ONE-POINT EXTENSIONS AND DERIVED EQUIVALENCE

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Abstract. Work of the first author with de la Peña [1], concerned with the class of algebras derived equivalent to a tubular algebra, raised the question whether a derived equivalence between two algebras can be extended to one-point extensions. The present paper yields a positive answer.

Let A be a finite-dimensional algebra (associative with 1) over a field k. Modules, for most of this paper, will be finite dimensional right modules, and $\operatorname{mod} A$ denotes the category of such modules over A. Each A-module M we may view as a $(k,A)-$bimodule ${}_kM_A$, and form the matrix algebra

$$\begin{bmatrix} k & M \\ 0 & A \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & m \\ 0 & a \end{bmatrix} \mid \ \alpha \in k,\, m \in M,\, a \in A \right\}$$

which is called the *one-point extension* of A by M. We denote this algebra by $\bar{A}$ if M is clear from the context; moreover $\bar{M}$ will denote the indecomposable projective $\bar{A}$-module formed by the first row $[k, M]$ of $\bar{A}$. Note that $\bar{M}$ has trivial endomorphism ring. Forming the module category (resp. the derived category) over the one-point extension algebra is in a sense inverse to forming the perpendicular category with respect to an exceptional object in a module category [4] (resp. in the derived category of a module category [2]). Both processes are important for induction arguments on the number of isomorphism classes of simple modules. Note that we view modules as stalk complexes concentrated in degree zero. A preprint version of the article has been used by a number of authors [3, 13, 8, 9].

Theorem 1. *Let A and B be two finite dimensional k-algebras, $M \in \operatorname{mod} A$, $N \in \operatorname{mod} B$ and denote by $\bar{A}$, $\bar{B}$ the respective one-point extensions. For any triangulated equivalence $\Phi : \mathrm{D}^b(\operatorname{mod} A) \to \mathrm{D}^b(\operatorname{mod} B)$ which maps the module M to the module N, there exists a triangulated equivalence $\bar{\Phi} : \mathrm{D}^b(\operatorname{mod} \bar{A}) \to \mathrm{D}^b(\operatorname{mod} \bar{B})$ which maps $\bar{M}$ to $\bar{N}$ and restricts to a triangulated equivalence from $\mathrm{D}^b(\operatorname{mod} A)$ to $\mathrm{D}^b(\operatorname{mod} B)$.*

For an abelian category $\mathcal{A}$ we denote by $\mathrm{K}^b(\mathcal{A})$ the homotopy category and by $\mathrm{D}^b(\mathcal{A})$ the derived category of bounded differential complexes in $\mathcal{A}$, see [12] for definitions and basic facts. Further, we denote by $\mathcal{P}_A$ the full subcategory of $\operatorname{mod} A$ given by the finitely generated projective A-modules. We identify $\mathrm{D}^b(\operatorname{mod} A)$ with the full subcategory $\bar{M}^\perp = \{X \mid \operatorname{Hom}(\bar{M}, X[i]) = 0 \text{ for all } i\}$ of $\mathrm{D}^b(\operatorname{mod} \bar{A})$.

Before entering the proof, we recall results from Rickard [10]. Any triangulated equivalence $\Phi : \mathrm{D}^b(\operatorname{mod} A) \to \mathrm{D}^b(\operatorname{mod} B)$ induces a triangulated equivalence $\varphi : \mathrm{K}^b(\mathcal{P}_A) \to \mathrm{K}^b(\mathcal{P}_B)$, where $\mathrm{K}^b(\mathcal{P}_A)$ refers to the homotopy category of bounded complexes in $\mathcal{P}_A$. In particular, $T = \varphi^{-1}(B[0])$ is a *tilting complex*, that is, for all $n \neq 0$ we have $\operatorname{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(\bar{T}, \bar{T}[n]) = 0$, and moreover $\operatorname{add}(\bar{T})$, the full subcategory of direct summands of finite direct sums of copies of $\bar{T}$, generates $\mathrm{K}^b(\mathcal{P}_{\bar{A}})$

as a triangulated category. Conversely, a given tilting complex T in $\mathrm{K}^b(\mathcal{P}_A)$ with endomorphism algebra B, gives rise to a triangulated equivalence from $\mathrm{D}^b(\mathrm{mod}\,A)$ to $\mathrm{D}^b(\mathrm{mod}\,B)$, sending T to $B[0]$.

Proof. Note that the canonical projection $\bar{A} \to A$ induces an embedding $\iota_A : \mathrm{mod}\,A \hookrightarrow \mathrm{mod}\,\bar{A}$ such that ($\star$) the two functors $\mathrm{Hom}_{\bar{A}}(\iota_A\_, \bar{M})$ and $\mathrm{Hom}_A(\_, M)$ from $\mathrm{mod}\,A$ to $\mathrm{mod}\,k$ are isomorphic and ($\star\star$) $\mathrm{Hom}_{\bar{A}}(\bar{M}, \iota_A\_)$ is the zero functor.

Let $\varphi : \mathrm{K}^b(\mathcal{P}_A) \to \mathrm{K}^b(\mathcal{P}_B)$ be the triangulated equivalence induced by Φ and set $T := \varphi^{-1}(B[0])$ and $\bar{T} := T \oplus \bar{M}[0]$. We are going to show that $\bar{T}$ is a tilting complex in $\mathrm{K}^b(\mathcal{P}_{\bar{A}})$. Further, we show that the endomorphism algebra of $\bar{T}$ is isomorphic to $\bar{B}$. It then follows from [10], as summarized before, that there is a triangulated equivalence $\bar{\Phi} : \mathrm{D}^b(\mathrm{mod}\,\bar{A}) \to \mathrm{D}^b(\mathrm{mod}\,\bar{B})$ sending T to $B[0]$ and $\bar{M}[0]$ to $\bar{N}[0]$, moreover, in view of ($\star\star$), $\bar{\Phi}$ extends Φ.

We get a sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(T, \bar{M}[0]) &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathrm{mod}\,A)}(T, M[0]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathrm{mod}\,B)}(B[0], N[0]) \\ &\cong \mathrm{Hom}_B(B, N) = N \end{aligned}$$

where the first one is due to ($\star$) and the second to Φ. By construction we have an isomorphism $\mathrm{End}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(T) \cong B$ and, passing to the homotopy categories, we derive from ($\star\star$) that $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(\bar{M}[0], T) = 0$. Since moreover $\mathrm{End}_{\bar{A}}\left(\bar{M}\right) = k$ this shows that $\mathrm{End}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(\bar{T})$ is in fact isomorphic to $\bar{B}$.

Because of ($\star\star$), we have $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(\bar{M}[0], T[n]) = 0$ for all n, and in view of ($\star$), we get an isomorphism $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(T, \bar{M}[n]) \to \mathrm{Hom}_{\mathrm{D}^b(\mathrm{mod}\,A)}(T, M[n])$. The latter term is isomorphic to $\mathrm{Hom}_{\mathrm{D}^b(\mathrm{mod}\,B)}(B[0], N[n])$ and thus is zero for all $n \neq 0$. Similarly, $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(T, T[n]) = \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_A)}(T, T[n]) = 0$ for all $n \neq 0$. Finally, $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}_{\bar{A}})}(\bar{M}[0], \bar{M}[n]) = \mathrm{Ext}^n_{\bar{A}}(\bar{M}, \bar{M}) = 0$ for all $n \neq 0$. Since, obviously, $\mathrm{add}\,\bar{T}$ generates $\mathrm{K}^b(\mathcal{P}_{\bar{A}})$ this proves that $\bar{T}$ is a tilting complex in $\mathrm{K}^b(\mathcal{P}_{\bar{A}})$.

Thus we obtain a triangulated equivalence $\bar{\varphi} : \mathrm{K}^b(\mathcal{P}_{\bar{A}}) \to \mathrm{K}^b(\mathcal{P}_{\bar{B}})$, which maps the tilting complex $\bar{T}$ to $\bar{B}[0]$ and its summand $\bar{M}$ to $\bar{N}$, and a corresponding triangulated equivalence $\bar{\Phi} : \mathrm{D}^b(\mathrm{mod}\,\bar{A}) \to \mathrm{D}^b(\mathrm{mod}\,\bar{B})$. Since $\bar{\Phi}(\bar{M}) = \bar{N}$, the functor $\bar{\Phi}$ further sends $\bar{M}^{\perp} = \mathrm{D}^b(\mathrm{mod}\,A)$ to $\bar{N}^{\perp} = \mathrm{D}^b(\mathrm{mod}\,B)$. □

Corollary 1. *Let A and H be two finite dimensional k-algebras such that there exists a triangulated equivalence $\Phi : \mathrm{D}^b(\mathrm{mod}\,A) \to \mathrm{D}^b(\mathrm{mod}\,H)$. We assume that H is hereditary. Then for every indecomposable A-module M, there exists an indecomposable H-module N such that there is a triangulated equivalence $\bar{\Phi} : \mathrm{D}^b(\mathrm{mod}\,\bar{A}) \to \mathrm{D}^b(\mathrm{mod}\,\bar{H})$, where $\bar{A}$ and $\bar{H}$ denote the respective one-point extensions of A and H, which restricts to a triangulated equivalence from $\mathrm{D}^b(\mathrm{mod}\,A)$ to $\mathrm{D}^b(\mathrm{mod}\,H)$.*

Proof. Since H is hereditary, every indecomposable object of $\mathrm{D}^b(\mathrm{mod}\,H)$ is given by a stalk complex $X[i]$ for some indecomposable H-module X. Modifying Φ by a suitable shift $[i]$, we may thus assume the existence of an H-module N with $\Phi(M) = N$. The assertion now follows from Theorem 1, observing that derived equivalences commute with the shift functors. □

We mention two further applications. Let A be a *derived canonical* algebra, that is, A is an algebra which is derived equivalent to a canonical algebra [11]. Note that this includes the case of an algebra derived equivalent to a tame hereditary or a tubular algebra. We call an A-module M *derived regular* if M belongs to a tube $\mathcal{T}$ in the derived category $\mathrm{D}^b(\mathrm{mod}\,A)$. If M has quasi-length n in $\mathcal{T}$, we say that M has *derived regular length* n. If moreover $n = 1$ we say that M is *derived regular simple*.

Corollary 2. *Let A be a derived canonical algebra, and let M be an A-module which is derived regular simple. Then the one-point extension of A by M is again derived canonical.*

Proof. The assertion holds for a canonical algebra [6], hence by Theorem 1 extends to the derived canonical situation. □

We recall that any tame hereditary algebra of type $\widetilde{\mathbb{D}}_n$ is in the same derived class as the canonical algebra of weight type $(2, 2, n-1)$.

Corollary 3. *Assume that A_1 and A_2 are derived canonical of type $(2, 2, n)$ and let M_i be an indecomposable A_i-module of derived regular length two taken from a rank n tube of $\mathrm{D}^b(\mathrm{mod}\,A_i)$, $i = 1, 2$. Then the resulting one-point extensions $\bar{A}_1$ and $\bar{A}_2$ are derived equivalent.* □

This implies, in particular, that the (strongly simply connected) *polynomial growth critical algebras* introduced by Nörenberg and Skowroński [7] with a fixed number of simple modules are in the same derived class, a result formerly requiring a case by case analysis.

Comments. **(a)** Assume that A (and hence also the one-point extension $\bar{A}$ with respect to the A-module M) has finite global dimension. Then the category $\mathrm{D}^b(\mathrm{mod}\,\bar{A})$ has Auslander-Reiten triangles [5]. We claim that the A-module M is isomorphic to the "middle term" E of the Auslander-Reiten triangle in $\mathrm{D}^b(\mathrm{mod}\,\bar{A})$

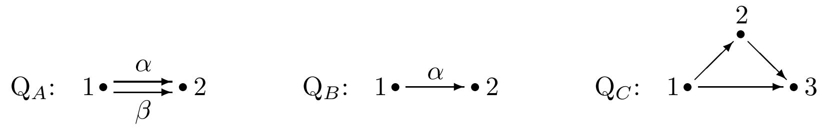
$$\tau\bar{M} \to E \to \bar{M} \to \tau\bar{M}[1]. \tag{1}$$

Moreover, if $r : \mathrm{D}^b(\mathrm{mod}\,\bar{A}) \to \mathrm{D}^b(\mathrm{mod}\,A)$ denotes the right adjoint functor to the inclusion $i : \mathrm{D}^b(\mathrm{mod}\,A) \hookrightarrow \mathrm{D}^b(\mathrm{mod}\,\bar{A})$ (cf. [2] for the existence of r), then $M = r\bar{M}$. Indeed, application of $\mathrm{Hom}\left(\bar{M}, -\right)$ to (1) yields a long exact homology sequence. Invoking Auslander-Reiten duality $\mathrm{Hom}\,(X, \tau Y[n]) = \mathrm{Hom}_k\,(\mathrm{Hom}(Y[n-1], X), k)$ and the exceptionality of $\bar{M}$, it follows that $\mathrm{Hom}\left(\bar{M}, E[n]\right) = 0$ holds for each $n \in \mathbb{Z}$, thus $E \in \mathrm{D}^b(\mathrm{mod}\,A)$. Moreover, for each $X \in \mathrm{D}^b(\mathrm{mod}\,A)$ the segment

$$\mathrm{Hom}\left(X, \tau\bar{M}\right) \to \mathrm{Hom}\,(X, E) \to \mathrm{Hom}\left(X, \bar{M}\right) \to \mathrm{Hom}\left(X, \tau\bar{M}[1]\right)$$

of the long exact homology sequence has vanishing end terms showing that the functors $\mathrm{Hom}\,(-, E)$ and $\mathrm{Hom}(-, \bar{M})$ agree on $\mathrm{D}^b(\mathrm{mod}\,A)$, hence implying $E \cong r\bar{M} \cong M$.

(b) The converse of Theorem 1 does not hold. Let A and B be the path algebras of the quivers Q_A and Q_B, respectively.



Let $M \in \operatorname{mod} A$ be given by $M(1) = M(2) = k$, $M(\alpha) = M(\beta) = 1_k$ and $N \in \operatorname{mod} B$ be given by $N(1) = k$, $N(2) = k^2$, $N(\alpha)$ the diagonal embedding. There does not exist a triangulated equivalence between $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b(\operatorname{mod} B)$, but $\mathrm{D}^b(\operatorname{mod} \bar{A})$ and $\mathrm{D}^b(\operatorname{mod} \bar{B})$ are equivalent as derived categories, since they are both tilted of the hereditary algebra C with quiver Q_C: let $T_A = P_1 \oplus P_3 \oplus S$ and $T_B = S \oplus I_1 \oplus I_3$, where P_x, respectively I_x denotes the projective cover, resp. injective hull of the simple in x and S is the indecomposable with $S(1) = S(3) = k$ and $S(2) = 0$. Then $\operatorname{End}(T_A) \cong \bar{A}$ and $\operatorname{End}(T_B) \cong \bar{B}$.

(c) We finally formulate an infinite variant of Theorem 1. For any ring A, denote by $\operatorname{Mod} A$ the category of *all* right A-modules.

Theorem 2. *Let A and B be two algebras over a commutative ring R with unit, $M \in \operatorname{Mod} A$, $N \in \operatorname{Mod} B$ and denote by $\bar{A}$, $\bar{B}$ the respective one-point extensions. For any triangulated equivalence $\Phi : \mathrm{D}^b(\operatorname{Mod} A) \to \mathrm{D}^b(\operatorname{Mod} B)$ which maps the module M to the module N, there exists a triangulated equivalence $\bar{\Phi} : \mathrm{D}^b(\operatorname{Mod} \bar{A}) \to \mathrm{D}^b(\operatorname{Mod} \bar{B})$ which maps $\bar{M}$ to $\bar{N}$ and restricts to a triangulated equivalence from $\mathrm{D}^b(\operatorname{Mod} A)$ to $\mathrm{D}^b(\operatorname{Mod} B)$.*

By Rickard [10], any triangulated equivalence from $\mathrm{D}^b(\operatorname{Mod} A)$ to $\mathrm{D}^b(\operatorname{Mod} B)$ induces a triangulated equivalence from $\mathrm{K}^b(\mathcal{P}_A)$ to $\mathrm{K}^b(\mathcal{P}_B)$. Thus, the proof of Theorem 1 extends to the present setting, replacing each occurrence of $\operatorname{mod} A$ ($\operatorname{mod} B$) by $\operatorname{Mod} A$ ($\operatorname{Mod} B$, respectively).

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