MODULE VARIETIES OVER CANONICAL ALGEBRAS

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Abstract. The main purpose of this paper is the study of module varieties over the class of canonical algebras, providing a rich source of examples of varieties with interesting properties. Our main tool is a stratification of module varieties, which was recently introduced by Richmond. This stratification does not require a precise knowledge of the module category. If it is finite, then it provides a method to classify irreducible components. We determine the canonical algebras for which this stratification is finite. In this case, we describe the algorithm for calculating the dimension of the variety and the number of irreducible components of maximal dimension. For an infinite family of examples we give easy combinatorial criteria for irreducibility, Cohen-Macaulay and normality.

1. Introduction and Main Results

1.1. **Canonical algebras.** Throughout, let k be an algebraically closed field. Any finite-dimensional k-algebra A is then Morita equivalent to kQ/I, where Q is the quiver of A and I an admissible ideal in the path algebra kQ, see [1] or [15] for details. We denote by Q_0 the set of vertices and by Q_1 the set of arrows of Q. For an arrow α of Q, we denote by $s(\alpha)$ its start point and by $e(\alpha)$ its end point.

An important class of algebras are the *canonical algebras*, introduced in [15]. Such an algebra depends on two data, the *type* $p = (p_1, \cdots, p_t)$ where $t \geq 3$ and the p_i's are integers with $p_i \geq 2$, and a *weight sequence* $\lambda = (\lambda_3, \cdots, \lambda_t)$ of pairwise different non-zero elements in k. Given p and λ, the associated canonical algebra $C(p, \lambda)$ equals kQ_p/I_λ. Here Q_p is the quiver with vertices

$$Q_0 = \{\alpha, \omega, (i,j) \mid 1 \leq i \leq t, 1 \leq j \leq p_i - 1\}$$

and arrows

$$Q_1 = \{\gamma_{ij} \mid 1 \leq i \leq t, 1 \leq j \leq p_i\},$$

where $s(\gamma_{ip_i}) = \alpha$, $s(\gamma_{ij}) = (i,j)$ if $j < p_i$, $e(\gamma_{i1}) = \omega$ and $e(\gamma_{ij}) = (i, j-1)$ if $j > 1$. The ideal I_λ of kQ_p is generated by

$$\{\gamma_{11} \cdots \gamma_{1p_1} + \lambda_i \gamma_{21} \cdots \gamma_{2p_2} - \gamma_{i1} \cdots \gamma_{ip_i} \mid 3 \leq i \leq t\}.$$

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Note that we may assume $\lambda_3 = 1$, see Remark 4.1 for details. Canonical algebras are *quasi-tilted*, i.e. their global dimension $\mathrm{gldim}(C)$ is at most 2 and each indecomposable finite-dimensional module M has projective dimension $\mathrm{projdim}(M)$ or injective dimension $\mathrm{injdim}(M)$ bounded by 1.

1.2. **Module varieties.** We are now going to define the objects of our study, which are certain module varieties over a finite-dimensional k-algebra $A = kQ/I$.

By mod_A we denote the category of finite-dimensional (left) A-modules. Recall that the vertices of Q correspond to the isomorphism classes of simple A-modules. For a vertex x of Q we denote the corresponding simple module by S_x. Hence, the Grothendieck group $\mathrm{K}_0(A)$ of A may be identified with $\mathbb{Z}^{Q_0}$. Namely, for an A-module M and $x \in Q_0$, let $(\underline{\dim} M)_x$ be the multiplicity of S_x in a composition series of M. We call $\underline{\dim} M : Q_0 \to \mathbb{Z}, x \mapsto (\underline{\dim} M)_x$ the *dimension vector* of M. A dimension vector $\mathbf{d}$ is called *sincere* if $d_x \geq 1$ for all x. Finally, we denote $|\mathbf{d}| = \sum_{x \in Q_0} d_x$.

If $\mathbf{d} = (d_x)_{x \in Q_0}$ is a dimension vector of some A-module, then let $\mathrm{mod}_A(\mathbf{d})$ be the subcategory of mod_A containing the modules with dimension vector $\mathbf{d}$. We identify $\mathrm{mod}_A(\mathbf{d})$ with the category $\mathrm{rep}_{(Q,I)}(\mathbf{d})$ of representations of the bounded quiver (Q, I) with dimension vector $\mathbf{d}$. Thus, we may view $\mathrm{mod}_A(\mathbf{d})$ as an affine variety, see, for example, [2] or [14].

1.3. **Main results.** Let $\mathcal{R}$ be a minimal set of relations which generate the ideal I, and for $x, y \in Q_0$ let r_{xy} be the number of relations from x to y in $\mathcal{R}$. It is well known that r_{xy} does not depend on the choice of $\mathcal{R}$. For a dimension vector $\mathbf{d}$ let

$$a(\mathbf{d}) = \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{e(\alpha)} - \sum_{x,y \in Q_0} r_{xy} d_x d_y.$$

It follows from a generalization of Krull's principal ideal theorem that each irreducible component of $\mathrm{mod}_A(\mathbf{d})$ has dimension at least $a(\mathbf{d})$. It is important to know when the dimension of $\mathrm{mod}_A(\mathbf{d})$ equals $a(\mathbf{d})$. In this case, one can prove in many situations additional properties like Cohen-Macaulay or normality, see [8] for the definitions of the geometrical concepts used here.

Theorem 1.1. *Let C be a canonical algebra, and let $\mathbf{d}$ be a sincere dimension vector. There exists a module M in $\mathrm{mod}_C(\mathbf{d})$ with $\mathrm{projdim}(M) \leq 1$ if and only if $\sum_{i=1}^{t} \max\{0, d_\alpha - d_{ij} \mid 1 \leq j \leq p_i - 1\} \leq 2d_\alpha$. In this case, the following hold:*

(1) *If $\dim \mathrm{mod}_C(\mathbf{d}) = a(\mathbf{d})$, then $d_\alpha + (m-2)d_\omega \leq 1 + \sum_{\ell=1}^{m} d_{i_\ell j_\ell}$ for $3 \leq m \leq t$, all $1 \leq i_1 < \cdots < i_m \leq t$ and all $j_1, \cdots, j_m$;*
(2) *If $\mathrm{mod}_C(\mathbf{d})$ is irreducible, then $d_\alpha + (m-2)d_\omega \leq \sum_{\ell=1}^{m} d_{i_\ell j_\ell}$ for $3 \leq m \leq t$, all $1 \leq i_1 < \cdots < i_m \leq t$ and all $j_1, \cdots, j_m$.*

Note that one can dualize this theorem by exchanging the values of d_α and d_ω and by replacing the condition $\mathrm{projdim}(M) \leq 1$ by $\mathrm{injdim}(M) \leq 1$.

Theorem 1.2. *Let C be a canonical algebra of type $(p_1, p_2, 2)$, and let $\mathbf{d}$ be a sincere dimension vector. Then the following hold:*

(1) *If $d_\alpha + d_\omega \leq d_{1j_1} + d_{2j_2} + d_{31} + 1$ for all j_1, j_2, then $\dim \mathrm{mod}_C(\mathbf{d}) = a(\mathbf{d})$;*
(2) *If $d_\alpha + d_\omega \leq d_{1j_1} + d_{2j_2} + d_{31}$ for all j_1, j_2, then $\mathrm{mod}_C(\mathbf{d})$ is irreducible and a complete intersection. In particular, it is Cohen-Macaulay;*
(3) *If $d_\alpha + d_\omega \leq d_{1j_1} + d_{2j_2} + d_{31} - 1$ for all j_1, j_2, then $\mathrm{mod}_C(\mathbf{d})$ is normal.*

If C is of type $(p_1, p_2, 2)$, and if M is a C-module with $\mathrm{projdim}(M) \leq 1$ or $\mathrm{injdim}(M) \leq 1$, then one can combine the above theorems in order to get a necessary and sufficient condition for $\dim \mathrm{mod}_C(\underline{\dim} M) = a(\underline{\dim} M)$ and for the irreducibility of $\mathrm{mod}_C(\underline{\dim} M)$. Compare this with the classical example given in 4.7. We expect that similar results can be proved by the same methods as used here for the other subfinite canonical algebras, see 1.5 and Theorem 2.16. However, the proofs will be considerably more technical.

1.4. **Remarks on previous works.** For small types p the module category mod_C over a canonical algebra $C = C(p, \lambda)$ is well known, that is if the type p equals $(p_1, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$, $(5, 3, 2)$, $(6, 3, 2)$, $(3, 3, 3)$, $(4, 4, 2)$ or $(2, 2, 2, 2)$. In these cases C is *tame*, see [7] or [9] for a precise definition. In all other cases C is *wild*, and a classification of the indecomposable modules is regarded to be impossible.

Previous work done on the study of module varieties involved the knowledge on the module category. The examples, which are studied in [2], [3] and [4], are mainly of the form $\mathrm{mod}_A(\mathbf{d})$, where all indecomposable A-modules are known, and one assumes additionally that there exists an indecomposable A-module with dimension vector $\mathbf{d}$. For example, it is shown in [3] that, if A is tame and quasi-tilted, and if there exists an indecomposable module in $\mathrm{mod}_A(\mathbf{d})$, then $\mathrm{mod}_A(\mathbf{d})$ is always of dimension $a(\mathbf{d})$, and the number of irreducible components is at most 2. It seems impossible to apply the methods, which are used in the proofs of these results, to situations where the indecomposable A-modules are not known. Also in the situations which are studied in [6], [11], [12] or [16] there exists a good knowledge of the corresponding module categories.

The presented work shows, that the above results do no longer hold for wild quasi-tilted algebras. In 4.5 we provide examples for the following phenomenon: For $m \geq 0$ let C_m be the canonical algebra of type $(m + 6, m + 6, 2)$. Then there exists an indecomposable C_m-module M such that $\dim \mathrm{mod}_{C_m}(\underline{\dim} M) = a(\underline{\dim} M) + m + 1$, and $\mathrm{mod}_{C_m}(\underline{\dim} M)$ is not *equidimensional*, i.e. there exist irreducible components of different dimensions.

1.5. **Richmond's theorem.** Since knowledge on the module category over a wild algebra A is scarce, we use a different strategy. Our main tool is a stratification of the module variety $\mathrm{mod}_A(\mathbf{d})$, which was introduced in [13] by Richmond and will be explained now.

Let $\mathbf{d}$ be a dimension vector with $|\mathbf{d}| = n$. Let $\mathcal{S}_A(\mathbf{d})$ be a set of representatives of isomorphism classes of submodules of A^n which have dimension vector $\underline{\dim}(A^n) - \mathbf{d}$. For each L in $\mathcal{S}_A(\mathbf{d})$ let $\mathrm{mod}_A(\mathbf{d})_L$ be the points M in $\mathrm{mod}_A(\mathbf{d})$ such that there exists a short exact sequence $0 \longrightarrow L \longrightarrow A^n \longrightarrow M \longrightarrow 0$ of A-modules. Such a set is called *stratum.* Note that $\mathrm{mod}_A(\mathbf{d})$ is the disjoint union of the $\mathrm{mod}_A(\mathbf{d})_L$'s where L runs through $\mathcal{S}_A(\mathbf{d})$. If U and V are in $\mathcal{S}_A(\mathbf{d})$, then define $\mathrm{mod}_A(\mathbf{d})_U \leq \mathrm{mod}_A(\mathbf{d})_V$ if $\mathrm{mod}_A(\mathbf{d})_U$ is contained in the closure of $\mathrm{mod}_A(\mathbf{d})_V$. This defines a partial order on the strata in $\mathrm{mod}_A(\mathbf{d})$. The following theorem can be found in [13] and plays a central role in all our proofs. We think that it can be applied in many other important situations as well.

Theorem 1.3 (Richmond)**.** $\mathrm{mod}_A(\mathbf{d})_L$ *is a smooth, irreducible affine variety of dimension* $\dim_k \mathrm{Hom}_A(L, A^n) - \dim_k \mathrm{End}_A(L) - n^2 + \sum_{x \in Q_0} d_x^2$. *Furthermore, it is locally closed in* $\mathrm{mod}_A(\mathbf{d})$*.*

Note that this is a slightly modified version of Richmond's theorem. We formulate her result for varieties of representations of quivers whereas she formulates it in terms of the variety of k-algebra homomorphisms from A to the set of $n \times n$-matrices. The precise connection between these points of view is described in [5].

It is easy to check that in case $\mathcal{S}_A(\mathbf{d})$ is finite, the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ are exactly the closures of the starta which are maximal with respect to the partial order $\leq$ as defined above. The algebra A is called *subfinite* if $\mathcal{S}_A(\mathbf{d})$ is a finite set for all $\mathbf{d}$. For $m \geq 0$ let $\mathcal{J}(A^m)$ be a set of representatives of isomorphism classes of indecomposable submodules of A^m. We assume $\mathcal{J}(A^m) \subseteq \mathcal{J}(A^{m+1})$ for all m and define $\mathcal{J}_A = \bigcup_{m \geq 1} \mathcal{J}(A^m)$.

If there exists some minimal integer $s(A)$ such that for all $\mathbf{d}$ each module in $\mathcal{S}_A(\mathbf{d})$ is isomorphic to a module of the form $\bigoplus_{i=1}^m U_i$ with $U_i \in \mathcal{J}(A^{s(A)})$ for all i, then we call A *subfinite of rank* $s(A)$. We call A $\mathbf{d}$-*subfinite*, if $\mathcal{S}_A(\mathbf{d})$ is a finite set.

In Section 2, the subfinite canonical algebras are classified (Theorem 2.16), and the submodules of free modules are described. Our main results are proved in Section 3 and follow as special cases from 3.7 and 3.8. In Section 4, we give some examples.

2. **Classification of subfinite canonical algebras**

Throughout this section, let $C = C(p, \lambda) = kQ_p/I_\lambda$ be a canonical algebra.

2.1. **General considerations.** We start with a number of simple observations.

Lemma 2.1. *If there exists an m such that $\mathcal{J}(A^m)$ is not finite, then A is not subfinite.*

We denote by P_x the projective cover of the simple module S_x, associated to the vertex $x \in Q_0$ and abbreviate $P_{ij} := P_{(i,j)}$. Note that for any vertex $x \in Q_0$, $x \neq \alpha$, all submodules of the projective module P_x are again projective. Therefore, if U is an indecomposable submodule of C^n, which admits a non-zero morphism to some P_x, $x \neq \alpha$, then U is of the form P_y for some vertex $y \neq \alpha$. If there is no such morphism U is either isomorphic to P_α or a submodule of $\mathrm{rad}P_\alpha^n$. Thus, by 2.1 a canonical algebra is subfinite if and only if for each natural number n the module $\mathrm{rad}P_\alpha^n$ admits only finitely many isomorphism classes of submodules. Define $R = \mathrm{rad}P_\alpha$. To simplify notations, and just for this section, we call a submodule U of a free module *exceptional* if U does not admit a non-zero projective direct summand.

Lemma 2.2. *Let U be an indecomposable exceptional C-module. Then either U is isomorphic to R, or U admits a non-zero morphism to a maximal submodule of R.*

For a t-tupel $h = (h_1, \cdots, h_t)$ with $0 \leq h_i \leq p_i - 1$ and $1 \leq i \leq t$ we define $U(h)$ to be the submodule of R given by

$$\begin{aligned} U(h)(i,j) &= \begin{cases} 0, & \text{if } j > h_i \\ R(i,j), & \text{if } j \leq h_i \end{cases} \\ U(h)(\omega) &= R(\omega), \\ U(h)(\gamma_{ij}) &= \begin{cases} 0, & \text{if } j > h_i \\ R(\gamma_{ij}), & \text{if } j \leq h_i \end{cases}. \end{aligned}$$

Let $t(h) = |\{i \mid h_i \neq 0, 1 \leq i \leq t\}|$. Observe that $U(h)$ is decomposable if and only if $t(h) \leq 2$ if and only if $U(h)$ is projective. Define $\mathcal{H}_{\text{ns}} = \{h \mid t(h) \geq 3\}$.

Lemma 2.3. *We have $\mathcal{J}(C) = \{P_x, U(h) \,|\, x \in Q_0,\, h \in \mathcal{H}_{\text{ns}}\}$.*

Proof. Show that any indecomposable non-projective submodule of R is of the form $U(h)$. This is a straightforward calculation. □

Corollary 2.4. *If $\mathbf{d}_\alpha = 1$, then C is $\mathbf{d}$-subfinite, and each module in $\mathcal{S}_C(\mathbf{d})$ is isomorphic to a module of the form $\bigoplus_{i=1}^m U_i$ with $U_i \in \mathcal{J}(C)$ for all i.*

Corollary 2.5. *If $U \in \mathcal{S}_C(\mathbf{d})$, then U admits at most d_α direct summands of the form $U(h)$ with $h \in \mathcal{H}_{\text{ns}}$.*

Lemma 2.6. *If $g, h \in \mathcal{H}_{\text{ns}}$, then we have* $\mathrm{Hom}_C(U(h), U(g)) = k$ *if $g_i \leq h_i$ for all i, and* $\mathrm{Hom}_C(U(h), U(g)) = 0$, *else.*

2.2. Canonical algebras with five arms are not subfinite.

Lemma 2.7. *If C is subfinite, then $t \leq 4$.*

Proof. Clearly, it is enough to show that the canonical algebra C of type $(2,2,2,2,2)$ with $\lambda = (1, \lambda_4, \lambda_5)$ is not subfinite. For $a \in k$ let M_a be the representation of Q_p given by $M_a(\alpha) = 0$, $M_a(i,1) = k^3$ for $1 \leq i \leq 4$, $M_a(5,1) = k^2$ and $M_a(\omega) = k^8$, and for $1 \leq i \leq 5$ the maps $M_a(\gamma_{i1}) : M_a(i,1) \to M_a(\omega)$ are given by the matrices

$$\begin{bmatrix} 0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&0\\0&0&0\\0&0&1\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\\lambda_4&0&0\\0&1&0\\0&\lambda_4&0\\0&0&1\\0&0&\lambda_4\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0\\\lambda_5&0\\0&1\\0&\lambda_5\\1&1\\\lambda_5&\lambda_5\\1&a\\\lambda_5&a\lambda_5 \end{bmatrix}.$$

With some patience, the reader may easily verify that M_a is not isomorphic to M_b whenever $a \neq b$. Furthermore, M_a can be embedded into $\mathrm{rad}P_\alpha^4$ for any a. □

2.3. Three arms.

Lemma 2.8. *Let C be of type (p_1, p_2, p_3), and let U be an exceptional module. For $i, j \in \{1,2,3\}$, $i \neq j$, we have $\mathrm{Im}U(\gamma_{i1}) \oplus \mathrm{Im}U(\gamma_{j1}) = U(\omega)$.*

Proof. Since U is a submodule of R^n for some n, we get $\mathrm{Im}U(\gamma_{i1}) \cap \mathrm{Im}U(\gamma_{j1}) = 0$. For simplicity assume that $i = 1$ and $j = 2$, and suppose that there exists some $v \in U(\omega) \setminus (\mathrm{Im}U(\gamma_{11}) + \mathrm{Im}U(\gamma_{21}))$. If $v \neq \mathrm{Im}U(\gamma_{31})$, then P_ω is a direct summand of U. Otherwise, if ℓ is maximal such that $v \in \mathrm{Im}U(\gamma_{31}) \cdots U(\gamma_{3\ell})$, then one easily checks that U is isomorphic to $P_{3\ell} \oplus U'$. In both cases, we get a contradiction. □

Corollary 2.9. *Let C be of type (p_1, p_2, p_3). If U is an exceptional module and n minimal such that there exists an embedding $f : U \to R^n$, then $f(1,1)$, $f(2,1)$, $f(3,1)$ and $f(\omega)$ are isomorphisms.*

Proof. Let $U_i = \mathrm{Im}f(i,1)$, $V = \mathrm{Im}f(\omega) = k^{2n}$, and let $\phi_i : U_i \to V$ be the morphism induced by $R^n(\gamma_{i1})$. Let $b_1, \cdots, b_s$ be a basis of U_1. By the previous lemma there exist $c_1, \cdots, c_s \in U_2$ and $d_1, \cdots, d_s \in U_3$ such that

$$\phi_1(b_j) = \phi_2(c_j) + \phi_3(d_j) \quad \text{for all } j,$$

that is

$$\begin{bmatrix} b_j \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c_j \end{bmatrix} + \begin{bmatrix} d_j \\ d_j \end{bmatrix}.$$

Thus, we get $b_j = c_j = d_j$ for all j. Again by the lemma, we have $\dim U_1 = \dim U_2 = \dim U_3$. This implies $U_1 = U_2 = U_3 \subseteq k^n = R^n(1,1)$, and

$$\begin{bmatrix} b_1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} b_s \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ b_s \end{bmatrix}$$

is a basis of $V = \mathrm{Im} f(\omega)$. Since n was chosen minimal, we must have $s = n$, hence the result. □

For any positive natural numbers n_1, n_2 and n_3 let $\Sigma(n_1, n_2, n_3)$ be the hereditary algebra whose quiver is a star with one sink σ and 3 branches with n_1, n_2 and n_3 points, respectively. More precisely, let Q' be the quiver of $\Sigma(n_1, n_2, n_3)$ with vertices $Q'_0 = \{\sigma, (i,j) \mid 1 \leq i \leq 3,\ 2 \leq j \leq n_i\}$, and the arrows are $Q'_1 = \{\gamma_{ij} \mid 1 \leq i \leq 3,\ 2 \leq j \leq n_i\}$ with $s(\gamma_{ij}) = (i,j)$ for all i, j, $e(\gamma_{i2}) = \sigma$ and $e(\gamma_{ij}) = (i, j-1)$ for $1 \leq i \leq 3$ and $3 \leq j \leq n_i$.

Proposition 2.10. *Let C be of type (p_1, p_2, p_3), and let $A = \Sigma(p_1 - 1, p_2 - 1, p_3 - 1)$. Then there exists an equivalence*

$$\Phi : \mathrm{mod}^{\iota}_A \to \mathrm{mod}^{\mathrm{exc}}_C,$$

where $\mathrm{mod}^{\mathrm{exc}}_C$ is the full subcategory of mod_C given by the exceptional C-modules, and mod^{ι}_A is the additive hull in mod_A given by the indecomposable A-modules M satisfying $M(\sigma) \neq 0$.

Proof. We give the explicit construction of Φ. For $U \in \mathrm{mod}^{\iota}_A$ define $M = \Phi(U)$ by setting $M(\omega) = U(\sigma)^2$, $M(i,1) = U(\sigma)$ for $1 \leq i \leq 3$, $M(i,j) = U(i,j)$ and $M(\gamma_{ij}) = U(\gamma_{ij})$ for $1 \leq i \leq 3$ and $2 \leq j \leq p_i - 1$ and finally

$$M(\gamma_{11}) = \begin{bmatrix} U(\gamma_{11}) \\ 0 \end{bmatrix}, \quad M(\gamma_{21}) = \begin{bmatrix} 0 \\ U(\gamma_{21}) \end{bmatrix}, \quad M(\gamma_{31}) = \begin{bmatrix} U(\gamma_{31}) \\ U(\gamma_{31}) \end{bmatrix}.$$

For $f \in \mathrm{Hom}_A(U, V)$ with $U, V \in \mathrm{mod}^{\iota}_A$, define $g = \Phi(f)$ by $g_\omega = f_\sigma^2$, $g_{i1} = f_\sigma$ for $1 \leq i \leq 3$ and $g_{ij} = f_{ij}$ for $1 \leq i \leq 3$ and $2 \leq j \leq p_i - 1$.

Clearly, Φ is full and faithful. By the previous corollary, Φ is also dense, hence an equivalence. □

As a direct consequence we obtain the following results.

Corollary 2.11. *If C is of type (p_1, p_2, p_3), then C is subfinite if only if $A = \Sigma(p_1 - 1, p_2 - 1, p_3 - 1)$ is Dynkin. In this case, C is subfinite of rank $\max\{\dim M(\sigma) \mid M \in \mathrm{mod}^{\iota}_A, M$ indecomposable$\}$.*

Corollary 2.12. *If C is of type $(p_1, p_2, 2)$, then C is subfinite of rank 1.*

2.4. **Four arms.** Finally, we deal with the case where C is a canonical algebra of type (p_1, p_2, p_3, p_4). In the following, we will encript the dimension of the morphism space between two indecomposable modules M and N in quivers with relations in the following way. Namely, $\dim_k \mathrm{Hom}_C(M, N) = \dim_k kQ/I(M, N)$ where Q is a quiver having M and N as vertices, and

I is an ideal generated by linearly independent relations indicated by dotted arrows. By $\tau = \mathrm{Hom}_k(\mathrm{Hom}_C(?, C), k)$ we denote the Auslander-Reiten translate.

Lemma 2.13. *If C is a canonical algebra of type $(2, 2, 2, 2)$, then C is subfinite of rank 2, and the elements of $\mathcal{J}_C$ are the vertices of the following quiver, whereas the dimension of the morphism spaces can be read off from the picture below.*

$$\begin{tikzcd}
& P_{11} \arrow[ddr] \arrow[rr, dotted] & & U(0,1,1,1) \arrow[ddr] & & \\
& P_{21} \arrow[dr] \arrow[rr, dotted] & & U(1,0,1,1) \arrow[dr] & & \\
P_\omega \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted] & & \tau^- P_\omega \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted, shift left] \arrow[rr, dotted, shift right] & & R \arrow[r] & P_\alpha \\
& P_{31} \arrow[ur] \arrow[rr, dotted] & & U(1,1,0,1) \arrow[ur] & & \\
& P_{41} \arrow[uur] \arrow[rr, dotted] & & U(1,1,1,0) \arrow[uur] & &
\end{tikzcd}$$

Proof. Let $U \in \mathcal{J}(C^m)$ for some m. Then U is either projective or exceptional. If U is exceptional, but not isomorphic to R, then by 2.2 there exists a non-zero morphism to a maximal submodule of R. The maximal submodules of R are $U(0, 1, 1, 1) = \tau^- P_{11}$, $U(1, 0, 1, 1) = \tau^- P_{21}$, $U(1, 1, 0, 1) = \tau^- P_{31}$ and $U(1, 1, 1, 0) = \tau^- P_{41}$. Observe that these are postprojective modules. Since the postprojective component is standard and directed, U is also postprojective, and $U = \tau^- P_x$ for some $x \neq \alpha$. A direct calculation shows that $\tau^- P_\omega \in \mathcal{J}(C^2)$, hence the result. $\square$

Denote by $\overline{C}$ the canonical algebra of type $(3, 2, 2, 2)$ having the same weights as C. Clearly, a submodule $\overline{U}$ of $\overline{C}^m$ with $\overline{U}(\alpha) = 0$ may be viewed as a module over $\overline{C}_\circ = \overline{C}/(\alpha)$. The restriction of such a module to $C_\circ = C/(\alpha)$ may be viewed as a submodule U of C^m satisfying $U(\alpha) = 0$.

Let $\mathcal{U}$ (resp. $\overline{\mathcal{U}}$) be the full subcategory of mod_C (resp. $\mathrm{mod}_{\overline{C}}$) given by submodules X of free modules satisfying $X(\alpha) = 0$. Then $\overline{\mathcal{U}}$ is equivalent to the subspace category $\mathcal{V}(\mathcal{U}, \mathrm{Hom}_C(P_{11}, ?))$, that is its objects are triples (V, f, X) consisting of a vector space V, an object $X \in \mathcal{U}$ and a linear map $f : V \to \mathrm{Hom}_C(P_{11}, X)$. A morphism $\varphi = (\varphi_0, \varphi_1) : (V, f, X) \to (V', f', X')$ is a pair consisting of a linear map $\varphi_0 \in \mathrm{Hom}_k(V, V')$ and a morphism $\varphi_1 \in \mathrm{Hom}_C(X, X')$ such that $\mathrm{Hom}_C(P_{12}, \varphi_1)f = f'\varphi_0$.

In the following, we abbreviate $U_{\hat{1}} := U(0, 1, 1, 1)$, $U_{\hat{2}} := U(1, 0, 1, 1)$, $U_{\hat{3}} := U(1, 1, 0, 1)$, $U_{\hat{4}} := U(1, 1, 1, 0)$ and $Z := \tau^- P_\omega$. Further, choose non-zero morphisms $\beta : P_{11} \to Z$, $\gamma_i : Z \to U_{\hat{i}}$ and $\delta_i : U_{\hat{i}} \to R$ for $1 \leq i \leq 4$.

Lemma 2.14. *Let C be a canonical algebra of type $(3,2,2,2)$. Then C is subfinite of rank 3, the indecomposable submodules of free modules are*

$$X^\circ := (0,0,X) \text{ for } X \in \mathcal{U},$$
$$\overline{P}_{11} := (k, \mathrm{id}, P_{11}),\ \overline{Z} := (k,\beta,Z),\ R = (k, \delta_2\gamma_2\beta, R),$$
$$\overline{U}_{\hat{i}} := (k, \gamma_i\beta, U_{\hat{i}}) \text{ for } 1 \le i \le 4,$$
$$Y_1 = (k, \begin{bmatrix} \gamma_2\beta \\ \gamma_3\beta \\ \gamma_4\beta \end{bmatrix}, U_{\hat2} \oplus U_{\hat3} \oplus U_{\hat4}),\ Y_2 = (k^2, \begin{bmatrix} \gamma_2\beta & 0 \\ 0 & \gamma_3\beta \\ \gamma_4\beta & \gamma_4\beta \end{bmatrix}, U_{\hat2} \oplus U_{\hat3} \oplus U_{\hat4}),$$
$$\overline{U}_{\hat{i}\hat{j}} := (k, \begin{bmatrix} \gamma_i\beta \\ \gamma_j\beta \end{bmatrix}, U_{\hat{i}} \oplus U_{\hat{j}}) \text{ for } i < j \in \{2,3,4\},$$

and the morphism spaces can be read off from the following picture.

$$\begin{tikzcd}
 & & \overline{P}_{11} \arrow[rrr, dotted] \arrow[dr] & & & U^\circ_{\hat1} \arrow[drr] & & & & \\
 & P^\circ_{11} \arrow[ur] \arrow[ddr] \arrow[rr, dotted] & & \overline{Z} \arrow[urr] \arrow[ddr] \arrow[rrrr, dotted] & & & & R^\circ \arrow[ddr] & & \\
 & P^\circ_{21} \arrow[dr] \arrow[rr, dotted] & & U^\circ_{\hat2} \arrow[dr] \arrow[rr, dotted] & & \overline{U}_{\hat3\hat4} \arrow[dr] \arrow[rr, dotted] & & \overline{U}_{\hat2} \arrow[dr] & & \\
P^\circ_\omega \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted] & & Z^\circ \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted] & & Y_1 \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted] & & Y_2 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dotted] & & R \arrow[r] & P_\alpha \\
 & P^\circ_{31} \arrow[ur] \arrow[rr, dotted] & & U^\circ_{\hat3} \arrow[ur] \arrow[rr, dotted] & & \overline{U}_{\hat2\hat4} \arrow[ur] \arrow[rr, dotted] & & \overline{U}_{\hat3} \arrow[ur] & & \\
 & P^\circ_{41} \arrow[uur] \arrow[rr, dotted] & & U^\circ_{\hat4} \arrow[uur] \arrow[rr, dotted] & & \overline{U}_{\hat2\hat3} \arrow[uur] \arrow[rr, dotted] & & \overline{U}_{\hat4} \arrow[uur] & &
\end{tikzcd}$$

Proof. Since $\mathcal{U}$ is well known, $\overline{\mathcal{U}}$ can be calculated explicitely by the well known technique of subspace categories. □

Proposition 2.15. *Let C be a canonical algebra of type $p = (p_1, p_2, p_3, p_4)$. Then C is subfinite if and only if p equals $(2,2,2,2)$ or $(3,2,2,2)$.*

Proof. The sufficiency follows from 2.13 and 2.14. In order to show that for all other types the canonical algebra is not subfinite, it is sufficient to show that canonical algebras $\overline{\overline{C}}$ of type $(4,2,2,2)$ and of type $(3,3,2,2)$ are not subfinite. Again, the full subcategory $\overline{\overline{\mathcal{U}}}$ of $\mathrm{mod}_{\overline{\overline{C}}}$ given by submodules X of free modules satisfying $X(\alpha) = 0$ is equivalent to $\mathcal{V}(\overline{\mathcal{U}}, F)$, where $F = \mathrm{Hom}_{\overline{C}}(P_{12}, ?)$ or $F = \mathrm{Hom}_{\overline{C}}(P_{21}, ?)$, respectively. In both cases, we have $\dim_k F(Y_2) = 2$. Thus, $\overline{\overline{\mathcal{U}}}$ is not finite, and thus $\overline{\overline{C}}$ is not subfinite. □

2.5. **Classification of subfinite canonical algebras.**

Theorem 2.16. *A canonical algebra C is subfinite if and only if it is of type $(p_1, p_2, 2)$, $(p_1, 3, 3)$, $(4,4,3)$, $(5,4,3)$, $(6,4,3)$, $(2,2,2,2)$ or $(3,2,2,2)$.*

It turns out that each canonical algebra is either subfinite of a certain rank or not subfinite at all. One might ask whether this holds for all finite-dimensional algebras.

By the above result, in particular by the description of all submodules of free modules in the subfinite case, we obtain an algorithm from 1.3 for computing the dimension of the variety and the irreducible components of maximal dimension. In fact, for those subfinite cases with $t = 3$ we realized this algorithm as a computer program. Note that the subfinite canonical algebras with $t = 3$ can be divided according to 2.11 into the cases $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$. For the rest of this article we mainly focus on the most simple case $(p_1, p_2, 2)$, which corresponds to the Dynkin type $\mathbb{A}_{p_1+p_2-3}$. However, we expect similar results for the remaining subfinite cases.

3. Proof of the main results

3.1. Existence of modules of projective dimension at most 1.

Proposition 3.1. *Let $A = kQ/I$, and let $\mathbf{d}$ be a dimension vector. There exists a projective module $P \in \mathcal{S}_A(\mathbf{d})$ if and only if $\operatorname{mod}_A(\mathbf{d})$ contains a module M with $\operatorname{projdim}(M) \leq 1$. If this is the case, and if Q has no oriented cycles, then P is uniquely determined, $\operatorname{mod}_A(\mathbf{d})_P = \{M \in \operatorname{mod}_A(\mathbf{d}) \mid \operatorname{projdim}(M) \leq 1\}$ and the closure of $\operatorname{mod}_A(\mathbf{d})_P$ is an irreducible component. If additionally $\operatorname{gldim}(A) \leq 2$, then $\dim\ \operatorname{mod}_A(\mathbf{d})_P = a(\mathbf{d})$.*

Proof. The first part is clear. Since the function $M \mapsto \operatorname{projdim}(M)$ is upper-semicontineous, we know that $\{M \in \operatorname{mod}_A(\mathbf{d}) \mid \operatorname{projdim}(M) \leq 1\}$ is an open set in $\operatorname{mod}_A(\mathbf{d})$. In case Q has no oriented cycles, it is obvious that this set is equal to $\operatorname{mod}_A(\mathbf{d})_P$ for some projective module $P \in \mathcal{S}_A(\mathbf{d})$. This set is irreducible by 1.3. Since it is additionally open, we get that its closure is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$. The last statement of the proposition follows from Proposition 2.2 in [2]. □

For the rest of this section, let $C = C(p = (p_1, \cdots, p_t), \lambda) = kQ_p/I_\lambda$ be a canonical algebra, and let $\mathbf{d}$ be a sincere dimension vector with $|\mathbf{d}| = n$.

Let $\mathbf{d}^{\mathrm{op}}$ be the dimension vector with $d^{\mathrm{op}}_\alpha = d_\omega$, $d^{\mathrm{op}}_\omega = d_\alpha$ and $d^{\mathrm{op}}_{ij} = d_{ip_i-j}$ for $1 \leq i \leq t$ and $1 \leq j \leq p_i - 1$. The following two lemmas are an easy consequence of the fact that C is isomorphic to its opposite algebra C^{op}.

Lemma 3.2. *The affine varieties $\operatorname{mod}_C(\mathbf{d})$ and $\operatorname{mod}_C(\mathbf{d}^{\mathrm{op}})$ are isomorphic.*

Lemma 3.3. *There exists a module of injective dimension at most 1 in $\operatorname{mod}_C(\mathbf{d})$ if and only if there exists a module of projective dimension at most 1 in $\operatorname{mod}_C(\mathbf{d}^{\mathrm{op}})$.*

We define a dimension vector $\mathbf{d}^*$ as follows: Let $d^*_\alpha = d^*_\omega = 0$, for $1 \leq i \leq t$ let $d^*_{ip_i-1} = \max\{0, d_\alpha - d_{ip_i-1}\}$, and for $1 \leq i \leq t$ and $1 \leq j \leq p_i - 2$ define $d^*_{ij} = \max\{0, d_\alpha - d_{ij} - \sum_{\ell=j+1}^{p_i-1} d^*_{il}\}$, see 4.3 for examples. Since $\mathbf{d}$ is sincere, we get $\sum_{j=1}^{p_i-1} d^*_{ij} \leq d_\alpha - 1$ for all $1 \leq i \leq t$. Thus, $|\mathbf{d}^*| \leq t(d_\alpha - 1)$ holds. For a dimension vector $\mathbf{d}$ let $P(\mathbf{d})$ be the projective module with $\underline{\dim}\,\mathrm{top}P(\mathbf{d}) = \mathbf{d}$.

Lemma 3.4. *There exists a projective module $P \in \mathcal{S}_C(\mathbf{d})$ if and only if $\sum_{i=1}^t \max\{0, d_\alpha - d_{ij} \mid 1 \le j \le p_i - 1\} \le 2d_\alpha$ if and only if $|\mathbf{d}^*| \le 2d_\alpha$.*

Proof. The equivalence of the second and the third statement follows from the fact that $\sum_{j=1}^{p_i-1} d^*_{ij} = \max\{0, d_\alpha - d_{ij} \mid 1 \le j \le p_i - 1\}$.

Recall that each module in $\mathcal{S}_C(\mathbf{d})$ has dimension vector $\underline{\dim}(C^n) - \mathbf{d}$. If their exists a projective module with this dimension vector, then it is isomorphic to

$$P_\alpha^{n-d_\alpha} \oplus P_\omega^{n-d_\omega-(t-2)d_\alpha+\sum_{i=1}^t d_{i1}} \oplus \bigoplus_{i=1}^t P_{ip_i-1}^{n-d_{ip_i-1}+d_\alpha} \oplus \bigoplus_{i=1}^t \bigoplus_{j=1}^{p_i-2} P_{ij}^{n-d_{ij}+d_{ij+1}}.$$

The existence of such a module is equivalent to the condition $n - d_\omega - (t - 2)d_\alpha + \sum_{i=1}^t d_{i1} \ge 0$. Assume that we are in this case. Denote the above module by P. We have to check under which condition P can be embedded into C^n. Obviously, we have to map the direct summand $P_\alpha^{n-d_\alpha}$ injectively to the direct summand P_α^n of C^n. Then we try to embed the remaining direct summands of P. Note that they are all uniserial. One checks easily that we can embed almost all of them, exept a direct summand isomorphic to $P(\mathbf{d}^*)$, into the uniserial part of C^n. Then the question is reduced to the problem to embed $P(\mathbf{d}^*)$ into $P_\alpha^{d_\alpha}$. But this can be done if and only if $|\mathbf{d}^*| \le 2d_\alpha$.

Finally, note that the condition $\sum_{i=1}^t \max\{0, d_\alpha - d_{ij} \mid 1 \le j \le p_i - 1\} \le 2d_\alpha$ implies immediately $n - d_\omega - (t - 2)d_\alpha + \sum_{i=1}^t d_{i1} \ge 0$. This finishes the proof. □

3.2. **Proof of Theorem 1.1 and Theorem 1.2.** For dimension vectors $\mathbf{e}$ and $\mathbf{f}$ denote by $\mathbf{e} \cdot \mathbf{f} = \sum_{x \in Q_0} e_x f_x$ the scalar product and by $\mathbf{e} + \mathbf{f}$ the vector sum. A dimension vector $\mathbf{s}$ is called a *section*, if $s_\alpha = s_\omega = 0$, $s_{ij} \le 1$ for all $1 \le i \le t$ and $1 \le j \le p_i - 1$, and if $s_{ij_i} = 1$ for some j_i, then $s_{ij} = 0$ for all $j \ne j_i$. A section is called *non-split* if it has at least 3 non-zero entries. Otherwise, it is called *split*. If $\mathbf{s}$ is a non-split section, then let $U(\mathbf{s}) = U(h_1, \cdots, h_t)$ such that $\mathbf{s}$ is the dimension vector of the top of $U(h_1, \cdots, h_t)$. For $1 \le m \le d_\alpha$ let $\mathbf{s} = (\mathbf{s}_1, \cdots, \mathbf{s}_m)$ be an m-tupel of non-split sections. Define

$$\begin{aligned} U_{\mathbf{s}} &= \bigoplus_{\ell=1}^m U(\mathbf{s}_\ell) \oplus P_\alpha^{n-d_\alpha} \oplus P_\omega^{n-d_\omega-(t-2)d_\alpha+\sum_i d_{i1}+\sum_\ell(|\mathbf{s}_\ell|-2)} \oplus \\ & \bigoplus_{i=1}^t P_{ip_i-1}^{n-d_{ip_i-1}+d_\alpha-(\sum_\ell \mathbf{s}_\ell)_{ip_i-1}} \oplus \bigoplus_{i=1}^t \bigoplus_{j=1}^{p_i-2} P_{ij}^{n-d_{ij}+d_{ij+1}-(\sum_\ell \mathbf{s}_\ell)_{ij}}, \end{aligned}$$

where the sum over i runs from 1 to t, and the sums over ℓ run from 1 to m.

It is easy to check that $\underline{\dim}(U_{\mathbf{s}}) = \underline{\dim}(C^n) - \mathbf{d}$. For $1 \leq \ell \leq m$ let $n(\ell)$ be the dimension of $\operatorname{Hom}_C(U(\mathbf{s}_\ell), \bigoplus_{i=1}^m U(\mathbf{s}_i))$. Note, that we can express $n(\ell)$ in combinatorial terms by 2.6.

Proposition 3.5. *The module $U_{\mathbf{s}}$ lies in $\mathcal{S}_C(\mathbf{d})$ if and only if $P(\mathbf{d}^*)$ embeds into $P_\alpha^{d_\alpha - m} \oplus P(\sum_{\ell=1}^m \mathbf{s}_\ell)$. In this case, we have*

$$\dim \operatorname{mod}_C(\mathbf{d})_{U_{\mathbf{s}}} = a(\mathbf{d}) + \sum_{\ell=1}^m \left[d_\alpha + (|\mathbf{s}_\ell| - 2) d_\omega - n(\ell) - \mathbf{s}_\ell \cdot \mathbf{d} \right].$$

Thus, for all canonical algebras we get a description of the modules U in $\mathcal{S}_C(\mathbf{d})$ such that U is isomorphic to a module of the form $\bigoplus_{i=1}^m U_i$ with $U_i \in \mathcal{J}(C)$. Furthermore, we get an easy formula for computing the dimension of the corresponding strata.

Proof. Let P be the projective module with dimension vector $\underline{\dim}(C^n) - \mathbf{d}$. The module $U_{\mathbf{s}}$ is obtained from P by deleting a direct summand isomorphic to $P(\sum_{\ell=1}^m \mathbf{s}_\ell)$ and by adding the module $P_\omega^{\sum_{\ell=1}^m (|\mathbf{s}_\ell| - 2)} \oplus \bigoplus_{\ell=1}^m U(\mathbf{s}_\ell)$. If we want to embed $U_{\mathbf{s}}$ into a free module, we have to embed the direct summand $\bigoplus_{\ell=1}^m U(\mathbf{s}_\ell)$ into a direct summand isomorphic to P_α^m. Taking this into account, the same considerations as in the previous lemma yield that $U_{\mathbf{s}}$ embeds into C^n if and only if $P(\mathbf{d}^* - \sum_{\ell=1}^m \mathbf{s}_\ell)$ embeds into $P_\alpha^{d_\alpha - m}$. This is the case if and only if $P(\mathbf{d}^*)$ embeds into $P_\alpha^{d_\alpha - m} \oplus P(\sum_{\ell=1}^m \mathbf{s}_\ell)$.

Let P' be indecomposable projective, and let $h = (h_1, \cdots, h_t) \in \mathcal{H}_{\mathrm{ns}}$. Recall that $\operatorname{Hom}_C(U(h), P') = k$ if $P' = P_\alpha$, and $\operatorname{Hom}_C(U(h), P') = 0$, else. Furthermore, $\operatorname{Hom}_C(P_\alpha, U(h)) = 0$, $\operatorname{Hom}_C(P_{ij}, U(h)) = k$ if $j \leq h_i$, and $\operatorname{Hom}_C(P_{ij}, U(h)) = 0$, else. Finally, we have $\operatorname{Hom}_C(P_\omega, U(h)) = k^2$. Computing the dimensions of the homomorphism spaces between indecomposable projective modules is left to the reader as a lengthy but elementary exercise. Using this information and 1.3, we get the dimension formula for $\operatorname{mod}_C(\mathbf{d})_{U_{\mathbf{s}}}$. □

Lemma 3.6. *If C is of type (p_1, p_2, p_3), and if $d_\alpha + d_\omega \leq \mathbf{s} \cdot \mathbf{d} + 1$ for all non-split sections $\mathbf{s}$, then $|\mathbf{d}^*| \leq 2d_\alpha$.*

Proof. From the inequality in the assumption and from the fact that $\mathbf{d}$ is sincere, we get $d_\alpha \leq \mathbf{s} \cdot \mathbf{d}$ for all non-split sections $\mathbf{s}$. Let $\mathbf{s}$ be a non-split section such that $\mathbf{s} \cdot \mathbf{d}$ is minimal. It follows from the definition of $\mathbf{d}^*$ that $|\mathbf{d}^*| \leq 3d_\alpha - \mathbf{s} \cdot \mathbf{d}$. Combining this with our inequality we get $|\mathbf{d}^*| \leq 2d_\alpha$. □

Corollary 3.7. *Assume that $|\mathbf{d}^*| \leq 2d_\alpha$. Then the following hold:*

(1) *If $d_\alpha + (|\mathbf{s}| - 2)d_\omega > \mathbf{s} \cdot \mathbf{d} + 1$ for some non-split section $\mathbf{s}$, then $\dim \operatorname{mod}_C(\mathbf{d}) > a(\mathbf{d})$, and $\operatorname{mod}_C(\mathbf{d})$ is not equidimensional;*
(2) *If $d_\alpha + (|\mathbf{s}| - 2)d_\omega > \mathbf{s} \cdot \mathbf{d}$ for some non-split section $\mathbf{s}$, then $\operatorname{mod}_C(\mathbf{d})$ is not irreducible.*

Proof. First, assume that $d_\alpha + (|\mathbf{s}|-2)d_\omega > \mathbf{s}\cdot\mathbf{d}+1$ for some non-split section $\mathbf{s}$. We can choose $\mathbf{s}$ such that the vector $\mathbf{d}^* - \mathbf{s}$ contains no negative entries.

Next, let $P = P(\mathbf{d}^*) \oplus Z$ be the projective module with dimension vector $\underline{\dim}(C^n) - \mathbf{d}$. Since $|\mathbf{d}^*| \leq 2d_\alpha$, there exists an embedding $\iota : P(\mathbf{d}^*) \oplus Z \longrightarrow C^n$. We can choose ι such that $P(\mathbf{d}^*)$ is mapped to a direct summand isomorphic to $P_\alpha^{d_\alpha}$.

Combining these facts, we get that the module $U_{\mathbf{s}} = P(\mathbf{d}^* - \mathbf{s}) \oplus U(\mathbf{s}) \oplus P_\omega^{|\mathbf{s}|-2} \oplus Z$ can be embedded into C^n as well. Since $d_\alpha + (|\mathbf{s}|-2)d_\omega > \mathbf{s}\cdot\mathbf{d}+1$, it follows from 3.5 that $\dim \operatorname{mod}_C(\mathbf{d})_{U_{\mathbf{s}}} > a(\mathbf{d})$. In particular, there exists an irreducible component of $\operatorname{mod}_C(\mathbf{d})$ which has dimension greater than $a(\mathbf{d})$. On the other hand, by 3.1 there exists an irreducible component of dimension $a(\mathbf{d})$ which is given by the closure of $\operatorname{mod}_C(\mathbf{d})_P$. It follows that $\operatorname{mod}_C(\mathbf{d})$ is not equidimensional. Next, assume $d_\alpha + (|\mathbf{s}|-2)d_\omega > \mathbf{s}\cdot\mathbf{d}$ for some non-split $\mathbf{s}$. The same argument as above shows that $\operatorname{mod}_C(\mathbf{d})$ is not irreducible. □

Corollary 3.8. *If $d_\alpha = 1$, or if C is of type $(p_1, p_2, 2)$, then the following hold:*

(1) *Each U in $\mathcal{S}_C(\mathbf{d})$ is isomorphic to a module of the form $\bigoplus_{i=1}^m U_i$ with $U_i \in \mathcal{J}(C)$ for all i;*
(2) *If $d_\alpha + (|\mathbf{s}|-2)d_\omega \leq \mathbf{s}\cdot\mathbf{d}+1$ for all non-split sections $\mathbf{s}$, then $\operatorname{mod}_C(\mathbf{d})$ has dimension $a(\mathbf{d})$;*
(3) *If $d_\alpha + (|\mathbf{s}|-2)d_\omega \leq \mathbf{s}\cdot\mathbf{d}$ for all non-split sections $\mathbf{s}$, then $\operatorname{mod}_C(\mathbf{d})$ is irreducible and a complete intersection. In particular, it is Cohen-Macaulay;*
(4) *If $d_\alpha + (|\mathbf{s}|-2)d_\omega \leq \mathbf{s}\cdot\mathbf{d}-1$ for all non-split sections $\mathbf{s}$, then $\operatorname{mod}_C(\mathbf{d})$ is normal.*

Proof. Part (1) holds by 2.4 and 2.12. Let $U_{\mathbf{s}} = \bigoplus_{\ell=1}^m U(\mathbf{s}_\ell) \oplus P$ be in $\mathcal{S}_C(\mathbf{d})$ where P is projective and $m \geq 1$. Since $d_\alpha + (|\mathbf{s}_\ell|-2)d_\omega \leq \mathbf{s}_\ell\cdot\mathbf{d}+1$ for all ℓ, it follows from the dimension formula in 3.5 that $\dim \operatorname{mod}_C(\mathbf{d})_{U_{\mathbf{s}}} \leq a(\mathbf{d})$. Thus, $\operatorname{mod}_C(\mathbf{d})$ has to have dimension $a(\mathbf{d})$, that is (2).

If we additionally have $d_\alpha + (|\mathbf{s}_\ell|-2)d_\omega \leq \mathbf{s}_\ell\cdot\mathbf{d}$, then we get $\dim \operatorname{mod}_C(\mathbf{d})_{U_{\mathbf{s}}} < a(\mathbf{d})$. This implies that $\operatorname{mod}_C(\mathbf{d})$ has only one irreducible component, namely the closure of $\operatorname{mod}_C(\mathbf{d})_P$ where P is projective. Thus, $\operatorname{Ext}^2_C(M,M)$ vanishes generically. It follows that the associated scheme of modules with dimension vector $\mathbf{d}$ is generically reduced. Together with the fact that $\operatorname{mod}_C(\mathbf{d})$ has dimension $a(\mathbf{d})$, we get that the scheme is a complete intersection. Thus it is Cohen-Macaulay by Proposition 18.13 in [8]. This implies that the scheme of modules is reduced and can be indentified with $\operatorname{mod}_C(\mathbf{d})$. Compare [5] for similar argumentations. This proves part (3).

Observe that each point M in $\operatorname{mod}_C(\mathbf{d})_P$ is smooth in $\operatorname{mod}_C(\mathbf{d})$, since $\operatorname{Ext}^2_C(M,M) = 0$, see, for example, [12]. If $d_\alpha + (|\mathbf{s}|-2)d_\omega \leq \mathbf{s}\cdot\mathbf{d}-1$

for all non-split sections $\mathbf{s}$, then we get that each stratum different from $\mathrm{mod}_C(\mathbf{d})_P$ has dimension as most $a(\mathbf{d})-2$. Thus, the set of singular points has codimension at least 2. Since under these conditons we know already that $\mathrm{mod}_C(\mathbf{d})$ is Cohen-Macaulay and irreducible, we can apply Serre's normality criterion and get that $\mathrm{mod}_C(\mathbf{d})$ is normal, see Theorem 8.22A in [10]. □

4. Examples and Remarks

4.1. **On the definition of canonical algebras.** In the literature, canonical algebras are often defined slightly more general. From the type $p = (p_1, \cdots, p_t)$ is only requested that $t \geq 2$, whereas the integers p_i might be 1. In case $t = 2$, the algebra is hereditary. Thus, the associated module varieties are just affine spaces. For $t \geq 3$ we may assume that $p_i \geq 2$ for all i and $\lambda_3 = 1$. Otherwise, we may consider, instead of $C(p, \lambda)$, an isomorphic canonical algebra $C(p', \lambda')$ of type $p' = (p'_1, \cdots, p'_{t'})$ with $t' = t-1$, and with $\lambda'_3 = 1$ if $t' \geq 3$.

4.2. **Non-sincere dimension vectors.** If C is a canonical algebra of type p and $\mathbf{d}$ a dimension vector such that the set $\{i \mid 1 \leq i \leq t, d_{ij} = 0$ for some $j\}$ contains more than one element, then we can describe $\mathrm{mod}_C(\mathbf{d})$ easily. If it contains exactly one element, say i_1, then we get this description only if $C_{\mathrm{comm}}(p \setminus \{i_1\})$ is subfinite. Here, the quiver of $C_{\mathrm{comm}}(p \setminus \{i_1\})$ is obtained by deleting the arm i_1 and the admissible ideal is generated by all commutativity relations. This algebra is subfinite if and only if it representation-finite.

4.3. **Concrete examples.** Let C be the canonical algebra of type $(3,3,2)$, and let

$$\mathbf{d} = \begin{matrix} & d_\alpha & \\ d_{12} & d_{22} & \\ d_{11} & d_{21} & d_{31} \\ & d_\omega & \end{matrix} = \begin{matrix} & 3 & \\ 2 & 2 & \\ 1 & 2 & 1 \\ & 2 & \end{matrix}, \quad \mathbf{e} = \begin{matrix} & 3 & \\ 2 & 2 & \\ 2 & 2 & 1 \\ & 2 & \end{matrix}, \quad \mathbf{f} = \begin{matrix} & 3 & \\ 2 & 2 & \\ 2 & 2 & 2 \\ & 2 & \end{matrix}.$$

We get

$$\mathbf{d}^* = \begin{matrix} & 0 & \\ 1 & 1 & \\ 1 & 0 & 2 \\ & 0 & \end{matrix}, \quad \mathbf{e}^* = \begin{matrix} & 0 & \\ 1 & 1 & \\ 0 & 0 & 2 \\ & 0 & \end{matrix}, \quad \mathbf{f}^* = \begin{matrix} & 0 & \\ 1 & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{matrix},$$

and therefore $|\mathbf{d}^*| \leq 2d_\alpha$. Thus, by 3.4 and 3.1 $\mathrm{mod}_C(\mathbf{d})$ contains an irreducible component of dimension $a(\mathbf{d})$. The same holds for $\mathrm{mod}_C(\mathbf{e})$ and $\mathrm{mod}_C(\mathbf{f})$. Furthermore, we have $d_\alpha + d_\omega \leq \mathbf{s} \cdot \mathbf{d} + 1$, $e_\alpha + e_\omega \leq \mathbf{s} \cdot \mathbf{e}$ and $f_\alpha + f_\omega \leq \mathbf{s} \cdot \mathbf{f} - 1$ for all non-split sections $\mathbf{s}$. It follows from 1.1 and 1.2 that the variety $\mathrm{mod}_C(\mathbf{d})$ has dimension $a(\mathbf{d}) = 23$ but is not irreducible,

$\mathrm{mod}_C(\mathbf{e})$ is irreducible of dimension $a(\mathbf{e}) = 27$, and $\mathrm{mod}_C(\mathbf{f})$ is normal of dimension $a(\mathbf{f}) = 32$. Using 3.5 one can show that $\mathrm{mod}_C(\mathbf{d})$ has 3 irreducible components.

4.4. **Tame examples.** For $m, n \geq 0$ let C_n be a canonical algebra of type $(n+2, 2, 2)$, and let $\mathbf{d}_{m,n}$ be a dimension vector with $d_\alpha = d_{ij} = 1$ for all i, j and $d_\omega = m + 3$. Then the dimension of $\mathrm{mod}_{C_n}(\mathbf{d}_{m,n})$ is $a(\mathbf{d}_{m,n}) + m$, and there are exactly $n + 2$ irreducible components, one of dimension $a(\mathbf{d}_{m,n})$ and $n+1$ of dimension $a(\mathbf{d}_{m,n}) + m$. This follows from our main results but can easily be checked directly.

4.5. **Wild examples.** For $m \geq 0$ let C_m be the canonical algebra of type $(m+6, m+6, 2)$, and let $\mathbf{d}$ be the dimension vector with $d_\alpha = 1$, $d_\omega = m+5$, $d_{31} = 2$ and $d_{ij} = m + 5 - j + 1$ for $i = 1, 2$ and $1 \leq j \leq p_i - 1$. Then there exists a module M in $\mathrm{mod}_{C_m}(\mathbf{d})$ such that $\mathrm{End}_{C_m}(M) = k$, $\mathrm{Ext}^1_{C_m}(M, M) = 0$, $\mathrm{Ext}^2_{C_m}(M, -) = 0$, and $\dim \mathrm{mod}_{C_m}(\mathbf{d}) = a(\mathbf{d}) + m + 1$. In particular, M is indecomposable, $\mathcal{O}(M)$ is open in $\mathrm{mod}_{C_m}(\mathbf{d})$ and its closure in $\mathrm{mod}_{C_m}(\mathbf{d})$ is an irreducible component of dimension $a(\mathbf{d})$.

We give now the explicit description of the module M as representation of the bounded quiver of C_m, and we leave the verification of the stated properties to the reader. Let $M(x) = k^{d_x}$ for $x \in Q_0$. For $i = 1, 2$ the maps $M(\gamma_{ip_i})$ and $M(\gamma_{i1})$ are the identity, and for $2 \leq j \leq p_i - 1$ the maps $M(\gamma_{1j})$ (resp. $M(\gamma_{2j})$) are the inclusions onto the first (resp. last) $m + 5 - j + 1$ coordinates. Finally, we have

$$M(\gamma_{32}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad M(\gamma_{31}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4.6. **Zero-roots for tame cases.** If C is of type $(p_1, p_2, 2)$ and tame concealed or tubular in the sense of [15], and if $\mathbf{d}$ is a sincere dimension vector with $q_C(\mathbf{d}) = \sum_{x \in Q_0} d_x^2 - a(\mathbf{d}) = 0$, then $d_\alpha + d_\omega \leq \mathbf{s} \cdot \mathbf{d} - 1$ for all non-split sections $\mathbf{s}$. This can be shown by using the description of the dimension vectors $\mathbf{d}$ with $q_C(\mathbf{d}) = 0$ as given in [15].

4.7. **A classical example.** Let $A = kQ/I$ with $Q_0 = \{a, b, c\}$ and $Q_1 = \{\alpha, \beta\}$ with $s(\alpha) = a$, $s(\beta) = e(\alpha) = b$ and $e(\beta) = c$. Assume that I is generated by the path $\beta\alpha$. Let $\mathbf{d} = (d_a, d_b, d_c)$ be a sincere dimension vector. Then $\mathrm{mod}_A(\mathbf{d})$ has dimension $a(\mathbf{d})$ if and only if $d_a + d_c \leq d_b + 1$, and it is irreducible if and only if $d_a + d_c \leq d_b$, see, for example, [5].

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