THE GROTHENDIECK GROUP OF A CLUSTER CATEGORY

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Abstract. For the cluster category of a hereditary or a canonical algebra, equivalently for the hereditary category of coherent sheaves on a weighted projective line, we study the Grothendieck group with respect to an admissible triangulated structure.

1. Introduction

The cluster category $\mathcal{C} = \mathcal{C}(A)$ of a finite dimensional hereditary algebra A was introduced by Buan, Marsh, Reineke, Reiten and Todorov [1], in order to realize the cluster algebras of Fomin and Zelevinsky [2] via tilting theory.

The construction of the orbit category $\mathcal{C}(A)$, see [7], generalizes to the situation where A is any k-algebra of finite global dimension. In this paper, *all* algebras will be unitary, associative and of finite dimension over an algebraically closed ground field k.

We call a triangulated structure $\mathcal{S}$ on $\mathcal{C}$ *admissible* if the canonical projection functor $\pi : \mathrm{D^b}(\mathcal{H}) \to \mathcal{C}$ is exact, that is, sends exact triangles to triangles from $\mathcal{S}$. We use the notation $\mathcal{C}_{\mathcal{S}}$ if we consider $\mathcal{C}$ as a triangulated category with triangulated structure $\mathcal{S}$.

By Keller [7], $\mathcal{C}$ admits an admissible triangulated structure in case $\mathrm{D^b}(\operatorname{mod} A)$ is triangle-equivalent to $\mathrm{D^b}(\mathcal{H})$ for some hereditary abelian k-category $\mathcal{H}$. Assuming $\mathcal{H}$ connected, by Happel's classification theorem this happens if and only if A is derived equivalent to a hereditary or a canonical algebra, see [5, 6]. In the first case, we can choose $\mathcal{H} = \operatorname{mod} A$ where A is hereditary and in the second $\mathcal{H} = \operatorname{coh} \mathbb{X}$, the category of coherent sheaves over a weighted projective line $\mathbb{X}$, see [3]. In the present paper we focus on the case $\mathcal{H} = \operatorname{coh} \mathbb{X}$, but also deal with the cases $\mathcal{H} = \operatorname{mod} A$ where A is the path algebra of a Dynkin or an extended Dynkin quiver.

Given an admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$ we study the Grothendieck group $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})$ with respect to all triangles in $\mathcal{S}$ and compare it with the Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C})$ with respect to all *induced* triangles, that is, the images of exact triangles of $\mathrm{D^b}(\operatorname{mod} A)$ under the projection π.

Assuming A of finite global dimension, we denote by Φ the Coxeter transformation on $\mathrm{K}_0(\mathrm{D^b}(\operatorname{mod} A))$, that is, the map induced by the Auslander-Reiten translation τ of $\mathrm{D^b}(\operatorname{mod} A)$. In Section 3 we show the following result.

Proposition 1.1. *If A is an algebra of finite global dimension and $\mathcal{C} = \mathcal{C}(A)$ then we have $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathrm{Coker}(1+\Phi)$.*

Let A be a hereditary algebra of finite representation type or a canonical algebra. In both cases $\overline{\mathrm{K}}_0(\mathcal{C})$ and $\mathrm{K}_0(\mathcal{C}_\mathcal{S})$ are shown to be free either over $\mathbb{Z}$ or over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (independently of the admissible triangulated structure $\mathcal{S}$). We define the *dual Grothendieck groups* $\overline{\mathrm{K}}_0(\mathcal{C})^*$ and $\mathrm{K}_0(\mathcal{C})^*$ as the respective $\mathbb{Z}$- or $\mathbb{Z}_2$-dual. In Section 4 we show our first main result.

Theorem 1.2. *We have $\mathrm{K}_0(\mathcal{C}_\mathcal{S}) = \overline{\mathrm{K}}_0(\mathcal{C})$ in each of the following three cases:*

- (i) *A is canonical with weight sequence $(p_1, \dots, p_t)$ having at least one even weight.*
- (ii) *A is tubular,*
- (iii) *A is hereditary of finite representation type.*

The remaining canonical cases are covered by the next result.

Theorem 1.3. *Assume $\mathcal{C} = \mathcal{C}(A)$ is the cluster category of a canonical algebra A with weight sequence $(p_1, \dots, p_t)$, where all weights p_i are odd. For any admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$ the Grothendieck group $\mathrm{K}_0(\mathcal{C}_\mathcal{S})$ is a non-zero quotient of $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Accordingly, if A is canonical (of any weight type), we have $\mathrm{K}_0(\mathcal{C}_\mathcal{S}) \neq 0$ and Proposition 3.7 yields an explicit basis of $\overline{\mathrm{K}}_0(\mathcal{C})$. Since each tame hereditary algebra is derived equivalent to a canonical one, Theorem 1.2 and 1.3 cover also the tame hereditary situation. To prove the two theorems our main device is to provide a categorification of suitable members of the dual Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C})^*$, that is, to realize them by additive functions on $\mathcal{C}_\mathcal{S}$ in categorical terms of $\mathcal{C}$.

B. Keller informed the authors that his student Y. Palu proved $\mathrm{K}_0(\mathcal{C}_\mathcal{S}) = \overline{\mathrm{K}}_0(\mathcal{C})$ for the admissible structure $\mathcal{S}$ constructed in [7].

In the last section, we consider the cluster category $\mathcal{C}(\mathcal{T})$ of an "isolated" tube $\mathcal{T}$. We show that there always exists an admissible triangulated structure on $\mathcal{C}(\mathcal{T})$ and determine its Grothendieck group explicitly.

2. Notations and definitions

Definition of cluster categories. We assume that A is an algebra (we recall that this means a unitary, associative algebra of finite dimension over $k = \overline{k}$) of finite global dimension. We denote by $\operatorname{mod} A$ the category of finitely generated (or equivalently finite-dimensional) right A-modules and by $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$. Since A has finite global dimension, $\mathcal{D}$ is a triangulated category, see [4], and we denote by T its suspension functor $\mathrm{T}M = M[1]$. Moreover, $\mathcal{D}$ has Auslander-Reiten triangles and the Auslander-Reiten translation τ is an auto-equivalence of $\mathcal{D}$.

Denoting $F = \tau^{-1} \circ \mathrm{T}$, the cluster category $\mathcal{C} = \mathcal{C}(A)$ is defined as the *orbit category* $\mathcal{C}(A) = \mathcal{D}/F^{\mathbb{Z}}$, whose objects are the objects of $\mathcal{D}$ and whose morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}(A)}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(X, F^i Y),$$

which are finite dimensional spaces if A is derived equivalent to a hereditary or to a canonical algebra. We denote by $\pi : \mathcal{D} \to \mathcal{C}(A)$ the canonical projection functor and write occasionally πX rather than X for objects in $\mathcal{C}$ for emphasis.

Admissible triangulated structures. We call a triangulated structure $\mathcal{S}$ on $\mathcal{C}$ *admissible* if the projection π is exact and denote by $\mathcal{C}_{\mathcal{S}}$ the category $\mathcal{C}$ equipped with $\mathcal{S}$. Keller [7] proves the existence of an admissible triangulated structure for $\mathcal{C}(A)$ if $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ is triangle equivalent to $\mathrm{D}^{\mathrm{b}}(\mathcal{H})$ for some hereditary abelian k-category $\mathcal{H}$. Then $\mathcal{H}$ has a tilting complex, hence by [6, Theorem 1.7] a tilting object. We may assume that $\mathcal{H}$ is connected. Passing to a derived equivalent hereditary category we may then assume by Happel's theorem [5] that $\mathcal{H} = \operatorname{mod} H$, where H is a hereditary algebra, or $\mathcal{H} = \operatorname{coh} \mathbb{X}$, where $\mathbb{X}$ is a weighted projective line [3]. In the first case A is derived equivalent to a hereditary, in the second case to a canonical algebra, see paragraph "Canonical algebras" below. Since $\mathcal{C}$ – up to equivalence – only depends on $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$, we can assume that A itself is hereditary or canonical. Often, we also shall write $\mathcal{C}(\mathcal{H})$ instead of $\mathcal{C}(A)$ if $\mathrm{D}^{\mathrm{b}}(\mathcal{H}) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Grothendieck groups. Any $\mathcal{C}$ as above is equipped with the autoequivalence $\tau : \mathcal{C} \to \mathcal{C}$, induced by the Auslander-Reiten translation of $\mathrm{D^b}(\mathrm{mod}\, A)$. A triangle $X \to Y \to Z \to \tau X$ in $\mathcal{C}$ is called *induced* if it is – up to isomorphism – the image under π of an exact triangle in $\mathrm{D^b}(\mathrm{mod}\, A)$. Note that τ takes the role of a suspension functor for $\mathcal{C}$, although the induced triangles usually will not define a triangulated structure on $\mathcal{C}$. We denote by $\overline{\mathrm{K}}_0(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$ with respect to all induced triangles.

If $\mathcal{S}$ is an admissible triangulated structure on $\mathcal{C}$ we denote by $\mathrm{K}_0(\mathcal{C}_\mathcal{S})$ the Grothendieck group of $\mathcal{C}$ with respect to all triangles from $\mathcal{S}$. Since each induced triangle lies in $\mathcal{S}$ we get a natural epimorphism

$$\overline{\mathrm{K}}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}_\mathcal{S}).$$

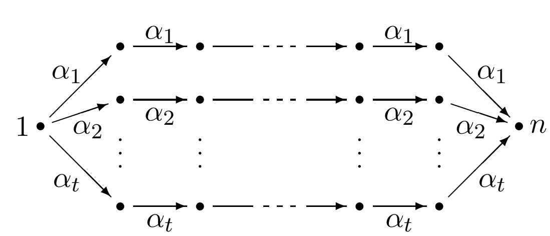
Hereditary categories. If $\mathcal{H}$ is hereditary then the derived category admits a simple description: the indecomposable objects of $\mathrm{D^b}(\mathcal{H})$ are of the form $\mathrm{T}^i X$ for $X \in \mathcal{H}$ indecomposable and some $i \in \mathbb{Z}$. The morphism spaces are given by

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(\mathrm{T}^i X, \mathrm{T}^j Y) = \mathrm{Ext}^{j-i}_{\mathcal{H}}(X, Y), \text{ for } X, Y \in \mathcal{H}. \tag{2.1}$$

In case $\mathcal{H} = \mathrm{coh}\, \mathbb{X}$, τ is an autoequivalence on $\mathcal{H}$ and therefore $\mathcal{H}$ is a fundamental region for the functor F, that is, for each indecomposable object $X \in \mathcal{D}$ there exists a unique $Y \in \mathcal{H}$ such that $X = F^i Y$ and therefore, we can identify the objects of $\mathcal{C}$ with the objects of $\mathcal{H}$ up to isomorphism.

We recall that the category $\mathrm{coh}\, \mathbb{X}$ has *Serre duality*, that is, there exists an autoequivalence τ for which $\mathrm{Ext}^1_{\mathcal{H}}(X, Y) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathcal{H}}(Y, \tau X)$ holds functorially in X and Y. Similarly the categories $\mathcal{D} = \mathrm{D^b}(\mathrm{coh}\, \mathbb{X})$ and $\mathcal{D} = \mathrm{D^b}(\mathrm{mod}\, A)$, for A hereditary, have also Serre duality in the sense that $\mathrm{Hom}_{\mathcal{D}}(X, Y[1]) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(Y, \tau X)$ holds functorially in X and Y.

Canonical algebras. Canonical algebras were introduced by C. M. Ringel in [11] as algebras $A = kQ/I$, where the quiver Q is obtained by joining a source 1 with a sink n by $t \geq 2$ arms consisting of $p_1, \ldots, p_t$ arrows respectively, all pointing from 1 to n:



The ideal I is generated by $t-2$ relations $\alpha_i^{p_i} = \alpha_2^{p_2} - \mu_i \alpha_1^{p_1}$ for some pairwise distinct $\mu_i \in k$ with $\mu_i \neq 0, 1$. The sequence $(p_1, \ldots, p_t)$ is called the *weight sequence* of A. If $\sum_{i=1}^{t} \frac{1}{p_i} = t-2$ then A is called *tubular*; this happens precisely for the weight sequences $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$. We usually omit weights $p_i = 1$ from the sequence, hence the weight sequence (3) means the sequence $(1,3)$. We recall that if A is canonical of weight type $(p_1, \ldots, p_t)$ then $\mathrm{D^b}(\operatorname{mod} A) \simeq \mathrm{D^b}(\operatorname{coh} \mathbb{X})$ for a weighted projective line $\mathbb{X}$ of weight type $(p_1, \ldots, p_t)$.

Tubes. Let $\mathbb{X}$ be a weighted projective line of weight type $(p_1, \ldots, p_t)$ and $\mathcal{H} = \operatorname{coh} \mathbb{X}$. We denote by $\mathcal{H}_0$ the full subcategory of $\mathcal{H}$ given by the objects of finite length and by $\mathcal{H}_+$ the full subcategory of direct sums of indecomposable objects of infinite length. It is known, see [3], that $\mathcal{H}_0 = \coprod_{x \in \mathbb{X}} \mathcal{T}_x$ is a coproduct of categories, where each $\mathcal{T}_x$ is a *tube of rank* q, that is a connected, hereditary, uniserial category, which in abstract form can be realized as $\operatorname{mod}_0^{\mathbb{Z}_q} k[[X]]$ (that is, as the category of $\mathbb{Z}_q$-graded $k[[X]]$-modules of finite length). Each tube of $\mathcal{H}_0$ has rank one except finitely many (exceptional) tubes having rank $p_1, \ldots, p_t$, respectively.

Furthermore $\operatorname{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0$ and for each non-zero object $M \in \mathcal{H}_+$ and each $x \in \mathbb{X}$, we have $\operatorname{Hom}_{\mathcal{H}}(M, \mathcal{T}_x) \neq 0$.

Formulas for $\mathrm{K}_0(\operatorname{coh} \mathbb{X})$. The Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of the abelian category $\mathcal{H} = \operatorname{coh} \mathbb{X}$ is described in detail in [9, 8]. It is equipped with the Euler form defined by

$$\langle [X], [Y] \rangle = \dim_k \operatorname{Hom}_{\mathcal{H}}(X, Y) - \dim_k \operatorname{Ext}^1_{\mathcal{H}}(X, Y)$$

on classes of objects X, $Y \in \mathcal{H}$. It follows from Serre duality that for all $\mathbf{x}$, $\mathbf{y} \in \mathrm{K}_0(\mathcal{H})$ we have $\langle \mathbf{y}, \mathbf{x} \rangle = -\langle \mathbf{x}, \Phi \mathbf{y} \rangle$, where Φ is the Coxeter transformation.

We denote by L the structure sheaf and for each $i = 1, \ldots, t$ the unique simple sheaf S_i belonging to the i-th exceptional tube such that $\operatorname{Hom}_{\mathcal{H}}(L, S_i) \neq 0$. Then $\operatorname{Hom}_{\mathcal{H}}(L, S_i)$ is one-dimensional, and $\operatorname{Hom}_{\mathcal{H}}(L, \tau^j S_i) = 0$ for $j = 1, \ldots, p_i - 1$. Furthermore, all simple sheaves from homogeneous tubes have the same class in $\mathrm{K}_0(\mathcal{H})$; we fix one, say S_0. Now define the following elements of $\mathrm{K}_0(\mathcal{H})$:

$$\mathbf{a} = [L], \quad \mathbf{s}_0 = [S_0], \quad \mathbf{s}_i = [S_i] \text{ for } i = 1, \ldots, t.$$

Define then the elements $\mathbf{s}_i(j) = \Phi^j \mathbf{s}_i$ for $j \in \mathbb{Z}_{p_i}$. For later use we reproduce some facts from [9].

Proposition 2.1. *Let* $\mathcal{H} = \operatorname{coh} \mathbb{X}$ *where* $\mathbb{X}$ *is of weight type* $(p_1, \ldots, p_t)$.

(a) *The abelian group* $\mathrm{K}_0(\mathcal{H})$ *is generated by the elements* $\mathbf{a}$, $\mathbf{s}_0$, $\mathbf{s}_i(j)$, $i = 1, \dots, t$ *and* $j = 0, \dots, p_i - 1$, *subject to the defining relations*

$$\sum_{j=0}^{p_i-1} \mathbf{s}_i(j) = \mathbf{s}_0, \quad \textit{for } i = 1, \dots, t. \tag{2.2}$$

(b) *Define* $p = \mathrm{lcm}(p_1, \dots, p_t)$, $\delta = p\left(t - 2 - \sum_{i=1}^{t} \frac{1}{p_i}\right)$ *and* $\mathrm{rk}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{s}_0 \rangle$. *Then for all* $\mathbf{x} \in \mathrm{K}_0(\mathcal{H})$, *we have*

$$\Phi^p \mathbf{x} = \mathbf{x} + \delta \cdot \mathrm{rk}(\mathbf{x}) \cdot \mathbf{s}_0$$

(c) *We have*

$$\Phi \mathbf{a} = \mathbf{a} - \sum_{i=1}^{t} \mathbf{s}_i + (t-2) \cdot \mathbf{s}_0$$

(d) *Furthermore, we have* $\langle \mathbf{s}_i(m), \mathbf{s}_j(n) \rangle = 0$ *for* $i \neq j$, *and*

$$\langle \mathbf{s}_i(m), \mathbf{s}_i(n) \rangle = \begin{cases} 1 & \textit{if } n \equiv m \bmod p_i \\ -1 & \textit{if } n \equiv m + 1 \bmod p_i \\ 0 & \textit{else} \end{cases}$$

for all $i = 1, \dots, t$. □

Beside the rank function $\mathrm{rk}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{s}_0 \rangle$ we also define the *degree* function by

$$\deg(\mathbf{x}) = \sum_{j=0}^{p-1} \langle \Phi^j \mathbf{a}, \mathbf{x} - \mathrm{rk}(\mathbf{x}) \mathbf{a} \rangle$$

where $p = \mathrm{lcm}(p_1, \dots, p_t)$. It is characterized by the properties $\deg(L) = 0$, $\deg(S_0) = p$ and $\deg(\tau^j S_i) = \frac{p}{p_i}$ for $i = 1, \dots, t$ and $j \in \mathbb{Z}$.

Discriminant and slope. Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ be of weight type $(p_1, \dots, p_t)$ and put $p = \mathrm{lcm}(p_1, \dots, p_t)$. The *discriminant*

$$\delta_{\mathcal{H}} = p\big((t-2) - \sum_{i=1}^{t} 1/p_i\big)$$

is an invariant of $\mathcal{H}$ deciding on the complexity of the classification problem for $\mathcal{H}$, hence for $\mathcal{C}(\mathcal{H})$, see [3]. For $\delta_{\mathcal{H}} < 0$ the category $\mathcal{H}$ is derived equivalent to the category mod A for the path algebra kQ of an extended Dynkin quiver, and each such algebra kQ has this property. For $\delta_{\mathcal{H}} = 0$ we are dealing with the tubular weights, and for $\delta_{\mathcal{H}} > 0$ the classification problem for $\mathcal{H}$ is wild. For this and the following statements we refer to [3].

Each bundle E has a *line bundle filtration* $0 = E_1 \subset E_1 \subset \cdots \subset E_r = E$ where each E_i/E_{i-1} is a line bundle. For each non-zero bundle E its *slope* $\mu(E) = \deg(E)/\operatorname{rk}(E)$ is a rational number such that

$$\mu(\tau E) = \mu(E) + \delta_{\mathcal{H}}$$

holds. By means of line bundle filtrations for E and F it follows that $\mathrm{Hom}_{\mathcal{H}}(E, F) = 0$ if $\mu(E) - \mu(F)$ is sufficiently large. In particular, for $\delta_{\mathcal{H}} > 0$ (resp. $\delta_{\mathcal{H}} < 0$) we have $\mathrm{Hom}_{\mathcal{H}}(\tau^n E, F) = 0$ (resp. $\mathrm{Hom}_{\mathcal{H}}(E, \tau^n F) = 0$ for $n \gg 0$.

3. Grothendieck group with respect to induced triangles

In this section, we describe the Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C})$ with respect to the induced triangles. Let $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$. Then the Coxeter transformation $\Phi : \mathrm{K}_0(\mathcal{D}) \to \mathrm{K}_0(\mathcal{D})$ is given by $\Phi([X]) = [\tau X]$ for any object X of $\mathcal{D}$.

Proof of Proposition 1.1. The projection $\pi : \mathcal{D} \to \mathcal{C}$ sends exact triangles to induced triangles, hence yields an epimorphism

$$\mathrm{K}_0(\mathcal{D}) \to \overline{\mathrm{K}}_0(\mathcal{C}), [X] \mapsto [\pi X].$$

We have $[FX] = -[\tau X]$ in $\mathrm{K}_0(\mathcal{D})$, hence $[\pi X] = [\pi F X] = -[\pi \tau X]$ in $\overline{\mathrm{K}}_0(\mathcal{C})$ showing that $\pi(1 + \Phi) = 0$. In order to prove the exactness of

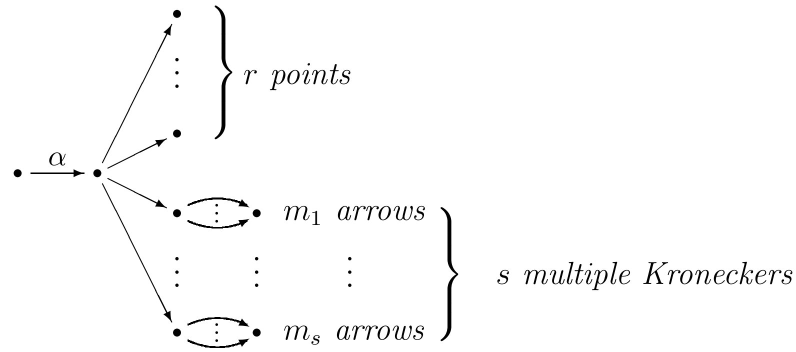
$$\mathrm{K}_0(\mathcal{D}) \xrightarrow{1+\Phi} \overline{\mathrm{K}}_0(\mathcal{C}) \xrightarrow{\pi} \overline{\mathrm{K}}_0(\mathcal{C}) \to 0$$

it therefore suffices to show that each morphism $\lambda : \mathrm{K}_0(\mathcal{D}) \to G$, for G an abelian group, with $\lambda(1+\Phi) = 0$ induces a morphism $\overline{\lambda} : \overline{\mathrm{K}}_0(\mathcal{C}) \to G$ with $\lambda = \overline{\lambda}\pi$.

By the assumption $\lambda(1 + \Phi) = 0$ the function $\lambda : \mathcal{D} \to G$ is constant on F-orbits and additive on exact triangles of $\mathcal{D}$, hence induces a function $\overline{\lambda} : \mathcal{C} \to G$ which is additive on induced triangles. □

Explicit description of $\overline{\mathrm{K}}_0(\mathcal{C})$. Write $\mathbb{Z}_m$ for $\mathbb{Z}/m\mathbb{Z}$. We have the following general description of $\overline{\mathrm{K}}_0(\mathcal{C})$.

Proposition 3.1. *Let A be any algebra of finite global dimension and let $\mathcal{C} = \mathcal{C}(A)$. Then $\overline{\mathrm{K}}_0(\mathcal{C})$ has a unique expression as $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} (\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i})$, for natural numbers r, s and positive $m_1, \ldots, m_s$ such that m_i divides m_{i+1} for all i. Moreover, any such group occurs as $\overline{\mathrm{K}}_0(\mathcal{C}(H))$, where H is the hereditary path algebra given by the following quiver.*

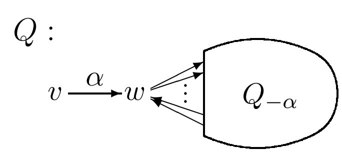


Before we enter the proof we need some preparatory lemmas. If Q is a quiver, we denote by B_Q the adjacency matrix of Q, that is, $(B_Q)_{ij}$ denotes the number of arrows in Q from i to j. Let C be the Cartan matrix of A, that is, a matrix representing the Euler form. Since A has finite global dimension, C has determinant ± 1 and $\Phi = -C^{-1}C^{\mathrm{tr}}$.

Lemma 3.2. *Let $A = kQ$ be a hereditary algebra and $\mathcal{C} = \mathcal{C}(A)$. Then we have $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathrm{Coker}(B_Q - B_Q^{\mathrm{tr}})$.*

Proof. Since A is finite-dimensional, Q can not contain an oriented cycle. Hence the vertices of Q can be ordered such that C is upper triangular. Thus we see that B_Q is nilpotent, hence $C = 1 + B_Q + B_Q^2 + B_Q^3 + \ldots$ is a finite sum and $C^{-1} = 1 - B_Q$. Therefore $1 + \Phi = (C^{-\mathrm{tr}} - C^{-1})C^{\mathrm{tr}} = ((1 - B_Q)^{\mathrm{tr}} - (1 - B_Q))C^{\mathrm{tr}} = (B_Q - B_Q^{\mathrm{tr}})C^{\mathrm{tr}}$, which shows that $\mathrm{Coker}(1+\Phi) = \mathrm{Coker}(B_Q - B_Q^{\mathrm{tr}})$, thus the result follows by Proposition 1.1. □

We call an arrow $\alpha : v \to w$ of a quiver Q a *source-arrow* if v is a source of Q and α is the unique arrow of Q starting in v. Similarly an arrow $\alpha : w \to v$ is a *sink-arrow* if v is a sink and α the unique arrow ending in v. In both cases we denote by $Q_{-\alpha}$ the quiver obtained from Q by removing the vertices v and w and all arrows starting or ending in v or w. The situation of a source-arrow is depicted as follows.



The next result is quite useful for calculating $\overline{\mathrm{K}}_0(\mathcal{C})$ in practice.

Lemma 3.3. *Let Q be a quiver with an arrow α, which is a source- or a sink-arrow. Denote $H = kQ$ and $H' = kQ_{-\alpha}$. Then we have $\overline{\mathrm{K}}_0(\mathcal{C}(H)) \simeq \overline{\mathrm{K}}_0(\mathcal{C}(H'))$.*

Proof. Assume that α is a source arrow (the case where α is a sink-arrow is similar). By renumbering the vertices, we can assume that α is the arrow $1 \to 2$. Then we have

$$B_Q - B_Q^{\mathrm{tr}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \rho^{\mathrm{tr}} \\ 0 & -\rho & B_{Q_{-\alpha}} - B_{Q_{-\alpha}}^{\mathrm{tr}} \end{bmatrix}.$$

Adding multiples of the first row to the rows $3, \dots, n$ and simultaneously adding (the same) multiples of the second column to the columns $3, \dots, n$ we obtain a transformation matrix T and a block diagonal matrix

$$T(B_Q - B_Q^{\mathrm{tr}})T^{\mathrm{tr}} = \operatorname{diag}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_{Q_{-\alpha}} - B_{Q_{-\alpha}}^{\mathrm{tr}})$$

and the result follows by Lemma 3.2. □

Proof of Proposition 3.1. If C denotes the Cartan matrix of A then $1 + \Phi = (C^{-\mathrm{tr}} - C^{-1})C^{\mathrm{tr}}$. Now $S = (C^{-\mathrm{tr}} - C^{-1})$ is skew-symmetric. Clearly, we have $\operatorname{Coker}(1 + \Phi) = \operatorname{Coker} S$.

Using the skew-normal form of S, see [10, Theorem IV.1], we obtain $S' = U^{\mathrm{tr}} S U$ for some $U \in \mathbb{GL}_n(\mathbb{Z})$, where $S' = \operatorname{diag}(B_0, B_1, \dots, B_s)$ is a block-diagonal matrix with the following blocks: B_0 is the zero matrix of size $r \times r$ and for $i = 1, \dots, s$,

$$B_i = \begin{bmatrix} 0 & m_i \\ -m_i & 0 \end{bmatrix}$$

where m_i divides m_{i+1} for all $i = 1, \dots, s-1$. Therefore $\operatorname{Im} S \simeq \operatorname{Im} S' \simeq \bigoplus_{i=1}^{s} (m_i \mathbb{Z})^2$ and we obtain $\overline{\mathrm{K}}_0(\mathcal{C}) \simeq \operatorname{Coker} S' \simeq \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} (\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i})$ as desired.

Let H be the hereditary algebra defined by the quiver in Proposition 3.1 and denote $H' = kQ_{-\alpha}$. By Lemma 3.3, we have $\overline{\mathrm{K}}_0(\mathcal{C}(H)) \simeq \overline{\mathrm{K}}_0(\mathcal{C}(H'))$. Now, the claim is obvious for H' since $B_{Q_{-\alpha}}^{-1} - B_{Q_\alpha}^{-\mathrm{tr}} = \operatorname{diag}(B_0, B_1, \dots, B_s)$ is the block-diagonal matrix as above. □

The hereditary case.

Proposition 3.4. *If A is a hereditary algebra whose quiver is a tree then $\overline{\mathrm{K}}_0(\mathcal{C}(A))$ is a free abelian group.*

Proof. Any tree can be reduced to a disjoint union of r vertices, for some r, by cutting off source- and sink-arrows. Hence, we get $\overline{\mathrm{K}}_0(\mathcal{C}(A)) \simeq \mathbb{Z}^r$ by Lemma 3.3. □

Proposition 3.5. *Let A be a connected hereditary representation-finite algebra, that is, the underlying graph of its quiver is a Dynkin diagram Δ. Then, we have the following description.*

$$\overline{\mathrm{K}}_0(\mathcal{C}) = \begin{cases} 0, & \text{if } \Delta = \mathbb{A}_n,\ \mathbb{E}_n \text{ with } n \text{ even} \\ \mathbb{Z}, & \text{if } \Delta = \mathbb{A}_n,\ \mathbb{D}_n,\ \mathbb{E}_7 \text{ with } n \text{ odd} \\ \mathbb{Z}^2, & \text{if } \Delta = \mathbb{D}_n \text{ with } n \text{ even} \end{cases}$$

Proof. This follows immediately using Lemma 3.3. □

The canonical case. We now assume that A is canonical of weight type $(p_1, \dots, p_t)$ and $\mathcal{H}$ is the associated category of coherent sheaves. We put $\mathcal{C} = \mathcal{C}(A) = \mathcal{C}(\mathcal{H})$ and start by describing $\overline{\mathrm{K}}_0(\mathcal{C})$ by generators and defining relations.

Proposition 3.6. *The abelian group $\overline{\mathrm{K}}_0(\mathcal{C})$ is generated by the elements $\overline{\mathbf{a}}, \overline{\mathbf{s}}_0, \overline{\mathbf{s}}_1, \dots \overline{\mathbf{s}}_t$ subject to the following defining relations.*

$$2\overline{\mathbf{s}}_0 = 0, \tag{3.1}$$

$$2\overline{\mathbf{a}} = \sum_{i=1}^{t} (\overline{\mathbf{s}}_i - \overline{\mathbf{w}}), \tag{3.2}$$

$$\overline{\mathbf{s}}_0 = \frac{1 - (-1)^{p_i}}{2} \overline{\mathbf{s}}_i, \text{ for } i = 1, \dots, t. \tag{3.3}$$

Proof. We recall from Proposition 2.1(a), that $\mathrm{K}_0(\mathcal{H})$ is the abelian group generated by $\{\overline{\mathbf{a}}, \overline{\mathbf{s}}_0, \overline{\mathbf{s}}_i(j) \mid i = 1, \dots, t \text{ and } j = 0, \dots, p_i - 1\}$ subject to the defining relations (2.2). Therefore $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathrm{K}_0(\mathcal{H}) / \operatorname{Im}(1 + \Phi)$ is the abelian group generated by the same generators with the relations (2.2) and the additional relations

$$\overline{\mathbf{a}} + \Phi\overline{\mathbf{a}} = 0, \tag{3.4}$$

$$\overline{\mathbf{s}}_0 + \Phi\overline{\mathbf{s}}_0 = 0 \text{ and} \tag{3.5}$$

$$\overline{\mathbf{s}}_i(j) + \Phi\overline{\mathbf{s}}_i(j) = 0 \text{ for } i = 1, \dots, t \text{ and } j = 1, \dots, p_i, \tag{3.6}$$

which altogether form a system of defining relations. Using Proposition 2.1(c), we can rewrite (3.4) as (3.2). Using $\Phi\mathbf{s}_0 = \mathbf{s}_0$ we rewrite (3.5) as (3.1). Using $\Phi\mathbf{s}_i(j) = \mathbf{s}_i(j+1)$ and (2.2) we obtain

$$\overline{\mathbf{s}}_0 = \sum_{j=0}^{p_1 - 1} (-1)^j \overline{\mathbf{s}}_i$$

which can be rewritten in the form (3.3). Thus, since $\Phi\mathbf{s}_i(j) = \mathbf{s}_i(j+1)$, the group $\overline{\mathrm{K}}_0(\mathcal{C})$ is generated by $\overline{\mathbf{a}}$, $\overline{\mathbf{s}}_0$, $\overline{\mathbf{s}}_i = \overline{\mathbf{s}}_i(0)$ for $i = 1, \dots, t$ subject to the defining relations (3.1), (3.2) and (3.3). □

Proposition 3.7. *Let* $\mathcal{H} = \operatorname{coh}\mathbb{X}$ *with weight sequence* $(p_1,\dots,p_t)$ *where* $p_1,\dots,p_r$ *are even and* $p_{r+1},\dots,p_t$ *are odd. Further let* $\mathcal{C} = \mathcal{C}(\mathcal{H})$.

(i) *If* $r \geq 1$ *then* $\overline{\mathrm{K}}_0(\mathcal{C})$ *is the free abelian group on* $\overline{\mathbf{a}}, \overline{\mathbf{s}}_2, \dots, \overline{\mathbf{s}}_r$.

(ii) *If* $r = 0$ *(that is, all weights* p_i *are odd) then* $\overline{\mathrm{K}}_0(\mathcal{C}) \simeq \mathbb{Z}\overline{\mathbf{a}} \oplus \mathbb{Z}\overline{\mathbf{s}}_0 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. Let first $r \geq 1$. Then, by (3.3), we have $\overline{\mathbf{s}}_0 = \frac{1-(-1)^{p_1}}{2}\overline{\mathbf{s}}_1 = 0$ and for $i > r$, we obtain $\overline{\mathbf{s}}_i = 0$, again by (3.3). Therefore $\overline{\mathbf{s}}_1 = 2\overline{\mathbf{a}} - \sum_{i=2}^{r} \overline{\mathbf{s}}_i$ because of (3.2). It follows that $\overline{\mathbf{a}}, \overline{\mathbf{s}}_2, \dots, \overline{\mathbf{s}}_r$ generate $\overline{\mathrm{K}}_0(\mathcal{C})$ without relations.

Now let $r = 0$. Then we obtain from (3.3) that $\overline{\mathbf{s}}_i = \overline{\mathbf{s}}_0$ for all $i = 1,\dots,t$. Therefore we get that $\overline{\mathrm{K}}_0(\mathcal{C})$ is generated by $\overline{\mathbf{a}}$ and $\overline{\mathbf{s}}_0$ with the remaining defining relations $2\overline{\mathbf{s}}_0 = 0$ and $2\overline{\mathbf{a}} = 0$. □

The dual Grothendieck groups. In the sequel the Grothendieck groups $\overline{\mathrm{K}}_0(\mathcal{C})$ and $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})$ are free over $\mathbb{Z}$ or $\mathbb{Z}_2$, respectively. We define *dual Grothendieck groups* $\overline{\mathrm{K}}_0(\mathcal{C})^*$ and $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})^*$ forming the respective $\mathbb{Z}$- or $\mathbb{Z}_2$-duals.

We first deal with the $\mathbb{Z}$-free case. Since the Cartan matrix has determinant ± 1, the Euler form induces an isomorphism $\mathrm{K}_0(\mathcal{H}) \xrightarrow{\sim} \mathrm{K}_0(\mathcal{H})^*$, $\mathbf{y} \mapsto \langle \mathbf{y}, - \rangle$. A linear form $\lambda : \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ induces a linear form $\overline{\lambda} : \overline{\mathrm{K}}_0(\mathcal{C}) \to \mathbb{Z}$ if and only if $\lambda \circ (1 + \Phi) = 0$.

Lemma 3.8. *A linear form* $\lambda = \langle \mathbf{y}, - \rangle$ *satisfies* $\lambda \circ (1+\Phi) = 0$ *if and only if* $\Phi\mathbf{y} = -\mathbf{y}$. *In particular, in this case* $\operatorname{rk}\mathbf{y} = 0$ *and* $\deg \mathbf{y} = 0$.

Proof. We have $\langle \mathbf{y}, - \rangle \circ (1+\Phi) = 0$ if and only if $\langle \mathbf{y}, \Phi^{-1}\mathbf{x} \rangle + \langle \mathbf{y}, \Phi\Phi^{-1}\mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathrm{K}_0(\mathcal{H})$, and since $\langle \mathbf{y}, \Phi\mathbf{x} \rangle = \langle \Phi^{-1}\mathbf{y}, \mathbf{x} \rangle$ this is equivalent to $\langle \mathbf{y} + \Phi\mathbf{y}, - \rangle = 0$. Since the Cartan matrix has determinant ± 1 the assertion follows. □

For any abelian group G define $G_2 = G \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Furthermore let $\operatorname{rk}_2, \deg_2 : \mathrm{K}_0(\mathcal{H})_2 \to \mathbb{Z}_2$ be the functions induced by rk and deg. Similarly define $\langle -, - \rangle_2 : \mathrm{K}_0(\mathcal{H})_2 \times \mathrm{K}_0(\mathcal{H})_2 \to \mathbb{Z}_2$ to be induced by the Euler form.

Proposition 3.9. *Let* $\mathcal{H} = \operatorname{coh}\mathbb{X}$ *with weight sequence* $(p_1,\dots,p_t)$ *and set* $\mathcal{C} = \mathcal{C}(\mathcal{H})$. *The group* $\langle \Phi \rangle$ *acts on* $\mathrm{K}_0(\mathcal{H})$ *by* $\Phi.\mathbf{y} = -\Phi\mathbf{y}$.

(i) *If there is at least one even weight* p_i *then there is an isomorphism*
$$\mathrm{K}_0(\mathcal{H})^{\langle \Phi \rangle} \xrightarrow{\sim} \overline{\mathrm{K}}_0(\mathcal{C})^*, \ \mathbf{y} \mapsto \langle \mathbf{y}, - \rangle$$

which gives rise to an exact sequence

$$0 \to \overline{\mathrm{K}}_0(\mathcal{C})^* \to \mathrm{K}_0(\mathcal{H}) \xrightarrow{1+\Phi} \mathrm{K}_0(\mathcal{H}) \to \overline{\mathrm{K}}_0(\mathcal{C}) \to 0.$$

(ii) *If all weights are odd, then there is an isomorphism*

$$\mathrm{K}_0(\mathcal{H})_2^{\langle\Phi\rangle} \xrightarrow{\sim} \overline{\mathrm{K}}_0(\mathcal{C})^*,\ \mathbf{y} \mapsto \langle \mathbf{y}, -\rangle_2$$

which gives rise to an exact sequence

$$0 \to \overline{\mathrm{K}}_0(\mathcal{C})^* \to \mathrm{K}_0(\mathcal{H})_2 \xrightarrow{1+\Phi} \mathrm{K}_0(\mathcal{H})_2 \to \overline{\mathrm{K}}_0(\mathcal{C}) \to 0.$$

Proof. Part (i) follows from Lemma 3.8 and the proof of (ii) is similar using reduction modulo 2. □

If $\mathbf{x} \in \mathrm{K}_0(\mathcal{H})$ is a Φ-periodic object with period $q_{\mathbf{x}}$, we define

$$v(\mathbf{x}) = \sum_{j=0}^{q_{\mathbf{x}}-1} (-1)^j \Phi^j \mathbf{x}$$

and if $q_{\mathbf{x}}$ is even, we define

$$h(\mathbf{x}) = \sum_{j=0}^{\frac{q_{\mathbf{x}}}{2}-1} \Phi^{2j}\mathbf{x}.$$

Proposition 3.10. *Let $\mathcal{H} = \operatorname{coh}\mathbb{X}$ with weight sequence $(p_1, \dots, p_t)$ where $p_1, \dots, p_r$ are even and $p_{r+1}, \dots, p_t$ are odd. Further let $\mathcal{C} = \mathcal{C}(\mathcal{H})$.*

(i) *If $r \geq 1$ then*

$$\langle v(\mathbf{s}_1), -\rangle, \langle h(\mathbf{s}_2) - h(\mathbf{s}_1), -\rangle, \dots, \langle h(\mathbf{s}_r) - h(\mathbf{s}_1), -\rangle \tag{3.7}$$

is a $\mathbb{Z}$-basis of $\overline{\mathrm{K}}_0(\mathcal{C})^*$.

(ii) *If $r = 0$ (that is, all weights p_i are odd) then* rk_2 *and* $\deg_2$ *is a $\mathbb{Z}_2$-basis of $\overline{\mathrm{K}}_0(\mathcal{C})^*$.*

Proof. (i) Clearly $(1+\Phi)v(\mathbf{s}_1) = 0$ since p_1 is even. Furthermore, $(1+\Phi)h(\mathbf{s}_i) = \mathbf{s}_0$ for $i = 1, \dots, r$ and therefore, by the Proposition 3.9, we get that (3.7) are indeed elements of $\overline{\mathrm{K}}_0(\mathcal{C})^*$. From the formulas

$$\langle v(\mathbf{s}_1), \mathbf{a}\rangle = 1, \qquad \langle v(\mathbf{s}_1), \mathbf{s}_h\rangle = 0$$
$$\langle h(\mathbf{s}_j) - h(\mathbf{s}_1), \mathbf{a}\rangle = 0, \qquad \langle h(\mathbf{s}_j) - h(\mathbf{s}_1), \mathbf{s}_h\rangle = \delta_{jh}$$

it follows that (3.7) forms a $\mathbb{Z}$-basis of $\overline{\mathrm{K}}_0(\mathcal{C})^*$.

(ii) We know that $\overline{\mathbf{a}}, \overline{\mathbf{s}}_0$ is a $\mathbb{Z}_2$-basis of $\overline{\mathrm{K}}_0(\mathcal{C})$ by Proposition 3.7. We have $\Phi\mathbf{s}_0 = \mathbf{s}_0$ and therefore $\mathrm{rk}_2 = \langle -, \mathbf{s}_0\rangle_2$ defines a linear form on $\overline{\mathrm{K}}_0(\mathcal{H})$.

Since $\overline{\mathbf{s}}_i = \overline{\mathbf{s}}_0$ for $i = 1, \ldots, t$, we get from Proposition 2.1(d) that $\Phi\overline{\mathbf{a}} = \overline{\mathbf{a}} \bmod 2$. Hence we get

$$\deg_2(\mathbf{x}) = \sum_{j=0}^{p} \langle \Phi^j \mathbf{a}, \mathbf{x} - \mathrm{rk}(\mathbf{x})\mathbf{a} \rangle_2 = \langle \mathbf{a}, \mathbf{x} - \mathrm{rk}(\mathbf{x})\mathbf{a} \rangle_2 = \langle \mathbf{a}, \mathbf{x} \rangle_2 + \mathrm{rk}_2(\mathbf{x}).$$

Thus, also deg induces a linear map $\deg_2 : \overline{\mathrm{K}}_0(\mathcal{C}) \to \mathbb{Z}_2$. Since $\mathrm{rk}_2(\mathbf{a}) = 1$, $\mathrm{rk}_2(\mathbf{s}_0) = 0$ and $\deg_2(\mathbf{a}) = 0$, $\deg_2(\mathbf{s}_0) = 1$, it follows that rk_2, $\deg_2$ form a $\mathbb{Z}_2$-basis of $\overline{\mathrm{K}}_0(\mathcal{C})^*$. □

4. Additive functions on $\mathcal{C}_{\mathcal{S}}$

Cutting technique. For a finite dimensional k-vector space V let $|V|$ (resp. $|V|_2$) denote its k-dimension (resp. its k-dimension modulo two). We put $\mu_E(X) = |\operatorname{Hom}_{\mathcal{C}}(E, X)|$ and write $\overline{\mu}_E(X)$ for $\mu_E(X)$ modulo two.

In the sequel we identify members λ from the dual Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C})^*$ with mappings λ defined on $\mathcal{C} = \mathcal{C}(\mathcal{H})$ with values in $\mathbb{Z}$, respectively in $\mathbb{Z}_2$, that are additive on induced triangles. We call λ *realizable* if, depending on the case considered, it has the form $\mu_E - \mu_F$ (resp. $\overline{\mu}_E$) with E and F from $\mathcal{C}$. The realizable functions form a subgroup of $\overline{\mathrm{K}}_0(\mathcal{C})^*$. Note that usually neither μ_E nor $\overline{\mu}_E$ or $\mu_E - \mu_F$ are realizable. Our next proposition shows how to construct realizable functions which additionally belong to $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})^*$ for an admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$.

For any object $U \in \mathcal{D} = \mathrm{D^b}(\mathcal{H})$ and any positive integer q define the function $\lambda_X^{(q)}$ on the objects Y of $\mathcal{C}$ by

$$\lambda_U^{(q)} : \mathcal{C} \to \mathbb{Z},\ \lambda_U^{(q)}(Y) = \sum_{i=0}^{q-1} (-1)^i |\operatorname{Hom}_{\mathcal{C}}(\pi U, \mathrm{T}^i Y)| \tag{4.1}$$

and set $\overline{\lambda}_U^{(q)} : \mathcal{C} \to \mathbb{Z}_2,\ Y \mapsto \lambda_U^{(q)}(Y) \bmod 2$

Proposition 4.1. *Suppose that U is an object in $\mathrm{D^b}(\mathcal{H})$ such that for some positive integer q we have $\tau^q X \simeq F^m X$ for some $m \in \mathbb{Z}$.*

(i) *If q is even then $\lambda_X^{(q)}$ is additive on each triangle of an admissible triangulated structure on $\mathcal{C}$.*
(ii) *If q is odd, then $\overline{\lambda}_X^{(q)}$ is additive on each triangle of an admissible triangulated structure on $\mathcal{C}$.*

Proof. Identify U with its image in $\mathcal{C}$. Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathrm{T}X$ be a triangle in $\mathcal{C}$ with respect to an admissible triangulated structure.

Application of the functor $\mathrm{Hom}_{\mathcal{C}}(U,-)$ gives a long exact sequence

$$\begin{aligned} 0\to K &\to \mathrm{Hom}_{\mathcal{C}}(U,X)\to \mathrm{Hom}_{\mathcal{C}}(U,Y)\to \mathrm{Hom}_{\mathcal{C}}(U,Z)\to \\ &\to \mathrm{Hom}_{\mathcal{C}}(U,\tau X)\to \mathrm{Hom}_{\mathcal{C}}(U,\tau Y)\to \cdots \\ \cdots &\to \mathrm{Hom}_{\mathcal{C}}(U,\tau^{q-1}Y)\to \mathrm{Hom}_{\mathcal{C}}(U,\tau^{q-1}Z)\to K'\to 0, \end{aligned}$$

where $K=\mathrm{Ker}(\mathcal{C}(U,\alpha))$ and

$$K'=\mathrm{Ker}(\mathcal{C}(U,\tau^q\alpha))\simeq \mathrm{Ker}(\mathcal{C}(\tau^{-q}U,\alpha))\simeq \mathrm{Ker}(\mathcal{C}(U,\alpha))=K. \tag{4.2}$$

The alternating sum of the dimensions of the spaces in the sequence equals zero. If q is even we hence get

$$\lambda_U^{(q)}(X)-\lambda_U^{(q)}(Y)+\lambda_U^{(q)}(Z)=0. \tag{4.3}$$

Therefore $\lambda_U^{(q)}$ is a linear form on $\mathrm{K}_0(\mathcal{C})$. If q is odd then this holds modulo 2. □

The even canonical case. We first study the case where $\mathcal{H}=\mathrm{coh}\,\mathbb{X}$ with weight sequence $(p_1,\dots,p_t)$. In this case we have linear forms which are defined by "periodic" elements which lie in tubes: If $U\in\mathcal{H}$ is indecomposable lying in a tube of rank q then $\tau^qU\simeq U$.

Assume that $p_1,\dots,p_r$ are even and $p_{r+1},\dots,p_t$ are odd. Let $\mathcal{T}_1,\dots,\mathcal{T}_r$ be the exceptional tubes in $\mathcal{H}_0$ of rank $p_1,\dots,p_r$, respectively, and recall that S_i is a simple object from $\mathcal{T}_i$. By Proposition 4.1 the functions $\lambda_i=\lambda_{S_i}^{(p_i)}$ are additive on the triangles of any admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$.

If $\mathbf{x}$ is an element in $\mathrm{K}_0(\mathcal{H})$, denote by $\widehat{\mathbf{x}}$ its image in $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})$.

Proposition 4.2. *Assume that the number r of even weights p_i is non-zero, then the linear forms λ_i $(i=1,\dots,r)$ are realizable, linearly independent over $\mathbb{Z}$ and $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})=\overline{\mathrm{K}}_0(\mathcal{C})\simeq\mathbb{Z}^r$.*

Proof. Linear independence of $\lambda_1,\dots,\lambda_r$ follows from $\lambda_i(S_j)=2\delta_{ij}$, where δ_{ij} denotes the Kronecker symbol. We conclude that $\widehat{\mathbf{s}}_1,\dots,\widehat{\mathbf{s}}_r$ are linearly independent and hence also $\widehat{\mathbf{a}}$, $\widehat{\mathbf{s}}_2,\dots,\widehat{\mathbf{s}}_r$, since $2\widehat{\mathbf{a}}=\sum_{i=1}^m\widehat{\mathbf{s}}_i$. Therefore $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})=\overline{\mathrm{K}}_0(\mathcal{C})\simeq\mathbb{Z}^r$ follows from Proposition 3.7 (i). □

The odd canonical case. We adopt the notations of the previous section. Recall that S_0 denotes a simple object from a homogeneous tube in $\mathcal{H}_0$.

Proposition 4.3. *Let $\mathcal{C}=\mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is the category of coherent sheaves on a weighted projective line of weight type $(p_1,\dots,p_t)$, where all weights p_i are odd. Then the following holds:*

(i) *Always* rk_2 *is a non-zero realizable member of* $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})^* \subseteq \overline{\mathrm{K}}_0(\mathcal{C})^*$.
(ii) *For* $\delta_{\mathcal{H}} \neq 0$ *the subgroup of realizable members of* $\overline{\mathrm{K}}_0(\mathcal{C})^*$ *agrees with the subgroup* $\langle \mathrm{rk}_2 \rangle$ *generated by the rank modulo two.*
(iii) *For* $\delta_{\mathcal{H}} = 0$*, that is for weight type* $(3,3,3)$*, we have equality*

$$\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})^* = \overline{\mathrm{K}}_0(\mathcal{C})^* = \mathbb{Z}_2 \,\mathrm{rk}_2 \oplus \mathbb{Z}_2 \deg_2,$$

and each member of $\overline{\mathrm{K}}_0(\mathcal{C})^*$ *is realizable.*

Proof. (i) and (iii): We invoke Proposition 3.10 and use that rk_2 can be realized as $\overline{\lambda}_0 = \overline{\lambda}_{S_0}^{(1)}$ where $\overline{\lambda}_0(L) = 1 \mod 2$ and $\overline{\lambda}_0(S_0) = 0$.
In the tubular case, only the weight type $(3,3,3)$ matters, thus the structure sheaf L lies in a tube of τ-period three. Hence $\deg_2$ is realized by $\overline{\lambda}_L^{(3)}$ where $\overline{\lambda}_L^{(3)}(S_0) = 1 \mod 2$.
(ii): Assume the function $\overline{\mu}_E(X) = |\mathrm{Hom}_{\mathcal{C}}(E,X)|_2$ with E from $\mathcal{H}$ is additive on induced triangles. We are going to show that $\overline{\mu}_E$ is a multiple of rk_2. Since $X \to X \to 0 \to \tau X$ is an induced triangle, we get

$$\overline{\mu}_E(X) = \overline{\mu}_E(\tau X) = \overline{\mu}_{\tau^{-1}E}(X)$$

for each object X of $\mathcal{H}$. Since $p = \mathrm{lcm}(p_1, \dots, p_t)$ is odd, setting $\overline{E} = \bigoplus_{j=0}^{p-1} \tau^j E$ we thus obtain $\overline{\mu}_E = \overline{\mu}_{\overline{E}}$.
Next, we use a decomposition $E = E_+ \oplus E_0$ of E into a bundle E_+ and an object E_0 of finite length. Invoking that τ^p acts as the identity on finite length objects of $\mathcal{H}$, we see that $\overline{E}_0 = \bigoplus_{j=0}^{p-1} \tau^j E$ is fixed under τ. The expression $\overline{\mu}_{\overline{E}_0}(X) = |\mathrm{Hom}_{\mathcal{H}}(\overline{E}_0, X)|_2 + |\mathrm{Ext}^1_{\mathcal{H}}(\overline{E}_0, \tau^{-1}X)|_2$ hence agrees with $\langle \overline{E}_0, X \rangle_2$, and $\overline{\mu}_{\overline{E}_0}$ is a multiple of rk_2. By part (i) the function rk_2 is additive on induced triangles, we conclude that the same holds for $\overline{\mu}_{E_+}$. From now on, we may hence assume that E is a bundle. Note that

$$\overline{\mu}_E(X) = \overline{\mu}_{\overline{E}}(X) = \langle\langle E, X \rangle\rangle_2 + \Delta_E(X),$$

where $\langle\langle E, X \rangle\rangle = \sum_{j=0}^{p-1} \langle \tau^j E, X \rangle$, $\langle\langle E, X \rangle\rangle_2 = \langle\langle E, X \rangle\rangle \mod 2$ and

$$\begin{aligned}\Delta_E(X) &= \sum_{j=0}^{p-1} \left(|\mathrm{Ext}^1_{\mathcal{H}}(\tau^j E, X)|_2 + |\mathrm{Ext}^1_{\mathcal{H}}(\tau^{j+1} E, X)|_2\right) \\ &= |\mathrm{Ext}^1_{\mathcal{H}}(E, X)|_2 + |\mathrm{Ext}^1_{\mathcal{H}}(\tau^p E, X)|_2.\end{aligned}$$

By the Riemann-Roch formula,

$$\langle\langle E, X \rangle\rangle = -\frac{p}{2}\, \delta_{\mathcal{H}} \,\mathrm{rk}(E)\, \mathrm{rk}(X) + \begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(X) \\ \deg(E) & \deg(X) \end{vmatrix}, \tag{4.4}$$

see [9], the function $\langle\langle E,-\rangle\rangle_2$ is a linear combination of rk_2 and $\deg_2$. Hence $\langle\langle E,-\rangle\rangle_2$ is a member of $\overline{\mathrm{K}}_0(\mathcal{C})^*$, implying that Δ_E also belongs to $\overline{\mathrm{K}}_0(\mathcal{C})^*$. By construction, Δ_E vanishes on S_0. By means of a line bundle filtration of E, Serre duality implies that $\Delta_E(L')=0$ for any line bundle L' of sufficiently large degree, and we deduce from Proposition 3.7 that $\Delta_E=0$. If E is of even rank, then the function $\langle\langle E,-\rangle\rangle_2$, hence also the function $\overline{\mu}_E$, is a multiple of rk_2, proving the claim in this case.

It remains to deal with the case that the rank of E is odd, where we deduce a contradiction from the assumption that $\overline{\mu}_E$ belongs to $\overline{\mathrm{K}}_0(\mathcal{C})^*$. Invoking $\overline{\mu}_E=\overline{\mu}_{\overline{E}}$, we have shown that $\Delta_{\overline{E}}=0$. Note that $\Delta_{\overline{E}}=0$ asserts that

$$|\operatorname{Ext}^1_{\mathcal{H}}(\overline{E},X)|_2=|\operatorname{Ext}^1_{\mathcal{H}}(\tau^{np}\overline{E},X)|_2 \tag{4.5}$$

for each $n\in\mathbb{Z}$ and each object X from $\mathcal{H}$.

Case $\delta_{\mathcal{H}}>0$: Clearly, the functions $\langle\overline{E},-\rangle=\langle\langle E,-\rangle\rangle$ and $\langle\tau^{np}\overline{E},-\rangle=\langle\langle\tau^{np}E,-\rangle\rangle$ are additive on induced triangles. They agree modulo two on S_0 and by formula (4.5) also on each line bundle L' of large negative degree. It then follows from Proposition 3.7 that $\langle\overline{E},-\rangle_2=\langle\tau^{np}\overline{E},-\rangle_2$.

By means of a line bundle filtration for $\overline{E}$ we obtain for each integer $n\gg 0$ two line bundles L_1 and L_2 of consecutive degrees d and $d+1$ such that

$$\operatorname{Hom}_{\mathcal{H}}(\tau^{np}\overline{E},L_i)=0 \text{ and } \operatorname{Ext}^1_{\mathcal{H}}(\overline{E},L_i)=0 \text{ for } i=1,2. \tag{4.6}$$

By (4.4) we get $\langle\overline{E},L_i\rangle_2=\alpha+\deg_2(L_i)$ for some $\alpha\in\mathbb{Z}_2$, only depending on E. We then choose one of the L_i such that (4.6) and further $\langle\overline{E},L_i\rangle_2=1$ holds. Invoking (4.5) we obtain the contradiction

$$\begin{aligned}1 &= \langle\overline{E},L_i\rangle_2=\langle\tau^{np}\overline{E},L_i\rangle_2\\ &= |\operatorname{Hom}_{\mathcal{H}}(\tau^{np}\overline{E},L_i)|_2+|\operatorname{Ext}^1_{\mathcal{H}}(\tau^{np}\overline{E},L_i)|_2=0.\end{aligned}$$

Case $\delta_{\mathcal{H}}<0$: The proof is similar, choosing $n\ll 0$. □

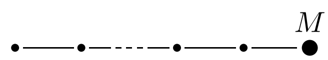
The Dynkin case. Now, let A be a connected hereditary representation-finite algebra whose quiver has as underlying graph the Dynkin diagram Δ. Then Δ is a star with length of the arms $p_1,\dots,p_t$ (where $t\le 3$) and the Auslander-Reiten quiver of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}A)$ is $\mathbb{Z}\Delta$ whose τ-orbits correspond to the vertices of Δ. In this case there are no tubes. Nevertheless we find “periodic” objects. Let m be the Coxeter number of Δ, that is, the order of the Coxeter transformation Φ. We have

$$m = \begin{cases} n+1 & \text{if } \Delta = \mathbb{A}_n, \\ 2(n-1) & \text{if } \Delta = \mathbb{D}_n, \\ 12,\ 18,\ 30 & \text{if } \Delta = \mathbb{E}_6,\ \mathbb{E}_7,\ \mathbb{E}_8,\ \text{respectively.} \end{cases}$$

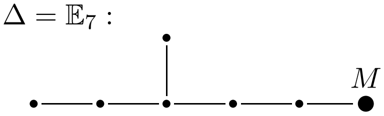
Since $\mathrm{K}_0(\mathcal{C}) = 0$ in the cases $\Delta = \mathbb{A}_n$ (n even), $\mathbb{E}_6$, $\mathbb{E}_8$ by Proposition 3.5, we restrict our attention to the remaining cases. Note that then m is always an even number.

Proposition 4.4. *In the cases $\Delta = \mathbb{A}_n$ with n odd or $\Delta = \mathbb{E}_7$, let M be an indecomposable object of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ lying in a τ-orbit as indicated in the following picture.*

$\Delta = \mathbb{A}_n$:



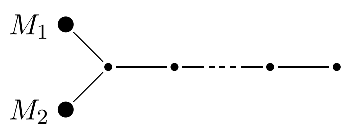
$\Delta = \mathbb{E}_7$:



Then $\lambda_M^{(m+2)}$ is a non-zero realizable function, which is additive on all triangles.

In the case $\Delta = \mathbb{D}_n$ ($n \geq 4$) one can choose indecomposable objects M_1 and M_2 of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ lying in the two τ-orbits as indicated in the following picture

$\Delta = \mathbb{D}_n$:



such that the functions $\lambda_i = \lambda_{M_i}^{(m+2)}$ for $i = 1, 2$ are non-zero realizable and, for n even, linearly independent.

Proof. For any indecomposable A-module U we have $\tau^m U \simeq \mathrm{T}^{-2} U$ in $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ which implies $\tau^{m+2} U \simeq F^{-2} U$. By Proposition 4.1 the function $\lambda_U^{(m+2)}$ is additive on triangles in $\mathcal{C}$ with respect to any admissible triangulated structure on $\mathcal{C}$.

Let $\mathcal{H} = \mathrm{mod}\, A$. For indecomposable objects M and N in $\mathcal{D}$ we have (identifying them with their images in $\mathcal{C}$)

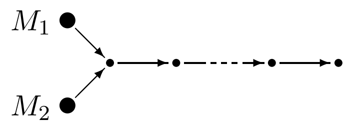
$$\begin{aligned} \lambda_M^{(m+2)}(N) &= \sum_{i=0}^{m+1} (-1)^i |\operatorname{Hom}_{\mathcal{C}}(\tau^{-i} M, N)| \\ &= \sum_{j \in \mathbb{Z}} \sum_{i=0}^{m+1} (-1)^i |\operatorname{Hom}_{\mathcal{D}}(M, \tau^{i-j} \mathrm{T}^j N)|. \end{aligned}$$

Setting $\mu_j(M,N) = \sum_{i=0}^{m+1}(-1)^i|\operatorname{Hom}_{\mathcal{D}}(M,\tau^{i-j}\mathrm{T}^jN)|$ we have $\mu_j(M,N)=0$ for $j<0$ and $M,\ N\in\mathcal{H}\cup\tau^-\mathcal{H}$.

In the case $\Delta=\mathbb{A}_n$ (n odd) we get $\lambda_M^{(m+2)}(M)=2$, where M is as indicated above. Indeed, $\mu_0(M,M)=1=\mu_1(M,M)$ and $\mu_j(M,M)=0$ for $j\geq 2$.

In the case $\Delta=\mathbb{E}_7$ we have $\lambda_M^{(m+2)}(M)=6$. Indeed, $\mu_0(M,M)=1$, $\mu_1(M,M)=3$, $\mu_2(M,M)=2$ and $\mu_j(M,M)=0$ for $j\geq 3$.

In the case $\Delta=\mathbb{D}_n$ let M_1 and M_2 be in the AR quiver lying in the following slice.



Let $\lambda_1=\lambda_{M_1}^{(m+2)}$ and $\lambda_2=\lambda_{M_2}^{(m+2)}$ where M_1 and M_2 are indicated as above. Let $M,\ M'\in\{M_1,\ M_2\}$ with $M\neq M'$. It is easy to see that $\operatorname{Hom}_{\mathcal{D}}(M,\tau^iM)\neq 0$ if and only if i is even and $-(n-2)\leq i\leq 0$. Similarly, $\operatorname{Hom}_{\mathcal{D}}(M,\tau^iM')\neq 0$ if and only if i is odd and $1\leq i\leq n-1$. Moreover,

$$\tau^{-(n-1)}M\simeq\begin{cases}\mathrm{T}M & n \text{ even},\\ \mathrm{T}M' & n \text{ odd}.\end{cases}$$

Using this we get $\mu_0(M,M)=1$, $\mu_0(M,M')=0$, and

	$\mu_1(M,M)$	$\mu_1(M,M')$	$\mu_2(M,M)$	$\mu_2(M,M')$
n even	$\frac{n}{2}$	$-\frac{n-2}{2}$	$\frac{n-2}{2}$	$-\frac{n-2}{2}$
n odd	$\frac{n-1}{2}$	$-\frac{n-1}{2}$	$\frac{n-3}{2}$	$-\frac{n-1}{2}$

and $\mu_j(M,M)=0=\mu_j(M,M')$ for $j\geq 3$. Consequently, for even n one has $\lambda_1(M_1)=n$, $\lambda_1(M_2)=-(n-2)$, $\lambda_2(M_1)=-(n-2)$ and $\lambda_2(M_2)=n$, and linear independence of λ_1 and λ_2 follows. If n is odd, then $\lambda_1(M_1)=n-1=\lambda_2(M_2)$ and $\lambda_1(M_2)=-(n-1)=\lambda_2(M_1)$. □

Proof of Theorem 1.2. For case (i) the assertion follows from Proposition 4.2, for case (ii) it follows from Proposition 4.3 and for (iii) it follows from the fact that by Proposition 1.1, $\mathrm{K}_0(\mathcal{C}_{\mathcal{S}})$ is a quotient of $\overline{\mathrm{K}}_0(\mathcal{C})$ and by Proposition 4.4 and 3.5 both are free of the same rank. □

Proof of Theorem 1.3. This follows immediately from Propositions 4.2 and 4.3. □

5. Cluster tubes

Existence of admissible structures. Let $\mathcal{T}$ be a tube of rank q. We consider the cluster category $\mathcal{C} = \mathcal{C}(\mathcal{T})$ as orbit category $\mathrm{D^b}(\mathcal{T})/F^{\mathbb{Z}}$, where again $F = \tau^{-1}\mathrm{T}$, where τ is the Auslander-Reiten translation and T the suspension functor. We call $\mathcal{C}(\mathcal{T})$ the *cluster tube* of rank q.

Since $\mathcal{T}$ has no tilting object, we can not invoke Keller's result [7] directly to conclude that $\mathcal{T}$ has an admissible triangulated structure. We now show that $\mathcal{T}$ admits an admissible structure anyway.

Proposition 5.1. *The cluster tube $\mathcal{C}(\mathcal{T})$ of rank q admits an admissible triangulated structure.*

Proof. Let $\mathbb{X}$ be a weighted projective line of weight type $(q) = (1, q)$ and let $\mathcal{H} = \operatorname{coh}\mathbb{X}$. Recall the definitions of $\mathcal{H}_0$ and $\mathcal{H}_+$ from Section 2. We may view $\mathcal{T}$ as a full subcategory of $\mathcal{H}_0$, which is even exact because of (2.1). Therefore $\mathcal{C}(\mathcal{T})$ is a full subcategory of $\mathcal{C}(\mathcal{H})$.

By [7], there exists an admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}(\mathcal{H})$. We denote by $\mathcal{S}'$ the subclass of $\mathcal{S}$ given by all triangles $X \to Y \to Z \to \mathrm{T}X$ such that $X, Y, Z \in \mathcal{C}(\mathcal{T})$. It is clear that once we show that $\mathcal{S}'$ is a triangulated structure on $\mathcal{C}(\mathcal{T})$ then it is admissible. Since $\mathcal{T}$, and then also $\mathcal{C}(\mathcal{T})$, is closed under direct sums and summands in $\mathcal{H}$, we only have to verify that $X, Y \in \mathcal{C}(\mathcal{T})$ implies $Z \in \mathcal{C}(\mathcal{T})$ for any triangle $X \to Y \to Z \to \mathrm{T}X$ in $\mathcal{S}$.

By the preceding remark, we can assume that $X, Y \in \mathcal{T}$ and $Z \in \mathcal{H}$. Write $Z = Z_+ \oplus Z_0$ where $Z_0 \in \mathcal{H}_0$ and $Z_+ \in \mathcal{H}_+$. Let $W \in \mathcal{H}$ be a simple object in some homogeneous tube $\mathcal{T}' \neq \mathcal{T}$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, W)$ to the triangle $X \to Y \to Z \to \mathrm{T}X$, we get an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{T}X, W) \to \mathrm{Hom}_{\mathcal{C}}(Z, W) \to \mathrm{Hom}_{\mathcal{C}}(Y, W)$$

whose end terms are zero, because $\mathcal{T}$ and $\mathcal{T}'$ are orthogonal in $\mathcal{H}$ and $\mathcal{C}(\mathcal{H})$. Therefore $\mathrm{Hom}_{\mathcal{C}}(Z, W) = 0$, in particular $\mathrm{Hom}_{\mathcal{H}}(Z, W) = 0$. Hence $Z_+ = 0$ and $Z_0 \notin \mathcal{T}'$. Since we can vary $\mathcal{T}' \subset \mathcal{H}_0$ we also see that $Z = Z_0 \in \mathcal{T}$. $\square$

The Grothendieck group of a cluster tube. Let $\mathcal{T}$ be a tube and $\mathcal{C} = \mathcal{C}(\mathcal{T})$ its cluster category. As in 1.1 one shows $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathrm{Coker}(1 + \Phi)$. We call an admissible triangulated structure on $\mathcal{T}$ an *induced* triangulated structure if it is obtained from an embedding of $\mathcal{T}$ in $\operatorname{coh}\mathbb{X}$ as explained in the previous paragraph.

Proposition 5.2. *Let $\mathcal{T}$ be a tube of rank q.*

(i) *If q is even then for any admissible triangulated structure $\mathcal{S}$ on $\mathcal{C} = \mathcal{C}(\mathcal{T})$ we have* $\mathrm{K}_0(\mathcal{C}_\mathcal{S}) = \overline{\mathrm{K}}_0(\mathcal{C}) \simeq \mathbb{Z}$.

(ii) *If q is odd then for any induced triangulated structure $\mathcal{S}$ on $\mathcal{C} = \mathcal{C}(\mathcal{T})$ we have* $\mathrm{K}_0(\mathcal{C}_\mathcal{S}) = \overline{\mathrm{K}}_0(\mathcal{C}) \simeq \mathbb{Z}_2$.

Proof. (i) If S is a simple object in $\mathcal{T}$ then $\mathrm{K}_0(\mathcal{T})$ is the free group generated by the elements $\mathbf{s}(j) = [\tau^j S]$, for $j \in \mathbb{Z}_q$. Therefore $\overline{\mathbf{s}}(j) = -\overline{\mathbf{s}}(j+1)$ in $\overline{\mathrm{K}}_0(\mathcal{C})$ and $\overline{\mathrm{K}}_0(\mathcal{C})$ is generated by $\overline{\mathbf{s}} = \overline{\mathbf{s}}(0)$ without relation. This shows $\overline{\mathrm{K}}_0(\mathcal{C}) = \mathbb{Z}\overline{\mathbf{s}} \simeq \mathbb{Z}$.

Finally, we can define $\lambda_S^{(q)} : \mathcal{C} \to \mathbb{Z}$ as in (4.1) which defines a linear form $\lambda : \mathrm{K}_0(\mathcal{C}_\mathcal{S}) \to \mathbb{Z}$ with $\lambda(S) = 2$. Thus $\mathrm{K}_0(\mathcal{C}_\mathcal{S})$ has at least rank one and (i) follows.

(ii) Let $S \in \mathcal{T}$ be a simple object. Then $\overline{\mathrm{K}}_0(\mathcal{C})$ is generated by $\overline{\mathbf{s}}$, where $\mathbf{s} = [S]$, and we have $2\overline{\mathbf{s}} = 0$. We show that $\overline{\mathbf{s}}$ induces a non-trivial element in $\mathrm{K}_0(\mathcal{C}_\mathcal{S})$. For any object X in $\mathcal{C}$ define

$$\lambda(X) = \sum_{j=0}^{q-1} |\operatorname{Hom}_\mathcal{C}(L, \tau^j X)|_2$$

For an object X in $\mathcal{T}$ we have $\lambda(\pi X) = \deg_2(X)$. Indeed, since $\tau^q X \simeq X$

$$\begin{aligned}\lambda(\pi X) &= \sum_{j=0}^{q-1} |\operatorname{Hom}_\mathcal{T}(L, \tau^j X)|_2 \pm \sum_{j=0}^{q-1} |\operatorname{Ext}^1_\mathcal{T}(L, \tau^{j-1} X)|_2 \\ &= \sum_{j=0}^{q-1} \langle L, \tau^j X\rangle_2 = \deg_2(X).\end{aligned}$$

In particular, $\lambda(\pi S) = 1 \neq 0$. Now, λ is additive on triangles in $\mathcal{C}$, which is shown with a version of the cutting technique similar to the proof of Proposition 4.1. In order to show that $K \simeq K'$ like in (4.2) we use that τ^q is the identity functor on $\mathcal{T}$. □

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