(\ast)$ together with $y$ forms a crown in $A$. 
Thus we have $y^* \in x_A^<$ and therefore $y^* < x < c_t$. On the other hand, since $x \to y$ is an arrow in $Q_{A^*}$, the vertex $y^*$ can not be smaller than $c_t$ in $A$. This contradicts the fact that $A$ is a poset algebra.

The rest of the proof follows as in (4.3).

6. Higher coranks

6.1 In the following we shall prove the following result which is related to the conjecture about algebras $B$ with corank $\chi_B > 2$, see (2.3.(5)).

**Proposition.** Let $A$ be a tree algebra or a strongly simply connected poset algebra with non-negative Euler form. Then any properly contained, convex algebra $B$ in $A$ whose Euler form has corank 2 is derived equivalent to a poset algebra $P(n)$.

6.2 We shall need the following result.

**Proposition.** Let $A$ be an algebra which is derived equivalent to a tubular algebra and let $M$ be an indecomposable $A$-module. Then the following hold.

1. $\chi_A(\dim M) \in \{0, 1\}$.
2. The Euler form of $A[M]$ is indefinite.

**Proof.** (i). By [11], the inclusion $\text{mod}_A \hookrightarrow D^b(A)$, $X \hookrightarrow X[0]$ induces an isometry $K_0(A) \to K_0(D^b(A))$. Hence we shall prove that $\chi_{D^b(A)}([M[0]]) \in \{0, 1\}$. By [12], for an indecomposable object $X \in D^b(A)$, there is a tubular algebra $B$ such that $X$ lies in the image of the composition mod $B \hookrightarrow D^b(B) \to D^b(A)$, of the inclusion with some triangular equivalence $F$, say $X = F(Y[0])$ for some indecomposable $B$-module $Y$. Hence $\chi_A(\dim M) = \chi_{D^b(A)}([M[0]]) = \chi_{D^b(B)}([Y[0]]) = \chi_B(\dim Y)$, and finally $\chi_B(\dim Y) \in \{0, 1\}$ by the results in [17].

(ii). Let $M$ be an indecomposable $A$-module and $A' = A[M]$. Then we have $\chi_A(\dim M) \in \{0, 1\}$. Assume first $\chi_A(\dim M) = 0$. As we have seen in the proof of Proposition 3, there exists a vector $v \in \text{rad} \chi_A$ such that $(\dim M, v)_A \neq 0$. Let $x$ be the extension vertex in $Q_{A'}$ such that $\text{rad} P_x = M$. Then $\langle v, e_x \rangle_{A'} = 0$ and $\langle e_x, v \rangle_{A'} = \langle \dim M, v \rangle_{A'} - \langle \dim M, v \rangle_A = -\langle \dim M, v \rangle_A \neq 0$ which implies that $\chi_{A'}$ is indeed indefinite.

Now assume $\chi_A(\dim M) = 1$. Suppose that $\chi_{A'}$ is non-negative. We shall show that $\dim M \in \chi_A^{-1}(1) \cap \chi_A^{-1}(0)^\perp$ in contradiction to (1.4). Indeed, if $v \in \chi_A^{-1}(0)$, then $\langle e_x, v \rangle_{A'} + \langle v, e_x \rangle_{A'} = 0$ (since otherwise $\chi_{A'}(2v \pm e_x) < 0$, a contradiction). Since $\langle v, e_x \rangle_{A'} = 0$, we have $0 = \langle e_x, v \rangle_{A'} = -\langle \dim M, v \rangle_A$.

6.3 Proof of the Proposition 6.1. Let $B$ be connected and convex in $A$ with $B \neq A$ and such that corank $\chi_B = 2$. By our Main Theorem, $B$ is derived equivalent to a tubular algebra or to a 2-tubular algebra. Since $B \neq A$, there exists a $B$-module $M$ such that $B[M]$ (or $[M]B$) is still convex in $A$. Since then $B[M]$ (resp. $[M]B$) is strongly simply connected, the module $M$
has to be indecomposable and by (6.2), the algebra $B$ can not be tubular.

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\square
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References


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