A characterization of positive unit forms, Part II

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Abstract

This paper concludes the work begun in [1]. It considers unit forms, i.e. positive definite integral quadratic froms with unitary coefficients in the quadratic terms. The equivalence classes of connected unit forms are given by Dynkin diagrams. The paper presents a characterization of positive unit forms which are equivalent to $\mathbb{D}_n$ for some integer n in terms of the associated bigraphs and gives a list for the case $\mathbb{E}_6$.

1 Introduction and Result

We consider *unit forms*, that are integral quadratic forms

$$q : \mathbb{Z}^n \to \mathbb{Z}, \quad q(x) = \sum_{i=1}^{n} q_i x(i)^2 + \sum_{i<j} q_{ij} x(i)x(j)$$

such that $q_i = 1$ for all i. Unit forms play an important role in the theory of representations of algebras as associated forms to a finite dimensional algebra over an algebraically closed field: the Tits form and in case the algebra has finite global dimension also the Euler form. Their properties, such as (weak) positivity or (weak) non-negativity, reflect properties of the algebras, see for example [4, 6, 2, 3].

A unit form is called *positive* if $q(x) > 0$ for all non-zero x. Two unit forms, p and q, are called $\mathbb{Z}$-*equivalent* if there exists a $\mathbb{Z}$-invertible linear transformation T such that $p = qT$. It is well known, that positive unit forms can be classified, up to $\mathbb{Z}$-equivalence, by Dynkin diagrams. Namely, one associates to each unit form q a bigraph $\mathrm{B}(q)$ with vertices $1, \ldots, n$ and edges of two types, full and broken ones, according to the following. Between i and j, there are $-q_{ij}$ full edges, if $q_{ij} < 0$, else there are q_{ij} broken edges. Conversely, to any bigraph B (without loops and not both, broken and full edges, between two fixed vertices) we may associate a unit form q_B such that $\mathrm{B}(q_B) = B$. A unit form is called *connected* if its bigraph is connected. Each connected, positive unit form q is $\mathbb{Z}$-equivalent to q_Δ, where Δ is a Dynkin diagram, called the *Dynkin-type* of q and denoted by $\mathrm{Dyn}(q)$. A bigraph is called a *cycle* if it is connected and every vertex has exactly two neighbours.

We denote by $\Phi(q)$ the *frame* of a unit form q, that is the graph obtained from $\mathrm{B}(q)$ by turning the broken edges into full ones. In [1] it was shown that a connected unit form q is positive of Dynkin type $\mathbb{A}_n$ if and only if $\mathrm{B}(q)$ *satisfies the cycle condition* (that is, each cycle contains an odd number of broken edges) and $\Phi(q)$ is *a tree assemblage of complete graphs* (that is, $\Phi(q)$ is obtained from the disjoint union of complete graphs $\Sigma_1, \dots, \Sigma_n$ by identifying $\sigma_i(\alpha)$ with $\sigma_j(\alpha)$, where $\alpha = \{i, j\}$ runs over all edges of a tree Γ with points $\{1, \dots, n\}$ and σ_i are injective maps from the set of edges in Γ ending in i to the vertex set of Σ_i.)

We now present two new constructions. First, for a given graph Γ and a vertex $x \in \Gamma$ we define a new graph $\Gamma[x]$, the *mirror extension* of Γ by x as follows. $\Gamma[x]$ has Γ as full subgraph plus one additional point x^* which is not connected to x by an edge but to any other point y of Γ by the same number of edges as so is x: $[x^*, y]_{\Gamma[x]} = [x, y]_{\Gamma[x]}$. In this situation, the points x and x^* are called *mirror points* of $\Gamma[x]$. If Γ is a tree assemblage of complete graphs we call $\Gamma[x]$ an $\mathbb{A}$-mirror extension.

The second construction is easier. Given a connected graph Γ and two vertices $x, y \in \Gamma$ with connecting distance $\mathrm{d}_\Gamma(x, y) > 2$ denote by $\Gamma/\{x = y\}$ the graph which is obtained from Γ by identifying x with y and call it *the cycling* of Γ in x and y. If x is a vertex of Γ, we denote by $\Gamma^{(x)}$ the full subgraph given by all vertices different from x. If Γ is a tree assemblage of complete graphs and $x, y \in \Gamma$ two vertices such, that $\Gamma^{(x)}$ and $\Gamma^{(y)}$ are still connected, we call $\Gamma/\{x = y\}$ an $\mathbb{A}$-cycling.

We are now ready to formulate the main result.

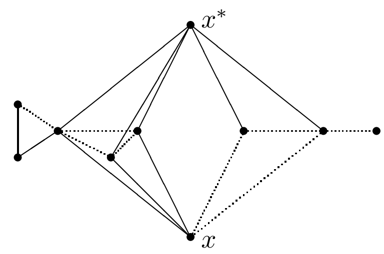
Main Theorem 1 *Let q be a unit form. Then q is positive of Dynkin type $\mathbb{D}_n$ if and only if the following three conditions are satisfied:*

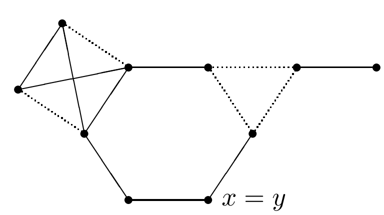
(i) $\mathrm{B}(q)$ *has more than 3 points,*

(ii) $\mathrm{B}(q)$ *satisfies the cycle condition and*

(iii) $\Phi(q)$ *is an $\mathbb{A}$-mirror extension or an $\mathbb{A}$-cycling.*

We give two examples of bigraphs which define positive unit forms of Dynkin type $\mathbb{D}_{10}$, where its frame is an $\mathbb{A}$-mirror extension (left side) and an $\mathbb{A}$-cycling (right side):





The frames of positive unit forms of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ were calculated completely using a computer program. We give here only the list of all frames for $\mathbb{E}_6$.

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2 Reduction of Frames

2.1

A graph Γ will be called *positive admissible* if there exists a positive unit form p such that $\Gamma = \Phi(p)$. In that case we call p a *positive presentation* of Γ. By [1, Theorem A], a positive admissible graph has a well determined *Dynkin type* $\mathrm{Dyn}(\Gamma)$ given by $\mathrm{Dyn}(\Gamma) = \mathrm{Dyn}(p)$ for any positive presentation p. Moreover, it follows from [2] that any full subgraph Γ' of a positive admissible graph Γ is again positive admissible and $\mathrm{Dyn}(\Gamma') \leq \mathrm{Dyn}(\Gamma)$, where the partial order is given by

$$\mathbb{A}_m \leq \mathbb{A}_n \leq \mathbb{D}_n \leq \mathbb{D}_p \text{ for } m \leq n \leq p$$

$$\mathbb{D}_p \leq \mathbb{E}_p \leq \mathbb{E}_q \text{ for } 6 \leq p \leq q \leq 8.$$

2.2

Let Γ be a graph and x, y two different vertices of Γ. We define a new graph Γ' with the same vertices as Γ by the following:

$$[r,s]_{\Gamma'} = \begin{cases} [r,s]_\Gamma & \text{if } r,s \neq x, \\ |[r,x]_\Gamma - [x,y]_\Gamma [r,y]_\Gamma| & \text{if } s = x, r \neq y, \\ [x,y]_\Gamma & \text{if } s = x, r = y. \end{cases}$$

We denote Γ' also by ΓT_{xy} and say that ΓT_{xy} is obtained by applying to Γ the *flation* T_{xy}. Note that if Γ has no double edge, then $\Gamma T_{xy} T_{xy} = \Gamma$. Two graphs are called *flation-equivalent* if one is obtained from the other by applying a sequence of flations. We recall that for $\varepsilon = \pm 1$ and $i \neq j$ we denote by T^{ε}_{ij} the invertible linear transformation given in the canonical base vectors e_i by $T^{\varepsilon}_{ij}(e_s) = e_s$ for all $s \neq i$ and $T^{\varepsilon}_{ij}(e_i) = e_i - \varepsilon e_j$.

Proposition 1 *Suppose that Γ and Γ' are two flation-equivalent graphs. Then Γ is positive admissible if and only if so is Γ'. Moreover, in this case we have $\mathrm{Dyn}(\Gamma) = \mathrm{Dyn}(\Gamma')$.*

Proof. Let Γ be positive admissible and $T = T_{x_1 y_1} \cdots T_{x_t y_t}$ be a sequence of flations for Γ. Let further q be a positive presentation of Γ and $\varepsilon = q_{x_1 y_1}$.

First, if $\varepsilon = 0$ then $\Gamma T_{x_1y_1} = \Gamma$. If $\varepsilon \neq 0$, we have either $\varepsilon = 1$ or $\varepsilon = -1$. We claim that $\Phi := \Phi(qT^{\varepsilon}_{x_1y_1}) = \Gamma T_{x_1y_1}$. By this the result follows then inductivley, since $\Gamma T_{x_1y_1}$ is again positive admissible and $\mathrm{Dyn}(\Phi) = \mathrm{Dyn}(\Gamma)$.

So, let $x = x_1$ and $y = y_1$. If $r, s \neq x$ we have $[r, s]_\Phi = |(qT^{\varepsilon}_{xy})_{rs}| = |q_{rs}| = [r, s]_\Gamma = [r, s]_{\Gamma T_{xy}}$. Further, $[x, y]_\Phi = | - q_{xy}| = [x, y]_{\Gamma T_{xy}}$. Finally let $r \neq x, y$. If $q_{rx} = 0$ or $q_{ry} = 0$, we again easily verify that $[r, x]_\Phi = [r, x]_{\Gamma T_{xy}}$. In the remaining case, we have $q_{rx}q_{xy}q_{yr} = 1$ since $B(q)$ satisfies the cycle-condition and hence $[r, x]_\Phi = 0$. On the other hand, by definition, we have $[r, x]_{\Gamma T_{xy}} = 0$. $\square$

2.3

The following Corollary will be very useful for the proof of the Main Theorem.

Corollary 1 *Suppose that q is a unit form such that* $\mathrm{B}(q)$ *satisfies the cycle-condition and Γ a graph wich is flation-equivalent to $\Phi(q)$. Then Γ is positive admissible if and only if q is positive.*

Proof. If Γ is positive admissible then so is $\Phi(q)$ (2.2). Hence there exists a positive presentation p of $\Phi(q)$. Since $B(q)$ satisfies the cycle-condition, we obtain by [1, Theorem A] that q is positive. The converse follows directly from (2.2). $\square$

2.4

For a given vertex x of a connected graph Γ let $\mathrm{v}_\Gamma(x)$ be the *connecting valence* of x, that is the number of connected components of $\Gamma^{(x)}$. A point of connecting valence bigger than one is called *knot* of Γ and a graph without knots is called *block*. We say that a flation T_{yz} for Γ *avoids* (resp. *strongly avoids*) x if $x \neq z$ (resp. if $x \neq z, y$) and we say that a sequence of flations (strongly) avoids x if each flation in the sequence (strongly) avoids x.

Notice, that if T is a flation for a graph Γ avoiding the vertices x and y and if $\mathrm{d}_\Gamma(x, y) > 2$ and $\mathrm{d}_{\Gamma T}(x, y) > 2$ then it does not matter if we first cyle Γ in x and y and apply then T or if we first apply T to Γ and then cycle in x and y, in both cases we obtain the same graph. Also, if T strongly avoids the vertex x of Γ then $\Gamma[x]T = (\Gamma T)[x]$.

For a sequence of points $x_1, \dots, x_t$ we set

$$T_{[x_1,\dots,x_t]} = T_{x_1x_2}T_{x_2x_3}\cdots T_{x_{t-1}x_t} \text{ and}$$
$$T_{x_1,[x_2,\dots,x_t]} = T_{x_1x_2}T_{x_1x_3}\cdots T_{x_1x_t}.$$

In the following, we will use the convention that the points of $\mathbb{A}_n$ (resp. of $\mathbb{D}_n$) are denoted by $1, \dots, n$ and the edges are $\{i, i+1\}$ for $i = 1, \dots, n-1$ (resp. $\{i, i+1\}$ for $i = 1, \dots, n-2$ and $\{n-2, n\}$).

Lemma 1 *Let A be a tree assemblage of complete graphs.*

(a) *For any two different non-knots x, y of A there exists a sequence T of flations for A which avoids x and y such that $AT = \mathbb{A}_n$ (and hence x and y are the end points of $\mathbb{A}_n$).*

(b) *For any vertex x of A there exists a sequence T of flations for A which strongly avoids x such that $AT = \mathbb{A}_n$ and x corresponds to an end point of $\mathbb{A}_n$ if and only of x is not a knot of A.*

Proof. (a) The proof is done by induction over the number t of complete graphs involved in the definition of A. If $t = 1$ then A is a block, and we verify easily that $AT_{[z_1,\dots,z_{n-1}]} = \mathbb{A}_n$, if the points of A are denoted by $x = z_1, z_2, \dots, z_n - 1, z_n = y$. Now, assume that A has at least one knot and let $B = \{z \in A \mid [x,z]_A \neq 0\} \cup \{x\}$. Since x is not a knot of A, we have that B is a complete graph.

If $[x,y]_A \neq 0$ then let $k_1, \dots, k_s$ denote the knots of A which belong to B. By our induction hypothesis there exists a sequence of flations T avoiding all points of B such that $A' = AT$ is obtained from the disjoint union of B with $\mathbb{A}_{n_1}, \dots, \mathbb{A}_{n_s}$ by identifying k_j with the point $1(j)$ of $\mathbb{A}_{n_j}$ (the points of $\mathbb{A}_{n_j}$ are denoted by by $1(j), \dots, n_j(j)$). We abreviate the resulting graph by $B[k_1,n_1]\cdots[k_s,n_s]$. Since $A'T_{k_1,[1(2),\dots,n_2(2)]} = B^{(k_1)}[k_2, n_1{+}n_2][k_3,n_3]\cdots[k_s,n_s]$, we obtain inductively a sequence of flations T' avoiding x and y such, that $A'' = A'T' = B'[k_s,n]$, where $n = n_1 + \cdots n_s$ and B' is a complete graph containing x, y and k_s and maybe some other vertices $a_1, \dots, a_r$. Since $A''T_{[x,a_1,\dots,a_r]}T_{y,[1(s),\dots,n(s)]} = \mathbb{A}_m$, we finish the proof in the case where $[x,y]_A \neq 0$.

If $[x,y]_A = 0$ then there exists a knot k such that x and y belong to different components of $A^{(k)} = B_1 \cup B_2$, we may assume $x \in B_1$ and $y \in B_2$. Denote by $\tilde{B}_i$ the full subgraph of A given by $B_i \cup \{k\}$. Since k is not a knot of $\tilde{B}_i$, for $i = 1, 2$ there exists, by induction hypothesis, a sequence of flations T_1 avoiding k and x such such that $\tilde{B}_1 T_1 = \mathbb{A}_{m_1}$ (hence k and x correspond to the end points of $\mathbb{A}_{m_1}$), and similarly, a sequence of flations T_2 for $\tilde{B}_2$ avoiding y and k such that $\tilde{B}_2 T_2 = \mathbb{A}_{m_2}$ (and hence y and k correspond to the end points of $\mathbb{A}_{m_2}$). Since A is obtained by the disjoint union of $\tilde{B}_1$ and $\tilde{B}_2$ identifying the two k in each part, and T_1, T_2 avoid k we obatin that $AT_1T_2 = \mathbb{A}_{m_1+m_2-1}$ and x and y correspond to the end points of $\mathbb{A}_{m_1+m_2-1}$.

(b) The proof is done by induction on the number of points. If $x \in A$ is a knot, we have $A^{(x)} = B_1 \cup B_2$ and let $\tilde{B}_i$ to be the full subgraph of A given by $B_i \cup \{x\}$. Then x is not a knot of $\tilde{B}_i$ and by hypothesis, there exists a sequence of flations T_i for $\tilde{B}_i$ strongly avoiding x such that $\tilde{B}_i T_i = \mathbb{A}_{n_i}$. Hence $AT = \mathbb{A}_n$ and x is a knot of AT.

If x is not a knot we denote by $y_1, \dots, y_r$ the neighbours of x in A. If $r = 1$, we can apply induction on $y_1 \in \Gamma^{(x)}$. Otherwise, we apply $T_{[y_1,\dots,y_r]}$ to Γ and obtain a graph in which x has exactly one neighbour, namely y_r. Hence we are back in the situation before. $\square$

2.5

Finally, we also need a positive result for graphs of type $\mathbb{D}_n$.

Lemma 2 *A cycle with $n \geq 4$ points is positive admissible and of Dynkin type $\mathbb{D}_n$.*

Proof. Let Γ be a cycle with $n \geq 4$ points $x_1, \ldots, x_n$ and edges $[x_i, x_{i+1}]_\Gamma = 1$ for $i = 1, \ldots, n-1$ and $[x_n, x_1]_\Gamma = 1$. Then $\Gamma T_{x_1,[x_2,\ldots,x_{n-1}]} = \mathbb{D}_n$, and hence the result follows from (2.2). $\square$

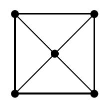
3 Forbidden subgraphs for $\mathbb{D}_n$

The proof the Main Theorem is combinatorial and bases on the technique of flations of graphs, by which given graphs will be reduced to special cases. Several cases have to be excluded. This is done in this section.

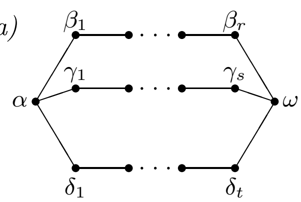
Proposition 2 *Let Γ be positive admissible graph of Dynkin type $\mathbb{D}_n$. Then Γ does not contain a full subgraph of the following list.*

(F1) *Extended Dynkin graphs.*

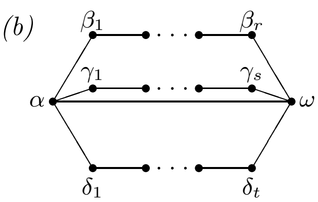
(F2)



(F3) *(a)*

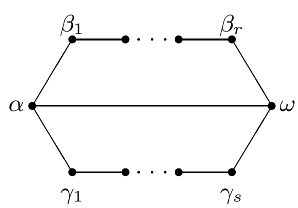


or *(b)*



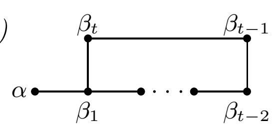
for $r, s, t \geq 1$

(F4)



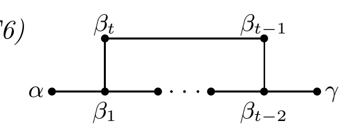
for $r, s \geq 2$

(F5)



for $t \geq 5$

(F6)



for $t \geq 4$.

Proof. We will show, that a graph of this list is either not positive admissible or it contains a graph which is positive admissible but has Dynkin type $\mathbb{E}_6$. The result follows then by (2.1) and (2.2).

Clearly no extended Dynkin graph is positive admissible. Since no bigraph B with frame $(F2)$ or $(F3a)$ can satisfy the cycle condition, those graphs do not admit a positive presentation. Let q be the unit form whose bigraph B has frame $(F3b)$ and where all edges are full except $\{\alpha, \omega\}$ which is broken. Clearly, B satisfies the cycle-condition, but for the vector v given by $v(x) = 1$ for all $x \in B$ we have $q(v) = 0$, hence, by (2.3), the graph of $(F3b)$ is not positive admissible. In case Γ is of the form $(F4)$, we observe that $\Gamma T_{\alpha,[\beta_1,\ldots,\beta_r,\gamma_1,\ldots,\gamma_s]} T_{\omega\alpha}$ contains $\mathbb{E}_6$. If Γ is of the form $(F5)$, we observe that $\Gamma T_{\beta_1,[\beta_2,\ldots,\beta_{t-1}]}$ contains $\mathbb{E}_6$. If Γ is of the form $(F6)$, then if $t > 4$ we may restrict to $\Gamma^{(\gamma)}$, which is of the form $(F5)$ and if $t = 4$, then $\Gamma T_{\beta_1\beta_4} T_{\beta_1\beta_3} = \mathbb{E}_6$. □

4 Reduction to Blocks

4.1

Lemma 3 *Let Γ be a positive admissible graph of Dynkin type $\mathbb{D}_n$ and x not a knot of Γ. Then there exists a sequence of flations T avoiding x for Γ such that $\Gamma T = \mathbb{D}_n$. If $\mathrm{Dyn}(\Gamma^{(x)}) = \mathbb{A}_{n-1}$ (resp. $\mathrm{Dyn}(\Gamma^{(x)}) = \mathbb{D}_{n-1}$) the sequence T may be chosen in such a way that x corresponds to the point n (resp. 1) of ΓT.*

Proof. Let q be a positive unit form with frame Γ. By [5, Theorem 6.2] the length of a sequence of inflations of q is bounded. Let T be be a non-prolongable sequence of inflations avoiding x for q and set $q' = qT$. Then $\mathrm{B}(q') = \mathbb{D}_n$, since otherwise there would exist a broken edge in $\mathrm{B}(q')$ between x and some other point i, showing that T could be prolonged with T^{+}_{ix}, a contradiction. It follows that $\Gamma T = \mathbb{D}_n$. □

The following situation will occur rather often: a given graph Γ is restricted to $\Gamma^{(x)}$ which is a disjoint union $\Gamma_1 \cup \ldots \cup \Gamma_v$, where $v = \mathrm{v}_\Gamma(x)$. We then will denote by $\tilde{\Gamma}_i$ the full subgraph of Γ given by $\Gamma_i \cup \{x\}$ without repeating it explicitly each time.

4.2

Lemma 4 *Let Γ be a positive admissible graph of Dynkin type $\mathbb{D}_n$ and suppose that there exists a knot x in Γ. Let $B_1, \ldots, B_{\mathrm{v}_\Gamma(x)}$ be the connected components of $\Gamma^{(x)}$ Then we have $\mathrm{v}_\Gamma(x) \leq 3$. Furthermore, if $\mathrm{v}_\Gamma(x) = 3$ then $\mathrm{Dyn}(\tilde{B}_i) = \mathbb{A}_{n_i}$ for all i, and if $\mathrm{v}_\Gamma(x) = 2$, we have $\mathrm{Dyn}(\tilde{B}_1) = \mathbb{A}_{n_1}$ and $\mathrm{Dyn}(\tilde{B}_2) = \mathbb{D}_{n_2}$. (or $\mathrm{Dyn}(\tilde{B}_1) = \mathbb{D}_{n_1}$ and $\mathrm{Dyn}(\tilde{B}_2) = \mathbb{A}_{n_2}$).*

Proof. Let $B(q)$ be a positive presentation of Γ. As in the proof of the previous lemma, let T be a non-prolongable sequence of inflations avoiding x for q and let Γ' be the frame of qT. It is easy to check that $\mathrm{v}_\Gamma(x) = \mathrm{v}_{\Gamma'}(x)$ and that the graph $\tilde{B}_i$ is flation-equivalent to $\tilde{C}_i$ if $C_1, \ldots C_{\mathrm{v}_\Gamma(x)}$ denote the connected components of ${\Gamma'}^{(x)}$ in the corresponding enumeration. Hence the result follows from $\Gamma' = \mathbb{D}_n$. $\square$

4.3

Proposition 3 *Let Γ be a positive admissible graph of Dynkin type $\mathbb{D}_n$ and suppose that there exists a vertex x of Γ with connecting valence 3. Then there exists a tree assemblage A of complete graphs and a vertex y of A such that $\Gamma = A[y]$.*

Proof. By (4.2) and (2.4) there exists a sequence of flations T strongly avoiding x such, that ΓT is a star with center x and 3 branches. Hence two of the branches consist of one single edge. Thus, by the observation in (2.4), if B_1, B_2 and B_3 are the connected components of $\Gamma^{(x)}$, we may assume that $B_1 = \{y\}$ and $B_2 = \{z\}$. Then we have that $\Gamma^{(z)}$ is a tree assemblage of full graphs and z is a mirror point of y. $\square$

5 $\mathbb{D}$-blocks

5.1

Lemma 5 *Let Γ be a block of Dynkin type $\mathbb{D}_n$. Then there exists a vertex x of Γ such that* $\mathrm{Dyn}(\Gamma^{(x)}) = \mathbb{A}_{n-1}$.

Proof. Suppose that this is not so and let Γ be a minimal such graph. Either there exists a vertex $x \in \Gamma$ such that $\Gamma^{(x)}$ is not block or for any $x \in \Gamma$ the restriction $\Gamma^{(x)}$ is a block but there exists $y \in \Gamma^{(x)}$ such, that $\mathrm{Dyn}(\Gamma^{(x)(y)}) = \mathbb{A}_{n-2}$. We will show that both cases lead to a contradiction.

Suppose first that $\Gamma^{(x)}$ is not a block. Then let y be a knot of $\Gamma^{(x)}$ and $\Gamma^{(x)(y)} = B_1 \cup \cdots \cup B_v$. Since $\Gamma^{(y)}$ is connected, we have $v = 2$ (otherwise $(F3)$ is contained in Γ). By (4.2), we may assume that $\mathrm{Dyn}(\tilde{B}_1) = \mathbb{D}_{n_1}$ and $\mathrm{Dyn}(\tilde{B}_2) = \mathbb{A}_{n_2}$. Let T_1 be a sequence of flations for $\tilde{B}_1$ avoiding y such that $\tilde{B}_1 T_1 = \mathbb{D}_{n_1}$.

Observe that the case, where $\mathrm{Dyn}(B_1) = \mathbb{A}_{n_1-1}$ with $n_1 \geq 5$, is impossible (choose $z \in B_2$ such that $[y,z]_\Gamma \neq 0$; then the restriction of ΓT_1 to $\{z,y\} \cup B_1$ contains $\mathbb{E}_6$). Thus $\mathrm{Dyn}((\tilde{B}_1 T_1)^{(y)})$ is either $\mathbb{A}_3$ or $\mathbb{D}_{n_1-1}$, we will assume that the y is the point 1 of $\tilde{B}_1 T_1 = \mathbb{D}_{n_1}$.

Let $x, z_1, \ldots, z_t, y$ be a shortest walk from x to y with $z_i \in B_2$ and let Γ' be the full subgraph of ΓT_1 given by $\{x, z_1, \ldots, z_t\} \cup \tilde{B}_1 T_1$. Since T_1 avoids y, we must have $[x,b]_{\Gamma'} \neq 0$ for some $b \in \tilde{B}_1 \setminus \{y\}$. Let C be the full subgraph of Γ' given by $\{x\} \cup \tilde{B}_1 T_1 \setminus \{y\}$. Applying flations of the form T_{xs}, for $s \in C$, $s \neq x$, we obtain a sequence of flations T_2 for C such that there exists only one edge in C which ends in x, say $\{x, i\}$. Notice that $(CT_2)^{(x)} = (\tilde{B}_1 T_1)^{(y)}$ which equals $\mathbb{A}_3$ or $\mathbb{D}_{n_1-1}$. If $i = n_1$ or $i = n_1 - 1$, then $\Gamma' T_2$ contains (F5), whereas if $3 \leq i \leq n_1 - 2$ then $\Gamma' T_2$ contains $\tilde{\mathbb{D}}_m$ for some m. It remains to consider the case $i = 2$. If there is an edge $\{x, y\}$ then $\Gamma' T_2 T_{x,[y,z_t,\ldots,z_2]} = \tilde{\mathbb{D}}_{n_1-1}$ and if not, then $\Gamma' T_2$ contains $\tilde{\mathbb{D}}_m$ for some m. In any case, a contradiction to the assumptions.

Assume now that for every $x \in \Gamma$, $\Gamma^{(x)}$ is a block and there exists a vertex $y \in \Gamma^{(x)}$ such, that $\mathrm{Dyn}(\Gamma^{(x)(y)}) = \mathbb{A}_{n-2}$. By (4.1), we may transform $\Gamma^{(x)}$ such that $\Gamma^{(x)} T = \mathbb{D}_{n-1}$ and $y = n - 1$. Again, by applying flations of the form T_{xz} with $z \neq y, x$, to Γ we may end up with a graph Γ' where ${\Gamma'}^{(x)} = \mathbb{D}_{n-1}$ and there is exactly one edge ending in x, either $\{x, 2\}$ or $\{x, n - 3\}$. In either case we must have $[x, y]_{\Gamma'} \neq 0$. If $[x, n - 3]_{\Gamma'} \neq 0$ we apply T_{xy} and observe that for $n > 5$ the resulting graph contains $\mathbb{E}_6$, whereas if $[x, 2]_{\Gamma'} \neq 0$ we must have $n = 5$ because of $(F6)$.

Since $\Gamma^{(x)}$ is a block for every $x \in \Gamma$ and $n = 5$, we have $\mathrm{Dyn}(\Gamma^{(x)(y)}) = \mathbb{A}_3$ for any $x \neq y$. However, the case $\Gamma^{(x)(y)} = \mathbb{A}_3$ is not possible: $\Gamma^{(2)}$ would be a cycle with four points, so by (F2) and (F3a) we would see that $\mathrm{Dyn}(\Gamma^{(1)}) = \mathbb{A}_4$. This shows that $\Gamma^{(x)(y)}$ is a complete graph with 3 points for every $x \neq y$. Thus Γ is complete and $\mathrm{Dyn}(\Gamma) = \mathbb{A}_5$, again a contradiction. □

5.2

Proposition 4 *Let q be a unit form such, that $\Phi(q)$ is a block. If q is positive of Dynkin type $\mathbb{D}_n$ then q satisfies (i), (ii) and (iii) of the Main Theorem.*

Proof. Properties (i) and (ii) are clear. By Proposition 5.1, there exists a vertex $x \in \Phi = \Phi(q)$ such that $\mathrm{Dyn}(\Phi^{(x)}) = \mathbb{A}_{n-1}$.

If $\Phi^{(x)}$ is a block, then it is a complete graph with vertices $y_1, \ldots, y_r, z_1, \ldots, z_s$ where $r + s = n-1$ and $[x, y_i] = 1$ for $i = 1, \ldots, r$ and $[x, z_i] = 0$ for $i = 1, \ldots, s$, see [1, Proposition 3.3]. Observe that $\Gamma = \Phi T_{[y_1,\ldots,y_r]} T_{[z_1,\ldots,z_s]}$ is a star with center y_r and branches $\{x, y_r\}$, $\{y_1, \ldots, y_r\}$ and $\{z_1, \ldots, z_s, y_r\}$. Since the Dynkin type of Γ is $\mathbb{D}_n$ we must have $r = 2$ or $s = 1$. In the first case, Φ is an $\mathbb{A}$-cycling and in the second case Φ is an $\mathbb{A}$-mirror extension of $\Phi^{(x)}$. This shows (iii) in the case where $\Phi^{(x)}$ is a block.

So, assume now that $\Phi^{(x)}$ is not a block, hence given as tree assemblage of complete graphs $\Gamma_1, \ldots, \Gamma_t$ defined by a tree T. If Γ_i is a *leaf*, that is $i \in T$ has exactly one neighbour or equivalently there exist exactly one knot k_i of $\Phi^{(x)}$ which belongs to Γ_i, then there exists a vertex $y_i \in \Gamma_i \setminus \{k_i\}$ such that $[x, y_i] = 1$, since otherwise k_i would be a knot of Φ. Let Σ be a minimal tree in $\Phi^{(x)}$ which contains $\{y_i \,|\, \Gamma_i \text{ is a leaf}\}$. Since $\mathrm{d}_{\Phi^{(x)}}(y_i, y_j) \geq 2$ for $i \neq j$ we obtain by (F3), that the number of leafs is at most 2, hence equals 2 since $\Phi^{(x)}$ is not a block. This implies that T is linear. Furthermore, by (F4) we have $[x, y] = 0$ whenever y does not belong to a leaf.

For simplicity, set $T = \mathbb{A}_t$, hence Γ_1 and Γ_t are the two leafs of $\Phi^{(x)}$. For $i = 1, t$ denote by $y_1^{(i)}, \ldots y_{r_i}^{(i)}, z_1^{(i)}, \ldots z_{s_i}^{(i)}$ the points of Γ_i such that $[x, y_j] = 1$

and $[x, z_j] = 0$ for any j. We suppose that either $k_i = y_1$ or $k_i = z_1$. Note that $r_i \geq 1$. Set

$$\Gamma = \Phi T_{[y_1^{(1)},\dots,y_{r_1}^{(1)}]} T_{[z_1^{(1)},\dots,z_{s_1}^{(1)}]} T_{[y_1^{(t)},\dots,y_{r_t}^{(t)}]} T_{[z_1^{(t)},\dots,z_{s_t}^{(t)}]}.$$

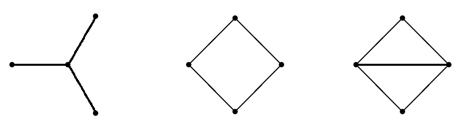
If $t > 2$ then $r_i = 1$ or $s_i = 0$ (for $i = 1, t$) since otherwise (F5) is contained in Γ, a contradiction. In all cases, we have that Φ is a cycling.

It remains to consider the case where $t = 2$, for which we have $k_1 = k_2$. If $[x, k_1]_\Phi = 0$, then $s_1 = s_2 = 1$ or $r_1 = r_2 = 1$ since otherwise (F5) is contained in Γ. In the first case Φ is a mirror extension and in the second a cycling. So finally assume that $[x, k_1]_\Phi = 1$. Then we have that $r_1 \geq 2$ and $r_2 \geq 2$ and hence $[x, k_1]_\Gamma = 1$. Thus we have $r_1 = 2$ or $r_2 = 2$, since otherwise (F4) is contained in Γ, assume $r_1 = 2$ (otherwise switch the roles of Γ_1 and Γ_2). Observe that for $r_2 > 2$ we must have $s_2 = 0$ (otherwise (F5) is contained in $\Gamma T_{k_1 y_2^{(1)}}$), hence Φ is a cycling, whereas if $r_2 = 2$ then $s_1 = 0$ or $s_2 = 0$ (otherwise $\Gamma T_{x,k_1}$ contains $\mathbb{E}_6$), again Φ is a cycling. □

6 Proof of the Main Theorem

First, let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected unit form which is positive and has Dynkin type $\mathbb{D}_n$. We have to show that q satisfies the property (iii) of the Main Theorem, since the first two conditions are trivially satisfied.

If $\Phi = \Phi(q)$ is a block, this follows from Proposition 5.2. If Φ contains a knot k of connecting valence 3, the result follows from Proposition 4.3. It remains the case where there is a knot k with $\mathrm{v}(k) = 2$. Let $\Phi^{(k)} = \Phi_1 \cup \Phi_2$. By Lemma 4.2, we may assume that $\mathrm{Dyn}(\tilde{\Phi}_1) = \mathbb{D}_m$ and $\mathrm{Dyn}(\tilde{\Phi}_2) = \mathbb{A}_p$. It is an easy exercise to check that in case $m = 4$ there are the 3 following possibilities for $\tilde{\Phi}_1$:



In each case Φ satisfies (iii). So let us assume that $m > 4$. Then by Lemma 4.1, we may exclude the case where $\mathrm{Dyn}(\Phi_1) = \mathbb{A}_{m-1}$. By induction we may then assume that $\tilde{\Phi}_1$ satisfies (iii). First suppose that $\tilde{\Phi}_1 = \Gamma[x]$ is a mirror extension. Since $\mathrm{Dyn}(\Phi_1) = \mathbb{D}_{m-1}$, we have that $k \neq x, x^*$ and hence we have that $\Phi = \Gamma'[x]$ is a mirror extension, where Γ' is the glueing of Γ and $\tilde{\Phi}_2$ in k which is a graph of Dynkin type $\mathbb{A}_{n-1}$.

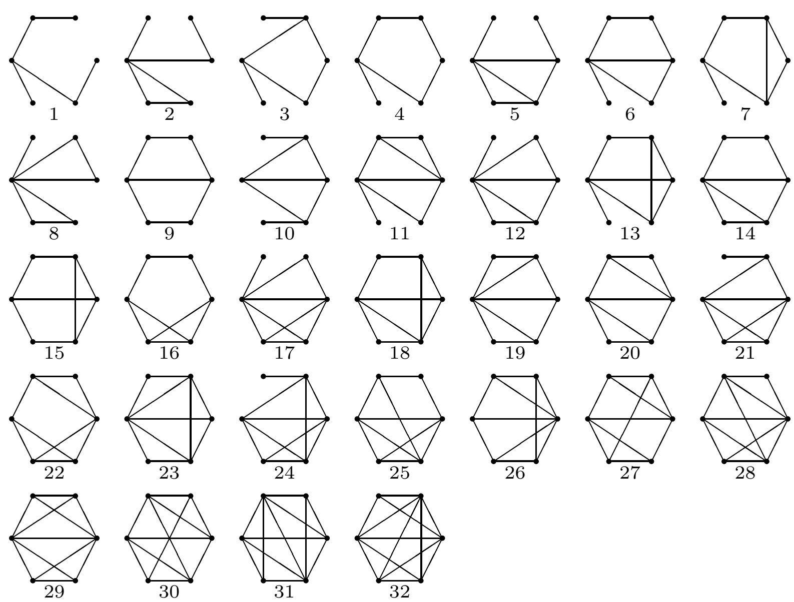
So assume now that $\tilde{\Phi}_1 = \Gamma/\{x = y\}$ is a cycling. Again, since $\mathrm{Dyn}(\Phi_1) = \mathbb{D}_{m-1}$ we have that $k \neq x = y$ and that k is not a knot of Γ. Hence we have that $\Phi = \Gamma'/\{x = y\}$ is a cycling, where Γ' is the glueing of Γ and $\tilde{\Phi}_2$ in k which is a graph of Dynkin type $\mathbb{A}_{n+1}$. In both cases, Φ satisfies (iii).

Conversely, let now q be a unit form such that (i), (ii) and (iii) are satisfied. Assume first that $\Phi := \Phi(q)$ is a cycling $\Phi = \Gamma/\{x = y\}$. By Lemma 2.4, there exist an iterated flation T avoiding x and y for Γ for Φ such that $\Gamma T = \mathbb{A}_{n+1}$ and x, y are the end points. Hence ΦT is a cycle. The result follows therefore by Lemma 2.5 and Lemma 2.3.

So assume that $\Phi = \Gamma[x]$ is a mirror extension. Let T be an iterated flation for Γ strongly avoiding x such that $\Gamma T = \mathbb{A}_{n-1}$. If x is not a knot of ΓT then $\Gamma[x]T = \Gamma T[x] = \mathbb{D}_n$ and we are done. If x is a knot denote by y_1, y_2 the two neighbours of x (and of x^*) in $\Gamma[x]T = \Gamma T[x]$. Then $\Gamma T T_{y_1 y_2}$ is still a mirror extension but contains a point of connecting valence 3, hence we are done also. $\square$

7 List of positive admissible graphs of Dynkin type $\mathbb{E}_6$

The following list shows all positive admissible graphs of Dynkin type $\mathbb{E}_6$. The list was obtained by a computer program in the most simple way: first a complete list of all connected graphs with 6 points was calculated and then for each such graph it was checked whether it admits a positive presentation, for which the Dynkin type was calculated.



There are 233 positive admissible graphs of Dynkin type $\mathbb{E}_7$ and 1242 positive admissible graphs of Dynkin type $\mathbb{E}_8$.

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