The repetitive partition of the repetitive category of a tubular algebra

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Abstract. In [8] it was shown that for a given tubular algebra A there exist only finitely many non-isomorphic tubular algebras T which are reflection-equivalent to A. We give a concrete recipe to construct all of them. The recipe also gives, up to isomorphism, all full and convex subalgebras of $\widehat{A}$ which are tame concealed. Finally we characterize the sets of points of $\widehat{A}$ whose corresponding projectives lie in a given tubular family of mod $\widehat{A}$.

1. Introduction and main results

In [8] it was shown that for a tubular algebra A there exist only a finite number of non-isomorphic tubular algebras T such that the repetitive categories $\widehat{A}$ and $\widehat{T}$ are isomorphic, or in other words, A and T are reflection-equivalent, i.e. T can be obtained from A by a sequence of reflections in the sense of [6]. We present a concrete recipe to construct all of them.

For this sake, we will need some notations. A branch is a tilted algebra of type $\mathbb{A}_n$, see for example [7]. The quiver Q_B with relations of a branch B is a finite, full and convex subquiver of the infinite fractal quiver given in the figure 1.1 containing the vertex b with the induced relations.

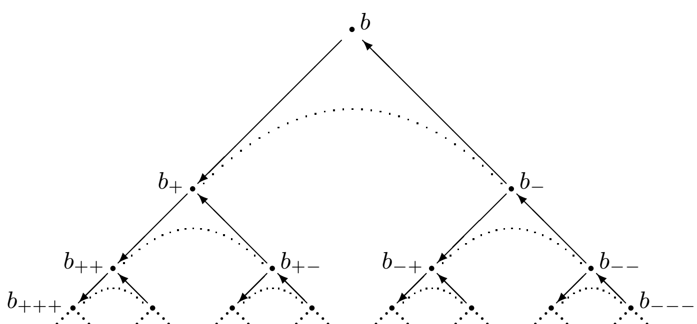

Figure 1.1

For each branch B we define the subset $B^- = \{b, b_{i_1\cdots i_n -} \in Q_B \,|\, i_j = \pm\}$ of vertices of Q_B.

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In [7] it was shown that a tubular algebra A is a tubular extension of a tame concealed algebra A_0 using modules $M_1, \dots, M_t$ from the tubular family of $\operatorname{mod} A_0$ and branches $B_1, \dots B_t$. We denote as in [7], $A = A_0[M_i, B_i]_{i=1}^t$.

Theorem 1. *Let A be a tubular algebra, $A = A_0[M_i, B_i]_{i=1}^t$. Then the algebra obtained from A by reflecting all points in the set $B_1^- \cup \cdots \cup B_t^-$ is tubular again. By iteration we obtain all tubular algebras which are reflection-equivalent to A.*

We recall from [4] that, for a tubular algebra A, the derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ and the stable category $\underline{\operatorname{mod}}\, \widehat{A}$ are equivalent and consist by [5] of a family $(\underline{\mathcal{T}}_q)_{q \in \mathbb{Q}}$ of tubular families $(\underline{\mathcal{T}}_q^{(\kappa)})_{\kappa \in \mathbb{P}^1 k}$ each of the same tubular type which is the extension type of A. We denote by $\mathcal{T}_q^{(\kappa)}$ the connected component of $\operatorname{mod} \widehat{A}$ which is sent to $\underline{\mathcal{T}}_q^{(\kappa)}$ under the projection of $\operatorname{mod} \widehat{A}$ to $\underline{\operatorname{mod}}\, \widehat{A}$. Thus, we may define $\mu : \widehat{A} \to \mathbb{Q}$ by the property $P_x \in \mathcal{T}_{\mu(x)}$, where P_x denotes the projective indecomposable $\widehat{A}$-module corresponding to the vertex $x \in \widehat{A}$. The calculation of the derived category of a tubular algebra A in [5] also shows, for a canonical tubular algebra A, the position of the projective $\widehat{A}$-modules in the category $\operatorname{mod} \widehat{A}$. The present paper generalizes part of the unstated result in [5] to arbitrary tubular algebras giving a description of the sets of indecomposable projectives which lie in the same tubular family of $\operatorname{mod} \widehat{A}$.

Theorem 2. *Let A be a tubular algebra and let $q \in \mathbb{Q}$ be such that $\mu^{-1}(q)$ is non-empty. Then there exists a tubular algebra $T = T_0[M_i, B_i]_{i=1}^t$, which is reflection-equivalent to A and such that $\mu^{-1}(q) = B_1^- \cup \cdots \cup B_t^-$.*

The partition of the points of $\widehat{A}$, for a tubular algebra A, by the fibres of μ will be called the repetitive partition of $\widehat{A}$. Finally we also prove the following result.

Theorem 3. *Let A be a tubular algebra. Then any full and convex subalgebra of $\widehat{A}$ which is tame concealed is isomorphic to T_0 for some tubular algebra T which is reflection-equivalent to A.*

For the definition of tubular algebras and their properties we refer to [7].

2. Example

Let A be the algebra given by the quiver and relations of the figure 2.1 on the left side, so we have $A = A_0[M_i, B_i]_{i=1}^2$. We enclose the tame concealed algebra A_0 with a square and we mark the points of $B_1^- \cup B_2^-$ by $\odot$.

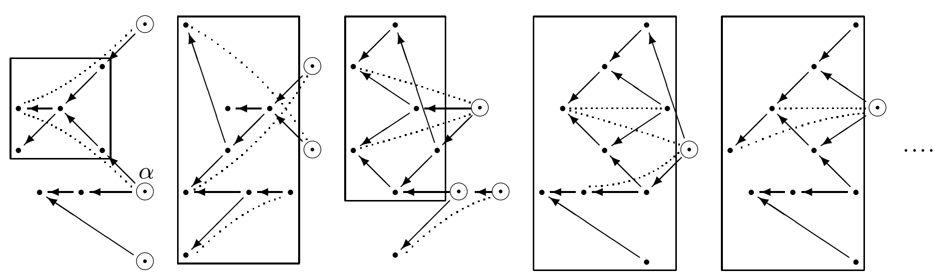

Figure 2.1

By reflecting the marked points we obtain the second algebra in the sequence where we proceed similar in order to obtain the third algebra and so on. The sixth algebra which we obtain this way is isomorphic to the first and hence, by Theorem

1, we constructed all tubular algebras which are reflection-equivalent to A. The construction gives us also the repetitive partition of $\widehat{A}$, as shown in the figure 2.2. where the vertical stripes indicate the points of $\widehat{A}$ whose corresponding projectives lie in the same tubular family of $\operatorname{mod}\widehat{A}$.

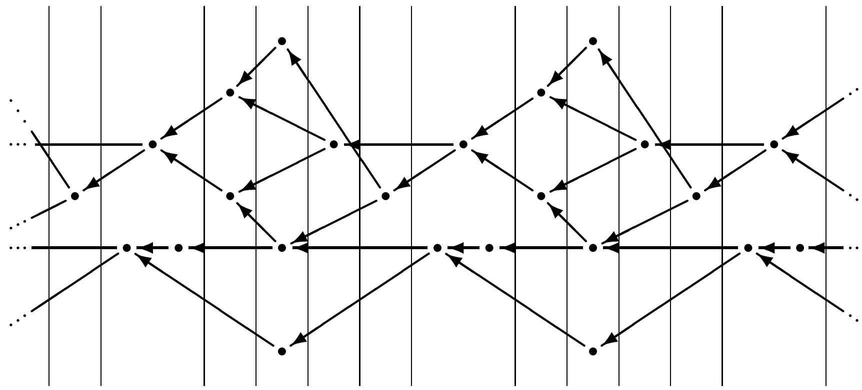

Figure 2.2

Finally, any full and convex subalgebra of $\widehat{A}$ which is tame concealed is isomorphic to one of the algebras given by a rectangle in the figure 2.1.

3. Proofs

A module M over an algebra A is called omnipresent if for all projective A-modules P we have $\operatorname{Hom}_A(P, M) \neq 0$.

PROPOSITION 1. *Let A be an algebra which is derived equivalent to a tubular algebra. Then A is tubular if and only if $\operatorname{mod} A$ contains a homogeneous tube with an omnipresent indecomposable module.*

PROOF. By [1], the algebras of infinite representation type which are derived equivalent to a tubular algebra are branch-enlargements of a tame concealed algebra T_0, i.e. obtained from a tubular algebra $T = T_0[M_i, B_i]_{i=1}^t$ by a sequence of reflections in points of $B_1 \cup \cdots \cup B_t$. If A is not tubular then $\operatorname{mod} A$ consists of a postprojective component, a preinjective component and a family $\mathcal{T} = (\mathcal{T}^{(\kappa)})_{\kappa \in \mathbb{P}^1 k}$ which is obtained from a stable tubular family by ray insertion and coray insertion, see [7]. Furthermore, $\mathcal{T}$ contains at least one projective P_x and at least one injective I_y and different components of $\mathcal{T}$ are pairwise orthogonal. Thus an omnipresent indecomposable module M satisfies $\operatorname{Hom}(M, \mathcal{T}) \neq 0 \neq \operatorname{Hom}(\mathcal{T}, M)$. This implies that M belongs to $\mathcal{T}$. By the orthogonality, we have that M, P_x and I_y belong to the same component $\mathcal{T}^{(\kappa)}$, which therefore is not a homogeneous tube. $\square$

For each $q \in \mathbb{Q}$ we denote by $\mathcal{S}_q$ the full subalgebra of $\widehat{A}$ given by the support of a chosen homogeneous tube of $\mathcal{T}_q$. Furthermore we set

$$\Pi = \{q \in \mathbb{Q} \mid \mathcal{T}_q \text{ contains a projective module}\}$$

and we define the bijective map $\sigma : \Pi \to \Pi$ by $\sigma(p) = \min\{r \in \Pi \mid r > p\}$.

LEMMA. *For any $q \in \mathbb{Q}$ we have $\mathcal{S}_q \subseteq \{s \in \widehat{A} \mid \mu(P_s) < q < \mu(I_s)\}$. If $q \notin \Pi$ even equality holds.*

Proof. Let $s \in \mathcal{S}_q$. Then there exists a non-zero morphism $P_s \to M$ for some $\widehat{A}$-module M in a homogeneous tube of $\mathcal{T}_q$. It is not hard to see that $\mathrm{Hom}_{\widehat{A}}(\mathcal{T}_q^{(\kappa)}, \mathcal{T}_{q'}^{(\lambda)}) = 0$ whenever $q > q'$ or $q = q'$ and $\kappa \neq \lambda$, since in those cases we have $\underline{\mathrm{Hom}}_{\widehat{A}}(\underline{\mathcal{T}}_q^{(\kappa)}, \underline{\mathcal{T}}_{q'}^{(\lambda)}) = 0$. Thus we have $\mu(s) < q$. This proves $\mathcal{S}_q \subseteq \{s \in \widehat{A} \mid \mu(P_s) < q < \mu(I_s)\}$. If $q \notin \Pi$ we even have equality since $\mathcal{T}_q$ inherits the separartion-property from $\underline{\mathcal{T}}_q$. $\square$

We recall from [6] the following facts. Let A be a finite-dimensional algebra. The objects of the category $\widehat{A}$ are pairs (q, i) where q is a point of the quiver Q_A of A and i an integer. A subcategory S of $\widehat{A}$ is called a slice (resp. complete slice) if for each point $q \in Q_A$ there exists at most one (resp. exactly one) integer i such that (q, i) belongs to S.

Proposition 2. *With the above notations, $\mathcal{S}_q$ is a critical (respectively tubular) algebra if $q \in \Pi$ (respectively if $q \notin \Pi$).*

Proof. Let first $q \notin \Pi$. Then $\mathcal{S}_q$ is a full complete and convex slice of $\widehat{\mathcal{S}}$ and hence it follows that $\widehat{S}_q$ is isomorphic to $\widehat{S}$, see [6]. In particular, $\mathcal{S}_q$ is derived equivalent to a tubular algebra and by definition it admits an omnipresent indecomposable module, hence by Proposition 1, the algebra $\mathcal{S}_q$ is tubular.

Observe that for $p \in \Pi$ only the stable tubes of $\mathcal{T}_p$ belong to the image of the embedding of $\mathrm{mod}\,\mathcal{S}_p$ in $\mathrm{mod}\,\widehat{\mathcal{S}}$. For a chosen $q \in (p, \sigma(p))$ we have that the homogeneous tubes of $\mathcal{T}_{q'}$ belong to the image of the embedding of $\mathrm{mod}\,\mathcal{S}_q$ in $\mathrm{mod}\,\widehat{\mathcal{S}}$ if and only if q' belongs to the closed interval $[p, \sigma(p)]$. Moreover, $\mathcal{S}_q$ properly contains $\mathcal{S}_p$. Hence if we embed $\mathrm{mod}\,\mathcal{S}_p$ into $\mathrm{mod}\,\mathcal{S}_q$, then correspondingly the homogeneous tubes of the tubular family of $\mathrm{mod}\,\mathcal{S}_p$ embeds into the homogeneous tubes of the first tubular family in $\mathrm{mod}\,\mathcal{S}_q$. Therefore, $\mathcal{S}_p$ is one of the two critical subalgebras of the tubular algebra $\mathcal{S}_q$. $\square$

Proposition 3. *For each tubular algebra T, which is reflection-equivalent to A, there exists a $q \in \mathbb{Q} \setminus \Pi$ such that T is isomorphic to the algebra $\mathcal{S}_q$.*

Proof. Choose a homogeneous tube $\mathcal{T}$ of $\mathrm{mod}\,T$ which contains an omnipresent module. Then by the embedding of $\mathrm{mod}\,T$ into $\mathrm{mod}\,\widehat{A}$ this tube is sent to a homogeneous tube in $\mathrm{mod}\,\widehat{A}$. Hence we have that T is isomorphic to $\mathcal{S}_q$ for some $q \in \mathbb{Q} \setminus \Pi$. $\square$

Proof of Theorem 1. Let A_0 be a triangulated algebra (that is, the quiver of A_0 does not contain an oriented cycle), M an A_0-module and B a branch. In [3] it was shown that for a branch-source extension $A = A_0[M, B]$ there exists exactly one branch-sink extension of A_0 which can be obtained from A by reflecting points in B, and the reflected points are exactly B^-. Clearly, this argument can be iterated for a tubular algebra $A = A_0[M_i, B_i]_{i=1}^t$ and thus we obtain a branch-sink extension $A' = {}_{i=1}^{t}[B_i', M_i']A_0$ by reflecting the points in $B_1^- \cup \cdots \cup B_t^-$. Since A' and A have the same extension type, A' is cotubular and hence tubular, see [7].

In order to prove the second part of the statement, show that for any $q \notin \Pi$, the algebra $\mathcal{S}_q$ can be obtained from A by a sequence of reflections. Let p be a number in Π and choose $q \in (p, \sigma(p))$ and $q' \in (\sigma^{-1}(p), p)$. Now, $\mathcal{S}_p$ is contained in

$\mathcal{S}_q$ and in $\mathcal{S}_{q'}$, moreover $\mathcal{S}_q$ is a branch-source extension of $\mathcal{S}_p$ and $\mathcal{S}_{q'}$ is a branch-sink extension of $\mathcal{S}_p$. So, $\mathcal{S}_{q'}$ can be obtained from $\mathcal{S}_q$ by a sequence of reflections. Iterating this process and applying Proposition 3 we get the desired result. □

Proof of Theorem 2. Let $p \in \Pi$. Choose $q \in (p, \sigma(p))$ and $q' \in (\sigma^{-1}(p), p)$. By Proposition 2 we know that $\mathcal{S}_q$ is tubular, $\mathcal{S}_q = \mathcal{S}_p[M_i, B_i]_{i=1}^t$. As in the proof of Theorem 1, we see that $\mathcal{S}_{q'}$ can be obtained from $\mathcal{S}_q$ by reflecting points from $B_1 \cup \cdots \cup B_t$. Hence [3] implies that the reflected points are $B_1^- \cup \cdots \cup B_t^-$. On the other hand we have $\mathcal{S}_q \setminus \mathcal{S}_{q'} = \mu^{-1}(p)$ and thus the result. □

Proof of Theorem 3. Let C be full and convex subalgebra of $\widehat{A}$ which is tame concealed. Then the tubes of the tubular family of $\operatorname{mod} C$ are embedded into a tubular family $\mathcal{T}_p$ of $\operatorname{mod} \widehat{A}$ and since there are only finitely many non-homogeneous tubes in $\mathcal{T}_p$ we obtain that C is isomorphic to $\mathcal{S}_p$ for some $p \in \mathbb{Q}$. By Proposition 2, we thus have $p \in \Pi$. Therefore we obtain the result by choosing $q \in (p, \sigma(p))$ and setting $T = \mathcal{S}_q$. □

4. Application

Once we have written down the so-called repetitive partition as in the introduction, it is very easy to get examples of derived tubular algebras of finite representation-type. Even more, we can give a recipe to construct all of them.

Proposition. *The set of all representation-finite derived tubular algebras is obtained as follows. First construct any tubular algebra A. Then take any full complete and convex slice in $\widehat{A}$ which does not contain any tame concealed algebras as given by Theorem 3.*

Proof. First, by [2], for any derived tubular algebra T there exists a tubular algebra A such that T is isomorphic to a full complete and convex slice in $\widehat{A}$. If T is representation-infinite then T is a branch-enlargement of a tame concealed algebra C, see [1]. Hence C is convex in $\widehat{A}$. So we are done by applying Theorem 3. □

We consider this procedure in our example. If we delete from $\widehat{A}$ all points $\alpha[i]$ for $i \in \mathbb{Z}$ (compare Figure 1.1) the resulting category is no longer connected (its quiver is painted in Figure 4.1).

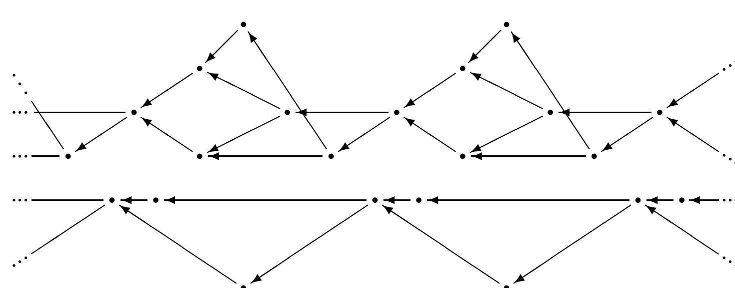

Figure 4.1

This makes it easier to list all algebras of finite representation type, which are derived equivalent to A. Namely, glue any algebra given by one of the quivers with relations of Figure 4.2

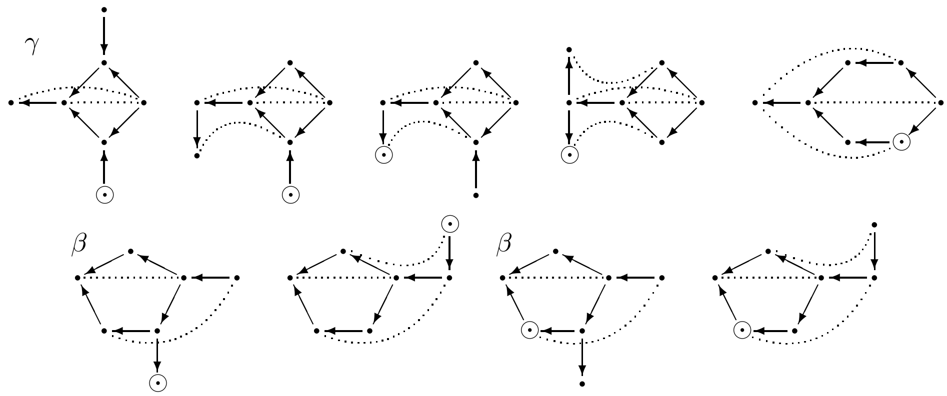

Figure 4.2

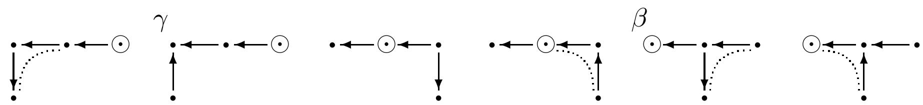

Figure 4.3

with one algebra of the list 4.3 in the exceptional points marked by $\odot$ and divide by the ideal consisting of all paths which start and end in different parts (the pairs γ-γ and β-β are excluded since they define representation-infinite algebras).

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