A characterization of positive unit forms of Dynkin type $\mathbb{A}_n$

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1 Introduction and Results

1.1 An integer quadratic form

$$q : \mathbb{Z}^n \to \mathbb{Z}, \quad q(x) = \sum_{i=1}^{n} q_i x(i)^2 + \sum_{i<j} q_{ij} x(i)x(j)$$

is called *unit form* provided $q_i = 1$ for all i. The form is *positive* if $q(x) > 0$ for all non-zero $x \in \mathbb{Z}^n$. Clearly, for a positive unit form q we must have $|q_{ij}| \leq 1$ for all $i < j$. Unit forms play an important role in the theory of representations of algebras as associated forms to a finite dimensional algebra over an algebraically closed field such as the Tits form and in case where the algebra has finite global dimension also the Euler form. Their properties such as (weakly) positivity or (weakly) non-negativity reflect properties of the algebras, see for example ...???.
To a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ we associate a bigraph $\mathbf{B}_q$ with n vertices and edges as follows. Two different vertices i and j are joined by $|q_{ij}|$ *full* edges if $q_{ij} \leq 0$ and by q_{ij} *broken* edges if $q_{ij} > 0$. Clearly, any *reduced* (that is, between two vertices i and j there are not both full and broken edges) Γ withot *loop* (that is an edge form one vertex to itself) is isomorphic to $\mathbf{B}_q$ for some unit form q, which we denote by $\mathbf{q}_\Gamma$. A unit form q is *connected* if so is $\mathbf{B}_q$. In the following we assume that bigraphs are reduced and without loop.

1.2 Two unit forms $q, q' : \mathbb{Z}^n \to \mathbb{Z}$ are called $\mathbb{Z}$-*equivalent* if there is a $\mathbb{Z}$-invertible linear map $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $q' = qT$. The following classical result is basic for this work.

Theorem. *A connected unit form is positive if and only if it is $\mathbb{Z}$-equivalent to $\mathbf{q}_\Delta$, where $\Delta = \mathbb{A}_m$, $\mathbb{D}_n$ or $\mathbb{E}_p$ ($1 \leq m$, $4 \leq n$ and $6 \leq p \leq 8$) is a Dynkin diagram.*

Clearly, for a given quadratic form q the Dynkin diagram Δ is uniquely determined up to isomorphism. We denote thus $\mathrm{Dyn}(q) := \Delta$ and call it the *Dynkin type* of q.

1.3 Denote by $\mathbf{F}_{m,m'}$ the bigraph with $m+m'$ vertices $(1,1)$,...,$(1,m)$,$(2,1)$, ...,$(2,m')$ and full edges $(1,i)$ —— $(2,j)$ and broken edges $(1,i)\cdots\cdots(1,i')$ and $(2,j)\cdots\cdots(2,j')$ for all i,i',j,j'.

Let T be a graph with t vertices and $\mathcal{B}_1,\dots,\mathcal{B}_t$ bigraphs. Further we assume that for any vertex i of T we have an injective map $\sigma_i : T(i) \to (\mathcal{B}_i)_\circ$, where $T(i)$ denotes the set of edges in T ending in i and $(\mathcal{B}-I)_\circ$ the vertex set of $\mathcal{B}_i$. With this data we define the *assemblage* of the bigraphs $\mathcal{B}_1,\dots,\mathcal{B}_t$ to be the bigraph obtained by the disjoint union of the $\mathcal{B}_1,\dots,\mathcal{B}_t$ by identifying $\sigma_i(\alpha)$ with $\sigma_j(\alpha)$ for any edge $i\overset{\alpha}{—}j$ of T. If T is a tree we call the assemblage a *tree assemblage*.

We are now ready to formulate the main result.

Theorem. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. Then q is positive with Dynkin type $\mathbb{A}_n$ if and only if $\mathbf{B}_q$ is a tree assemblage of bigraphs of the form $\mathbf{F}_{m,m'}$.*

Example. The bigraph in Figure 1 defines a positive quadratic form of Dynkin type $\mathbb{A}_{19}$. We have marked the hinges by a big dot.

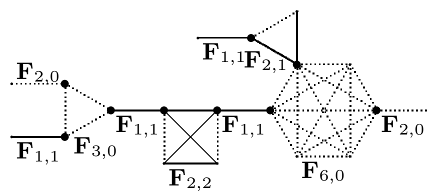

Figure 1

2 Basic Tools

2.1 We denote by $\mathbf{e}_i$ the i-th canonical basic vector of $\mathbb{Z}^n$. For $\varepsilon = \pm$ and $1 \le i,j \le n$, $i \ne j$ let $T_{ij}^{\varepsilon} : \mathbb{Z}^n \to \mathbb{Z}^n$ be the linear transformation given by $T_{ij}^{\varepsilon}(\mathbf{e}_s) = \mathbf{e}_s$ for all $s \ne i$ and $T_{ij}^{\varepsilon}(\mathbf{e}_i) = \mathbf{e}_i - \varepsilon\mathbf{e}_j$.

It easy to check, that for a positive unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ with $q_{ij} = \varepsilon 1$, the quadratic form $q' = qT_{ij}^{\varepsilon}$ is again a positive unit form given by the formulas $(\star)$ $q'_{rs} = -q_{rs}$ for $r \ne i$, $q'_{is} = q_{is} - \varepsilon q_{js}$ for $s \ne j$ and $q'_{ij} = -q_{ij}$.

Therefore, T_{ij}^{+} (resp. T_{ij}^{-}) is called an *inflation* (resp. *deflation*) for q if $q_{ij} = 1$ (resp. $q_{ij} = 1$).

We call two bigraphs *equivalent* if the corresponding unit forms are so and for a bigraph $\mathcal{B}$ with n points and a $\mathbb{Z}$-invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ we also write $\mathcal{B}T$ instead of $\mathbf{B}_{\mathbf{q}_{\mathcal{B}}T}$.

Lemma. *Let $\mathcal{B}$ be a bigraph and $b, b' \in \mathcal{B}$ two different vertices. Let $\mathcal{C}_1$ (resp $\mathcal{C}_2$) be the bigraph which is obtained form the disjoint union of $\mathcal{B}$ with a point x by joining b with x by a full (resp. broken) line and b' with x by a broken (resp. full) line.*

Then the two bigraphs $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent.

Proof: Using the formulas ($\star$) it is easy to verify that $\mathcal{C}_2 = \mathcal{C}_1 T^-_{bx} T^+_{b'x}$. □

2.2 We say that q' is a *restriction* of q if $\mathbf{B}_{q'}$ is isomorphic to a full subbigraph of $\mathbf{B}_q$. The following result was shown in [1] using the technique of deflations and inflations.

Theorem. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a positive unit form of Dynkin type $\mathbb{A}_n$. Then any restriction of q has Dynkin type $\mathbb{A}_m$ for some $m \leq n$.*

3 Cycles and Blocks

3.1 A bigraph $\mathcal{B}$ is called a *cycle* if it is connected and any vertex is connected to exactly two other vertices. For a bigraph $\mathcal{B}$ and a vertex x of $\mathcal{B}$ we denote by $\mathcal{B}^{(x)}$ the full subbigraph of $\mathcal{B}$ which is given by all points different from x.

Lemma. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a positive unit form of Dynkin type $\mathbb{A}_n$. Then $\mathbf{B}_q$ is a cycle if and only if $\mathbf{B}_q$ is isomorphic to $\mathbf{F}_{3,0}$ or to $\mathbf{F}_{2,1}$.*

Proof: Using deflations it is easy to see that $\mathbf{F}_{3,0}$ and $\mathbf{F}_{2,1}$ define positive definite unit forms of Dynkin type $\mathbb{A}_3$. Clearly they are cycles.

Suppose that $q : \mathbb{Z}^n \to \mathbb{Z}$ be a positive unit form of Dynkin type $\mathbb{A}_n$ such that $\mathbf{B}_q$ is a cycle. By reordering the vertices and (2.1), we may assume that $q_{i\,i+1} = 1$ for all $i = 1, \ldots, r$ and $q_{i\,i+1} = -1$ for all $i = r+1, \ldots, n$, where $q_{n\,n+1} := q_{1n}$. Denote this bigraph shortly with $C(n, r)$. We will show that $n = 3$ and that r is odd.

If r is even we have $q(a) = 0$, where $a \in \mathbb{Z}^n$ is defined by $a(i) = (-1)^{i+1}$ for $i = 1, \ldots, r+1$ and by $a(i) = a(r+1)$ for $i \geq r+1$, a contradiction.

If $r > 2$ then we have $\mathbf{B}_q T^+_{r\,r+1} T^+_{r\,r-1} = C(n, r-2)$. By induction it is thus enough to study the bigraphs $C(n, 1)$.

First, we verify directly that $\mathrm{Dyn}(\mathbf{q}_{C(4,1)}) = \mathbb{D}_4$. For $n > 4$ we observe that $(C(n,1)T^+_{12})^{(1)}$ is isomorphic to $C(n-1, 1)$. Thus by induction and (2.2) we infer that $\mathrm{Dyn}(\mathbf{q}_{C(n,1)}) \neq \mathbb{A}_n$ for $n > 3$. This shows $n = 3$. □

3.2 Let $\mathcal{B}$ be a bigraph. A sequence of points $x_1, \ldots, x_n$ such that $q_{x_i x_{i+1}} \neq 0$ for $i = 1, \ldots, n-1$ is called a *walk* in $\mathcal{B}$. Further, we say that a connected bigraph $\mathcal{B}$ is a *block* if $\mathcal{B}^{(x)}$ is connected for all $x \in \mathcal{B}$.

Lemma. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a positive unit form of Dynkin type $\mathbb{A}_n$ such that $\mathbf{B}_q$ is a block. Then $q_{ij} \neq 0$ for all $i \neq j$.*

Proof: Clearly, it is enough to show that if $\mathbf{B}_q$ is a block then $\mathbf{B}_q^{(x)}$ is a block for any $x \in \mathbf{B}_q$. Suppose this is not so, that is $\mathbf{B}_q$ is a block, but $\mathbf{B}_q^{(x)}$ is not a block for some $x \in \mathbf{B}_q$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two connected components of $\mathbf{B}_q^{(x)(y)}$. Since

$\mathbf{B}_q^{(y)}$ is connected, there exist vertices $z_i \in \mathcal{B}_i$ such that $q_{xz_i} \neq 0$ for $i = 1, 2$. Thus there exist walks $x, z_{i,1}, z_{i,2}, \ldots, z_{i,n_i}, y$ with $z_{i,j} \in \mathcal{B}_i$ for $j = 1, \ldots, n_i$ and $i = 1, 2$. If we take those walks such that n_1 and n_2 are minimal, we infer by (3.1), that $q_{xy} \neq 0$ and that $n_1 = n_2 = 1$. Let $\mathcal{B}$ be the restriction of $\mathbf{B}_q$ to $\{x, z_{1,1}, z_{2,1}, y\}$. There are four possible cases and each of them satisfies $\mathrm{Dyn}(\mathbf{q}_{\mathcal{B}}) = \mathbb{D}_4$ in contradiction to (2.2). □

Remark. It follows directly form (2.1) that if q satisfies the hipothesis of the lemma, then for all pairwise different i, j, k we have $(**)$: $q_{ij}q_{jk}q_{ki} = 1$.

3.3 Proposition. *The following conditions are equivalent for a bigraph $\mathcal{B}$.*

(i) *$\mathcal{B}$ is a block and $\mathbf{q}_{\mathcal{B}}$ is positive of Dynkin type $\mathbb{A}_n$.*

(ii) *$\mathcal{B}$ is isomorphic to $\mathbf{F}_{m,m'}$ for some $m \geq 1, m' \geq 0$ with $m + m' = n$.*

Proof: First assume that $\mathbf{B}_q$ is a block. Let $x \in \mathbf{B}_q$ be a fixed vertex. Denote $(1,1) = x$ and denote the vertices of the set $\{y \in \mathbf{B}_q \mid q_{x,y} > 0\}$ by $(1,2), \ldots, (1,m)$ and the vertices of the set $\{y \in \mathbf{B}_q \mid q_{x,y} > 0\}$ by $(2,1), \ldots, (2,m')$.
It follows from $(**)$ that for any two different $i, i' \in \{2, \ldots, m$ we have a broken edge between $(1,i)$ and $(1,i')$. and for any two different $j, j' \in \{1, \ldots, m'\}$ we have also a broken edge between $(2,j)$ and $(2,j')$. Finally it follows by the same argument that for any i, j as above we have a full edge between $(1,i)$ and $(2,j)$. Thus B_q is of the form $\mathbf{F}_{m,m'}$.
Clearly, the bigraphs $\mathcal{B} = \mathbf{F}_{m,m'}$ are blocks. It remains thus to show that the associated unit forms are positive of Dynkin type $\mathbb{A}_{m+m'}$. Indeed, it is easy to see, that $\mathcal{B}T^+_{(1,2)(1,3)}T^+_{(1,3)(1,4)} \cdots T^+_{(1,m-1)(1,m)}T^+_{(2,1)(2,2)}T^+_{(2,2)(2,3)} \cdots T^+_{(2,m'-1)(2,m')}$ is isomorphic to $\mathbb{A}_{m+m'}$. □

In view of Proposition 3.3, we call a bigraph an $\mathbb{A}$-*block* if it is ismorphic to $\mathbf{F}_{m,m'}$ for some $m \geq 1, m' \geq 0$.

4 Proof of the main result

4.1 Let $A(n)$ be the Dynkin graph with points $1, \ldots, n$ and edges between the vertices i and $i+1$ for $i = 1, \ldots, n-1$.

Lemma. *Let $\mathcal{B}$ be a tree assemblage of $\mathbb{A}$-blocks $\mathcal{C}_1, \ldots, \mathcal{C}_t$. Let x be a vertex of $\mathcal{B}$ which is not a hinge. Then there exists a sequence of deflations $T^+_{y_i z_i}$, $z_i \neq x$, with composition T such that $\mathcal{B}T$ is isomorphic to $A(n)$ and x corresponds to the vertex 1 of $A(n)$.*

Proof: The proof is done by induction on t. If $t = 1$ then the assertion follows by the argument given in the proof of Proposition 3.3 by setting $x_1 = x$.
In the following we say that a deflation T^+_{yz} *avoids* x if $z \neq x$.
Now, let $t > 1$ assume that x belongs to $\mathcal{C}_i$ and let $h_1, \ldots, h_s$ be the hinges of $\mathcal{B}$ in $\mathcal{C}_i$. By induction hypothesis, there exist a sequence of deflations avoiding

x with composition T such that $\mathcal{B}_1 = \mathcal{B}T$ is obtained from the disjoint union of $\mathcal{C}_i = \mathbf{F}_{m,m}$ with $A(n_1), \dots, A(n_s)$ by identifying h_j with the vertex 1 of $A(n_j)$. We abreviate the resulting bigraph by the symbol $\mathbf{F}_{m,m'}[h_1, n_1] \cdots [h_s, n_s]$.

Without loss of generality, we may assume that $x, h_1, \dots, h_r$ (resp. $h_{r+1}, \dots, h_s$) belong to the first (resp. second) part of $\mathcal{C}_i$. If $r > 1$ we denote the vertices of $A(n_2)$ by $a_1, \dots, a_{n_2}$. The bigraph of $\mathcal{B}_1 T^+_{h_1 a_1} T^+_{h_1 a_2} \cdots T^+_{h_1 a_{n_2}}$ is isomorphic to the bigraph $\mathbf{F}_{m-1,m'}[h_2, n_1 + n_2][h_3, n_3] \cdots [h_s, n_s]$. So, inductively we obtain a sequence of deflations avoiding x with composition S_1 such that $\mathcal{B}_2 = \mathcal{B}_1 S_1 = \mathbf{F}_{u,u'}[h_r, v][h_s, v']$ where $u = m - r$, $u' = m' - s + r$, $v = n_1 + \cdots + n_r$ and $v' = n_{r+1} + \cdots + n_s$.

If y is vertex of $\mathbf{F}_u$ with $y \neq x, h_r$ then $\mathcal{B}_2 T^+_{h_r y}$ is isomorphic to $\mathbf{F}_{u-1,u'}[h_r, v + 1][h_s, v']$. Inductively we obtain a sequence of deflations avoiding x with composition S_2 such that $\mathcal{B}_3 = \mathcal{B}_2 T_2$ is isomorphic to $\mathbf{F}_{2,u'}[h_r, v{+}u{-}2][h_s, v']$ where the two points of $\mathbf{F}_2$ are x and h_r. Finally, verify that $\mathcal{B}_4 = \mathcal{B}_3 T^+_{y1} T^+_{y2} \cdots T^+_{y\, v+u-2}$ is isomorphic to $\mathbf{F}_{1,u'}[h_r, v + u - 1][h_s, v']$ and the point x corresponds to the vertex $v + u - 1$ of $A(v + u - 1)$.

Similarly, we show that there is a sequence of x-admissible deflations with composition S_4 such that $\mathcal{B}_4 S_4$ is isomorphic to $\mathbf{F}_{1,1}[h_r, v + u - 1][h_s, v' + u' - 2] \simeq A(u + v + u' + v' - 1)$. □

4.2 *Proof of the Main Result:* In the preceeding section we have seen that each tree assemblage of $\mathbb{A}$-blocks defines a positive unit form of Dynkin type $\mathbb{A}_n$ for some n. Conversely, let now $q : \mathbb{Z}^n \to \mathbb{Z}$ be a positive unit form with Dynkin type $\mathbb{A}_n$. We will show that $\mathbf{B}_q$ is a tree assemblage of $\mathbb{A}$-blocks.

If $\mathbf{B}_q$ is a block, then then the assertion follows by (3.3). Assume now that there exists a point $x \in \mathbf{B}_q$ such that $\mathbf{B}_q^{(x)}$ is not connected. By induction on the number of vertices we may assume that $\mathbf{B}_q$ is obtained from the disjoint union of $\mathcal{C}_1, \dots, \mathcal{C}_t$ by identifying the points $x_i \in \mathcal{C}_i$, that each $\mathcal{C}_i$ is itself a tree assemblage of blocks and that $\mathcal{C}_i^{(x_i)}$ is connected with $n_i \geq 1$ vertices. Thus, x_i is not a hinge of $\mathcal{C}_i$ and by (4.1), there is a sequence of x-admissible deflations such that the bigraph $\mathcal{B}$ of qT is a star with t branches of length $n_1, \dots, n_t$ respectively. Since q is positive and $\mathrm{Dyn}(q) = \mathbb{A}_n$ we infer that $t = 2$ and thus that $\mathcal{B}$ is a tree assemblage of $\mathbb{A}$-blocks. This finishes the proof of the main result. □

References

[1] M. Barot and J. A. de la Peña: *The Dynkin-type of a non-negative unit form.* To appear.