ROOT-INDUCED INTEGRAL QUADRATIC FORMS

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ABSTRACT. Given an integral quadratic unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ and a finite tuple of q-roots $r = (r^j)_{j\in J}$ the induced q-root form q_r is considered as in [3]. We show that two non-negative unit forms are of the same Dynkin type precisely when they are root-induced one from the other. Moreover, there are only finitely many unit forms without double edges of a given Dynkin type. Root-induction yields an interesting partial order on the Dynkin types, which is studied in the paper.

1. Introduction and Results

We study integral quadratic forms

$$q : \mathbb{Z}^I \to \mathbb{Z}, v \mapsto q(v) = \sum_{i\in I} q_i v_i^2 + \tfrac{1}{2} \sum_{i\neq j\in I} q_{ij} v_i v_j$$

where I is a finite set and its cardinality will be called *the number of variables of* q. Often we will have $I = [n] = \{1, \dots, n\}$.

Further, q is a *unit form* (*semiunit form*) if $q_i = 1$ ($q_i \in \{0, 1\}$, respectively) for all $i \in I$. We say that q is *positive* (*non-negative*) if $q(v) > 0$ ($q(v) \geq 0$, respectively) for all $v \neq 0$.

A vector v is called a *q-root* if $q(v) = 1$. For instance, for a unit form the canonical base vectors e^i are roots. Given a finite tuple of q-roots $r = (r^j)_{j\in J}$ a new unit form $q_r : \mathbb{Z}^J \to \mathbb{Z}$ can be defined as in [3] by

$$q_r(y) = q(\sum_{j\in J} y_j r^j),$$

which we shall call the *q-root form induced by the tuple r*.

Two forms $q : \mathbb{Z}^I \to \mathbb{Z}$ and $q' : \mathbb{Z}^J \to \mathbb{Z}$ are called *equivalent* if they describe the same maps up to a change of basis, that is, if there exists a linear $\mathbb{Z}$-invertible transformation $T : \mathbb{Z}^I \to \mathbb{Z}^J$ such that $q = q'T$. It was shown in [1], see also 2.2, that the equivalence classes of non-negative unit forms are parametrized by two data: the *corank*, a natural number, and the *Dynkin*

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type, that is the disjoint union of some of the following Dynkin diagrams: $\mathbb{A}_n$ ($n \geq 1$), $\mathbb{D}_n$ ($n \geq 4$) or $\mathbb{E}_n$ ($n = 6, 7, 8$). In this work we show the following result.

Theorem A. *Two non-negative unit forms are of the same Dynkin type precisely when they are root-induced one from the other.*

In order to study more closely the relationship between p and p_r, we introduce some more notions. Given two forms $q : \mathbb{Z}^I \to \mathbb{Z}$ and $q' : \mathbb{Z}^J \to \mathbb{Z}$, we define a new form $q \oplus q' : \mathbb{Z}^{I \dot\cup J} \to \mathbb{Z}$ called the *direct sum* of q and q', by

$$q \oplus q'(x, y) = q(x) + q'(y).$$

A form q is called *connected* if $q = q' \oplus q''$ implies $q = q'$ or $q = q''$.

In Section 2, we will give precise conditions on the tuple of p-roots r for which $p > 0$ and p_r share the same Dynkin type. As a consequence, we get the following result for unit forms *without double edges*, that is, forms q whose non-square coefficients q_{ij} satisfy $|q_{ij}| < 2$.

Theorem B. *There are only finitely many unit forms without double edges of a given Dynkin type Δ. The number of variables of any such unit from is (strictly) bounded by the number of positive roots of* $\mathbf{qf}(\Delta)$.

We show in Section 4 that double edges of non-negative unit forms may be reduced in a straightforward way and such forms are thus of little interest from a combinatorial point of view.

The article is organized as follows: In Section 2, we explain the basic facts of root induction and recall some results about non-negative unit forms. In Section 3, we determine the equivalence classes of non-negative unit forms defined by root induction, in particular we prove Theorem A. In Section 4 we show Theorem B and in Section 5, we determine the order on the equivalence classes of non-negative unit forms defined by root-induction, which provides an interesting order on the Dynkin types.

2. Preparatory results

2.1. **Transitivity of root-induction.**

Lemma 2.1. *Root-induction is transitive.*

Proof. Suppose that $q : \mathbb{Z}^I \to \mathbb{Z}$ and $r = (r^j)_{j \in J}$ is a finite tuple of q-roots and $s = (s^k)_{k \in K}$ is a finite tuple of $q_r - roots$. We show that there exists a finite tuple $t = (t^h)_{h \in H}$ such that $(q_r)_s = q_t$. Now, $s^k = \sum_j s^k_j e^j \in \mathbb{Z}^J$ and hence we have for $x \in \mathbb{Z}^K$ that

$$\begin{aligned}(q_r)_s(x) =& q_r(\sum\nolimits_k x_k s^k)\\ =& q(\sum\nolimits_j \left(\sum\nolimits_k x_k s^k\right)_j r^j)\\ =& q(\sum\nolimits_k x_k \left(\sum\nolimits_j s^k_j r^j\right).\end{aligned}$$

So, by setting $t^k = \sum_j s^k_j r^j \in \mathbb{Z}^I$ we get $(q_r)_s = q_t$ provided the vectors t^k are q-roots. This is easily seen: $q(t^k) = q(\sum_j s^k_j r^j) = q_r(s^k) = 1$. □

2.2. **Equivalence classes of non-negative semiunit forms.** It is useful to associate to a semiunit form $q : \mathbb{Z}^I \to \mathbb{Z}$ a bigraph $\mathbf{bg}(q)$, having I as vertex set and $-q_{ij}$ *full* edges (respectively q_{ij} *broken* edges) between i and j if $q_{ij} < 0$ (respectively $q_{ij} \geq 0$) and one full loop at i if $q_i = 0$. It is clear that in this way we obtain a bigraph with at most one full loop in each point, no broken loops and no mixed edges between two points. Conversely to any bigraph Γ with such properties we may associate a semiunit form $\mathbf{qf}(\Gamma)$. Notice that q is connected if and only if $\mathbf{bg}(q)$ is so.

If $q : \mathbb{Z}^I \to \mathbb{Z}$ is a unit form then for any subset $J \subseteq I$ and $r = (e^j)_{j\in J}$ the q-root induced form $q' = q_r$ is called *restriction* of q and conversely q is called *extension* of q'. The fact will be denoted by $q' \subseteq q$ and happens if and only if $\mathbf{bg}(q')$ is a full subbigraph of $\mathbf{bg}(q)$. In this case we will have a canonical inclusion $\iota : \mathbb{Z}^J \to \mathbb{Z}^I$ and we will identify $v \in \mathbb{Z}^J$ with its image under ι if no confusion can arise.

Let $q : \mathbb{Z}^I \to \mathbb{Z}$ be a semiunit form. The free abelian subgroup rad $q = \{v \in \mathbb{Z}^I \mid q(v + w) = q(w), \forall w \in \mathbb{Z}^I\}$ is called the *radical* of q, its rank the *corank* of q. If q is non-negative then the radical coincides with the zero fibre of q. We denote by ζ the semiunit from $\mathbb{Z} \to \mathbb{Z}, v \mapsto 0$. It was shown in [1] that any connected non-negative unit form q is equivalent to $\mathbf{qf}(\Delta) \oplus \zeta^c$, where $\zeta^c = \zeta \oplus \dots \zeta$ (c copies), c is the corank of q and Δ a Dynkin diagram, uniquely determined by q, called the *Dynkin type* of q and denoted by $\mathrm{Dyn}(q) = \Delta$ in the sequel.

Lemma 2.2. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form and $r = (r^1, \dots, r^n)$ an n-tuple of q-roots which form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Then q and q_r are equivalent.*

Proof. This follows immediatly from the fact that $q_r = q \circ T$, where T is the linear $\mathbb{Z}$-invertible map defined by $T(e^i) = r^i$. □

2.3. **Omissible variables.** For a unit form $q : \mathbb{Z}^I \to \mathbb{Z}$, we call $i \in I$ an *omissible variable* or just *omissible* if there exists a vector $v \in \mathrm{rad}\, q$ such that $v_i = 1$. It was shown in [2], that a non-negative unit form with non-zero radical always admits an omissible variable and that the restriction of q to the remaining variables $I \setminus \{i\}$ has the same Dynkin type as q.

Furthermore, it has been shown in [2], that, if q is a non negative unit form there exists a positive restriction $p \subseteq q$ with $\mathrm{Dyn}(p) = \mathrm{Dyn}(q)$. In the sequel we will call p a *core* of q. Notice that cores are not uniquely determined in general, but just up to equivalence.

Suppose that $p \subseteq q$ is a core, then for any q-roots v there exists a unique p-root v_1 and a unique radical vector v_0 such that $v = v_0 + v_1$. This follows from the fact that cores may be obtained by iteratively deleting omissible variables. In the following we explain how q may be recovered from its core.

2.4. **One-point extensions.** Given q a unit form and a v a q-root we define

$$q[v] := q_{e(v)}, \quad \text{where} \quad e(v) = (e^1, \dots, e^n, v)$$

and call it the *one-point extension of q by v*. We can calculate the new coefficients explicitely:

$$q[v]_{i,n+1} = 2v_i + \sum_{j \neq i} q_{ij} v_j, \tag{1}$$

where, for convenience, we set $q_{ji} = q_{ij}$ for $i < j$.

Lemma 2.3. *If $q : \mathbb{Z}^{n-1} \to \mathbb{Z}$ is a connected non-negative unit form and v is a q-root, then $q[v]$ is again a connected non-negative unit form and* $\mathrm{Dyn}(q[v]) = \mathrm{Dyn}(q)$. *Moreover, the last variable of $q[v]$ is omissible and* $\mathrm{rad}\ q[v] = \mathrm{rad}\ q \oplus \mathbb{Z}(-v + e^n)$.

Proof. Clearly, $q[v]$ is a non-negative unit form. Further, if $n-1$ is the number of variables of q then $q[v](-v+e^n) = q(-v+v) = 0$ shows that the last variable is omissible. Consequently, $\mathrm{Dyn}(q[v]) = \mathrm{Dyn}(q)$ and $\tilde{v} = -v + e^n \in \mathrm{rad}\ q[v]$. If $w \in \mathrm{rad}\ q[v]$ then $w = (w - w_n\tilde{v}) + w_n\tilde{v} \in \mathrm{rad}\ q + \mathbb{Z}\tilde{v}$. Clearly $\mathrm{rad}\ q \cap \mathbb{Z}\tilde{v} = 0$. Suppose that $q[v]$ is not connected, that is $q[v]_{i\ n+1} = 0$ for all $i = 1, \dots, n$. Then $0 = q[v](-\iota v + e^{n+1}) = q(v) + q[v](e^{n+1}) = 2$, a contradiction. $\square$

For a unit form q and a tuple of q-roots $s = (s^1, \dots, s^t)$, we denote by $q[s] = q[s^1][s^2]\cdots[s^t]$ the *multi-point extension* of q.

Lemma 2.4. *An iterated one-point extension of a non-negative unit form is a multi-point extension, more precisely if q is a non-negative unit form and $q = q^0, q^1, \dots q^t$ is a sequence of unit forms such that $q^{i+1} = q^i[w^i]$ for some q^i-root w^i, then there exists a t-tuple $s = (s^1, \dots, s^t)$ such that $q^t = q[s]$.*

Proof. By induction on t. The cases $t = 0, 1$ are clear. For $t > 1$ assume $q : \mathbb{Z}^m \to \mathbb{Z}$ and let $s' = (s^1, \dots, s^{t-1})$ be such that $q^{t-1} = q[s']$.

Using Lemma 2.3, we see that $\tilde{s}^i = -s^i + e^{m+i}$ $(i = 1, \dots, t-1)$ is a radical vector of $q[s']$. Define $s^t = w^t - v \in \mathbb{Z}^m$, where $v = \sum_{i=1}^{t-1} w^t_{m+i}\tilde{s}^i \in \text{rad } q^{t-1}$. The assertion follows now from the fact that $p[w] = p[w+v]$ for any $v \in \text{rad } q$. Indeed, since $q(v) = 0$, the function q assumes a global minimum in v and therefore all partial derivatives $\frac{\partial q(x)}{\partial x_i} = 2x_i + \sum_{j \neq i} q_{ij}x_j$ vanish in $x = v$. Thus by the formula (1), we have $q[w+v]_{i,n+1} = q[w]_{i,n+1}$. $\square$

2.5. **Unit forms with the same Dynkin type.**

Proposition 2.5. *Let Δ be a Dynkin diagram and $p = \mathbf{qf}(\Delta) : \mathbb{Z}^m \to \mathbb{Z}$ be the associated quadratic form. Then the following are equivalent for $q : \mathbb{Z}^n \to \mathbb{Z}$.*

(i) *q is a connected, non-negative unit form with Dynkin type Δ.*
(ii) *There exists an n-tuple $r = (r^1, \dots, r^n)$ of p-roots such that $q = p_r$ and there exists a subsequence $1 \leq i_1 < \dots < i_m \leq n$ such that $r^{i_1}, \dots, r^{i_m}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^m$.*
(iii) *There exists a unit form $p' : \mathbb{Z}^m \to \mathbb{Z}$ which is equivalent to p and there exists a $(n-m)$-tuple $s = (s^1, \dots, s^{n-m})$ of p'-roots such that $q = p'[s]$ up to a permutation of the indices.*

Proof. **(i)⇒(iii).** By induction on the corank c of q. If $c = 0$, that is q is positive, then take $p' = p$ and for Π the identity matrix. In case $c > 0$, let i be an omissible variable of q and let q' be the restriction of q to the other variables. By induction hypothesis, we have $q' = p'[s']$ up to permutation of the indices for some tuple $s' = (s^1, \dots, s^{c-1})$ of p'-roots. Since i is omissible there exists a radical vector $v \in \text{rad } q$ with $v_i = 1$. Then $w = v - e^i$ is a q'-root and $q = q'[w]$ again up to permutation of the indices. Altogether, we have $q = p'[s'][w] = p'[s]$ by Lemma 2.4.

(iii)⇒(i). It follows by induction from Lemma 2.3, that p'_s is connected, non-negative and of Dynkin-type Δ. These properties clearly do not change under reordering of the variables.

(ii)⇒(iii). In order to keep notations simple, we assume first that $i_j = j$ for $j = 1, \dots, m$ and adjust to the general case in the end. For $i > m$, there exists integers s^i_j such that

$$r^i = s^i_1 r^1 + \dots s^i_m r^m,$$

defining hence vectors $s^i = (s^i_1, \dots, s^i_m) \in \mathbb{Z}^m$. Define $p' := p_{r'} : \mathbb{Z}^m \to \mathbb{Z}$, where $r' = \{r^1, \dots, r^m\}$. Then p' is equivalent to p, by Lemma 2.2.

Let $s = \{s^{m+1}, \dots, s^n\}$ and calculate

$$\begin{aligned} p'[s](y) =& p_{r'}\left(y_1e^1 + \dots + y_me^m + y_{m+1}s^{m+1} + \dots + y_ns^n\right) \\ =& p_{r'}\left(\sum\nolimits_{a=1}^m (y_a + y_{m+1}s_a^{m+1} + \dots + y_ns_a^n)e^a\right) \\ =& p\left(\sum\nolimits_{a=1}^m (y_a + y_{m+1}s_a^{m+1} + \dots + y_ns_a^n)r^a\right) \\ =& p\left(y_1r^1 + \dots + y_mr^m + \sum\nolimits_{j=m+1}^n y_j(s_1^jr^1 + \dots + s_m^jr^m)\right) \\ =& p\left(y_1r^1 + \dots + y_mr^m + y_{m+1}r^{m+1} + \dots + y_nr^n\right) \\ =& p_r(y) = q(y). \end{aligned}$$

Furthermore, the vectors s^j are p'-roots, since $p'(s^j) = p_{r'}(s_1^jr^1 + \dots + s_m^jr^m) = p(r^j) = 1$.

Now, if $r^{i_1}, \dots, r^{i_m}$ generate $\mathbb{Z}^m$, then let σ be a permutation of $\{1, \dots, n\}$ such that $\sigma(i_j) = j$ for $j = 1, \dots, m$. Hence $q = p'_s\Pi$, where Π is the permutation matrix associated with σ.

(iii)$\Rightarrow$ (ii) We have $q = p'_t$ where $t = (e^1, \dots, e^m, s^1, \dots, s^{n-m})$. Since p is equivalent to p' there exists a $\mathbb{Z}$-invertible T such that $p = p'T$. The vectors $r^i = T^{-1}t^i \in \mathbb{Z}^m$ are p-roots and $p_r = p'_t$. Clearly $r^1, \dots, r^m$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^m$. □

Proposition 2.6. *A non-negative unit form q is root induced from* $\mathbf{qf}(\mathrm{Dyn}(q))$.

Proof. Let q be a non-negative unit form and denote by p its core. Clearly p, as a restriction of q, has not more variables than q and it is a q-root induced form. Since p is equivalent to $p' = \mathbf{qf}(\mathrm{Dyn}(q))$, we have that $p' = p_s$ and hence by Proposition 2.5 (ii), we have $q = p'_r = p_{r'}$ by the transitivity of root-induction. □

We shall need the following result in the last section.

Corollary 2.7. *If $q : \mathbb{Z}^n \to \mathbb{Z}$ is a non-negative unit form of corank one with $v \in \mathbb{Z}^n$ such that* $\mathrm{rad}\, q = \mathbb{Z}v$*, then $|v_i| \le 6$ for any $i = 1, \dots, n$.*

Proof. Let $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ be a core of q and w a p-root such that $q = p[w]$. Then $v = \pm(w + e^n)$. The assertion follows now from the fact that $|w_i| \le 6$, see [3]. □

3. Root-equivalence of non-negative unit forms

We call two unit forms p and q *root-equivalent* if p is a q-root induced form and q is a p-root induced form. Notice that by Lemma 2.1 this is indeed an equivalence relation on the unit forms, and that this equivalence relation generates the usual equivalence under change of basis. We also recall from the proof of Proposition 2.6 that a non-negative unit form is root-induced from its core.

Proof of Theorem A. Suppose that $p : \mathbb{Z}^I \to \mathbb{Z}$ and $q : \mathbb{Z}^J \to \mathbb{Z}$ are root-equivalent, that is $p = q_r$ for some tuple of q-roots $r = (r^i)_{i \in I}$ and $q = p_s$ for some tuple of p-roots $s = (s^j)_{j \in J}$. By Proposition 2.5, p and its core are root-equivalent and so are q and its core. Hence we can suppose that p and q are positive.

Hence the vectors r^i $(i \in I)$ are linearly independent (otherwise there would exist $\sum_i x_i r^i = 0$ for some non-zero $x \in \mathbb{Z}^I$ and hence $p(x) = q_r(x) = q(0) = 0$ in contradiction with the positivity of p). This implies that $|I| \le |J|$. Similarly we have $|J| \le |I|$. This argument can be refined to hold even for each connected component. Indeed, if we assume that $p = p^1 \oplus \ldots \oplus p^m$ and $q = q^1 \oplus \ldots \oplus q^n$ where each p^a and each q^b is connected, then any p-root s^j is in fact a p^{a_j}-root for some $a_j = 1, \ldots m$ (extended by zero to the other components). If $s^{(h)} = (s^j)_{j \in J_h}$ denotes the tuple of all such roots where $a_j = h$, we have $q = p_s = (p^1)_{s^{(1)}} \oplus \ldots \oplus (p^m)_{s^{(m)}}$ and $J = \bigcup_{h=1}^m J_h$. Thus it follows from $|I| = |J|$ that $s^{(h)}$ is a $\mathbb{Q}$-basis of the domain of p^a and therefore $q = p_s$ has at least as many connected components as p. By interchanging the roles of p and q, we see that p and q have the same number of components and there is a permutation π such that p^h and $q^{\pi(h)}$ are root-equivalent. Hence we can suppose from the beginning that p and q are positive and connected.

Again, since the vectors r^i $(i \in I)$ are linearly independent, we get an injective linear map

$$\varphi : \mathbb{Z}^I \to \mathbb{Z}^J, x \mapsto \sum\nolimits_{i \in I} x_i r^i.$$

which induces an injective function on the roots $p^{-1}(1) \to q^{-1}(1)$ since $q(\varphi(x)) = q_r(x) = p(x)$. Hence p and q are positive connected unit forms with the same number of variables and the same number of roots. This implies that they are equivalent, see for example [2]. In particular, p and q must have the same Dynkin type.

Suppose now that $\mathrm{Dyn}(p) = \mathrm{Dyn}(q)$. Then, by Proposition 2.5, we see that p (respectively q) and its core p' (respectively q') are root-equivalent and have the same Dynkin type. Therefore p' and q' are positive unit forms with the same Dynkin type and therefore equivalent, in particular root-equivalent. □

4. Unit forms without double edges

Recall that a non-negative unit form q satisfies $|q_{ij}| \leq 2$ for any $i \neq j$. We say that a q has a *double edge* if there exists $i \neq j$ such that $|q_{ij}| = 2$. The following result shows that non-negative unit forms with double edges of are not very interesting from a combinatorial point of view.

Lemma 4.1. *Suppose that $q : \mathbb{Z}^I \to \mathbb{Z}$ is a non-negative unit form with a* double edge *$q_{ij} = 2\varepsilon$, for $\varepsilon = \pm 1$. Then $q_{ih} = \varepsilon q_{jh}$ for any $h \neq i, j$. Furthermore, if $q' = q^{I \setminus i}$ is the restriction to $I \setminus i$ then $q = q'[\varepsilon e^j]$.*

Proof. Since $e^j - \varepsilon e^i$ is a radical vector of q, we have $q(e^j + e^h) = q(\varepsilon e^i + e^h)$ and therefore $q_{jh} = q(e^j + e^h) - q(e^j) - q(e^h) = q(\varepsilon e^i + e^h) - 2 = \varepsilon q_{ih}$.

By (1), we have $(q'[\varepsilon e^j])_{ij} = 2\varepsilon$, which implies that q and $q'[\varepsilon e^j]$ are both non-negative unit forms whose restriction to $I \setminus i$ coincides and which have a double edge between the vertices i and j. The remaining coefficients are therefore completely determined and must coincide also. □

The following result implies immediatly Theorem B.

Proposition 4.2. *Let q be a non-negative unit form of Dynkin type Δ and $p = \mathbf{qf}(\Delta)$. Then q has no double edge if and only if there exists $\Sigma \subset p^{-1}(1)$ such that $\Sigma \cap -\Sigma = \varnothing$ and $q = p_\Sigma$.*

Proof. Suppose first that q has no double edge. By Theorem A, there exists a tuple $r = (r^i)_{i \in I}$ of p-roots such that $q = p_r$. It remains to show that $r^i \neq r^j$ and $r^i \neq -r^j$ for $i \neq j$. But, if $r^i = \varepsilon r^j$ for $\varepsilon = \pm 1$, then $\varepsilon (q_r)_{ij} = q_r(e^i + \varepsilon e^j) - q_r(e^i) - q_r(e^j) = q(2r^i) - 2 = 2$, which shows that q would have a double edge.

If, conversely there is a tuple $r = (r^i)_{i \in I}$ of p-roots satisfying $r^i \neq r^j$ and $r^i \neq -r^j$ for $i \neq j$, then $q = p_r$ can not have a double edge. Indeed, $r^i \pm r^j \neq 0$ implies $0 < q(r^i \pm r^j) = q_r(e^i \pm e^j) = 2 \pm (q_r)_{ij}$. □

5. Order of the Dynkin types

5.1. **Basic properties of the order.** By Proposition 2.5, the partial order on the equivalence classes defined by root induction yields a partial order of the Dynkin types. In this section we investigate this order in detail. We start with some simple observations.

Proposition 5.1. *Let Γ, Δ and Σ be Dynkin diagrams.*

(i) *Any predecessor of $\Delta \coprod \Sigma$ is of the form $\Delta' \coprod \Sigma'$ where $\Delta' \leq \Delta$ and $\Sigma' \leq \Sigma$, possibly one of Δ' or Σ' empty.*
(ii) *If $\Gamma \coprod \Sigma \leq \Delta \coprod \Sigma$, then $\Gamma \leq \Delta$.*

(iii) *If* Γ *is an immediate predecessor of* Δ*, then for any Dynkin type* Σ*, we have that* $\Gamma \coprod \Sigma$ *is an immediate predecessor of* $\Delta \coprod \Sigma$*.*

(iv) *If* Γ *is an immediate predecessor of* Δ*, then either* $\Delta = \Gamma \coprod \mathbb{A}_1$ *or* $\Gamma = \Gamma' \coprod \Gamma''$*,* $\Delta = \Gamma' \coprod \Delta''$ *where* Δ'' *is connected and* Γ'' *is an immediate predecessor of* Δ''*.*

Proof. **(i)** Suppose that $\Theta \leq \Delta \coprod \Sigma$. Then we can find positive unit forms $p : \mathbb{Z}^I \to \mathbb{Z}$ and $q : \mathbb{Z}^J \to \mathbb{Z}$ such that $\Delta = \mathrm{Dyn}(p)$ and $\Sigma = \mathrm{Dyn}(q)$ and a tuple $r = (r^h)_{h\in H}$ of $(p \oplus q)$-roots such that $(p \oplus q)_r = \Theta$. Let $s^h \in \mathbb{Z}^I$ and $t^h \in \mathbb{Z}^J$ be such that $r^h = s^h \oplus t^h$. Then either $s^h = 0$ and t^h is a q-root or $t^h = 0$ and s^h is a p-root. Set $H' = \{h \in H \mid t^h = 0\}$ and $H'' = H \setminus H'$. Further denote $s = (s^h)_{h\in H'}$ and $t = (t^h)_{h\in H''}$. Then we have $(p \oplus q)_r = p_s \oplus q_t$ and therefore $\Theta = \Delta' \coprod \Sigma'$ with $\Delta' = \mathrm{Dyn}(p_s)$ and $\Sigma' = \mathrm{Dyn}(q_t)$.

(ii) and **(iii)** follow directly from (i) (without using any other property of the order) in very similar way. We shall only show here part (iii) and leave (ii) to the interested reader. It is enough to consider the case where Σ is connected. Suppose $\Gamma \coprod \Sigma < \Theta < \Delta \coprod \Sigma$. We shall show that we can find a diagram Ξ such that $\Gamma < \Xi < \Delta$.

Let $\Theta = \Delta' \coprod \Sigma'$ with $\Delta' \leq \Delta$ and $\Sigma' \leq \Sigma$ (not both equalities) and $\Gamma \coprod \Sigma = \Delta'' \coprod \Sigma''$ with $\Delta'' \leq \Delta'$ and $\Sigma'' \leq \Sigma'$ (not both equalities).

The connected diagram Σ must be a component of Σ'' or of Δ''. In the first case, we obtain $\Sigma'' = \Sigma$ and therfore $\Gamma = \Delta'' < \Delta' < \Delta$ both inequalities being strict. In the second case, we have $\Delta'' = \Delta''' \coprod \Sigma$ and consequently

$$\Gamma = \Delta''' \coprod \Sigma'' \leq \Delta''' \coprod \Sigma' \leq \Delta''' \coprod \Sigma = \Delta'' \leq \Delta' \leq \Delta.$$

Now, observe that the first or the third inequality is strict and similarly the second or the fourth must be strict. In any case we find a Ξ with $\Gamma < \Xi < \Delta$.

(iv) This follows now directly from (i) and (iii). □

5.2. **Immediate predecessors of Dynkin diagrams.** Proposition 5.1 (iv) shows that in order to understand the partial order defined by root-induction, it is enough to describe the immediate predecessors of Dynkin diagrams. This is done here. Given a Dynkin diagram Δ we denote by $\tilde{\Delta}$ the corresponding extended Dynkin diagram, see for example [3].

Theorem 5.2. *If* Γ *is an immediate predecessor of a Dynkin diagram* Δ*, then* Γ *is a restriction (by one point) of* Δ *or of* $\tilde{\Delta}$*.*

Proof. Let $p : \mathbb{Z}^I \to \mathbb{Z}$ and $q : \mathbb{Z}^J \to \mathbb{Z}$ be a non-negative unit forms such that $\mathrm{Dyn}(p) = \Gamma$ and $\mathrm{Dyn}(q) = \Delta$. Let $r = (r^i)_{i\in I}$ be a tuple of q-roots such that $p = q_r$. By restricting, if necessary, to the core of p, we can assume that p is positive.

Since $p' = \mathbf{qf}(\Gamma) = p_s$ for some tuple s of p-roots we can, by the transitivity of root-induction, also assume that $p = \mathbf{qf}(\Gamma)$. We have to distinguish two cases: (i) $|\Gamma| < |\Delta|$ and (ii) $|\Gamma| = |\Delta|$. For the first case, we observe that the roots r^i are linearly independent for $i \in I$ since q_r is positive. Hence we can extend r to a $\mathbb{Q}$-basis r' of $\mathbb{Z}^J$. Therefore q_r is a proper restriction of $q_{r'}$. Since Γ is an immediate predecessor of Δ, we conclude from $\Gamma = \mathrm{Dyn}(q_r) < \mathrm{Dyn}(q_{r'}) \leq \Delta$, that $\mathrm{Dyn}(q_{r'}) = \Delta$ and therefore Γ is a restriction of Δ.

In the second case, where $|\Gamma| = |\Delta|$, we can assume that p is a restriction of q, since p is a restriction of $q[r]$ and $q[r]$ has the same Dynkin type than q. So suppose that $q : \mathbb{Z}^J \to \mathbb{Z}$ is a non-negative unit form with $\mathrm{Dyn}(q) = \Delta$ and $p = q^I$ is the restriction to $I \subset J$.

In the next step we show that it is enough to consider the case where $|J| = |I| + 1$. Write $J \setminus I = \{j_1, \dots, j_t\}$ and define $I_a = I \cup \{j_1, \dots, j_a\}$ for $a = 0, 1, \dots, t$. Then we have $p = q^{I_0}$ and $q = q^{I_t}$. We get the sequence

$$\Gamma = \mathrm{Dyn}(p) \leq \mathrm{Dyn}(q^{I_1}) \leq \dots \leq \mathrm{Dyn}(q^{I_{t-1}}) \leq \mathrm{Dyn}(q[r]) = \mathrm{Dyn}(q) = \Delta,$$

and by hypothesis there exists a unique index a such that $\mathrm{Dyn}(q^{I_{a-1}}) < \mathrm{Dyn}(q^{I_a})$. Hence $\mathrm{Dyn}(q) = \mathrm{Dyn}(q^{I_a})$ and we can at once assume that $a = t$. If $t > 1$, then we have for $q' = q^{I_1}$ that $\mathrm{Dyn}(q') = \mathrm{Dyn}(p)$ and therefore the vertex j_1 is ommisible for q'. That is, there exists a vector v' in the radical of q' such that $q'(v') = 0$ and $v'_{j_1} = 1$. Hence $q(v) = 0$, where v denotes the extension of v' by zero entries to the remaining vertices. Since q is non-negative this implies that v is a radical vector of q with $v_{j_1} = 1$. Thus j_1 is omissible for q and $\mathrm{Dyn}(q^{J \setminus j_1}) = \mathrm{Dyn}(q)$. Clearly p is a restriction of $q^{J \setminus j_1}$. Thus a reordering of the vertices in $J \setminus I$ shows that indeed we can assume that $|J| = |I| + 1$. Set $j = j_1 \in J \setminus I$.

Resuming the above, we have $I = J \setminus \{j\}$ and $p = q^I$ and $p = \mathbf{qf}(\Gamma)$. Since $|\Gamma| = |\Delta|$, we see that q has corank one, so q is equivalent to $\mathbf{qf}(\tilde{\Delta})$.

In the following, we show that there exists a linear $\mathbb{Z}$-invertible transformation $T : \mathbb{Z}^J \to \mathbb{Z}^J$ such that the bigraph of $q \circ T$ is a diagram (that is $(q \circ T)_{hi} \leq 0$ for any $h \neq i$) and $T(e^i) = e^i$ for any $i \in I$. This implies the result, since on one hand we have that the bigraph of $q \circ T$ is an extended Dynkin diagram, since q is non-negative, connected and of corank one. On the other hand p is a restriction of $q \circ T$.

Indeed, let v be the unique vector $v \in \mathbb{Z}^J$ such that $v_j > 0$ and $\mathrm{rad}\, q = \mathbb{Z}v$. Suppose there exists an index $i \neq j$ such that $q_{ij} > 0$. Now consider T_1 the linear function given by $T_1(e^s) = e^s$ for $s \neq j$ and $T_1(e^j) = e^j - e^i$. A simple calculation shows that $q^1 = q \circ T_1$ is again a unit form. Of course it is non-negative of corank one. The unique vector v^1 such that $v^1_j > 0$ and $\operatorname{rad} q^1 = \mathbb{Z}v^1$ is $T_1^{-1}v = v + v_j e^i$. In particular we have $v^1_h = v_h$ for any $h \neq i$ and $v^1_i > v_i$. We proceed with q^1 instead of q and obtain iteratively a sequence of equivalent unit forms $q^u = q \circ T_u$ for $u = 1, 2 \dots$ and where T_u is linear $\mathbb{Z}$-invertible and $T_u(e^i) = e^i$ for any $i \in I$. Parallel to this sequence we obtain a sequence of vectors $v^u \in \mathbb{Z}^J$ (each unique for q^u) with $\sum_{h \in I} v^{u-1}_h < \sum_{h \in I} v^u_h$. Since $\sum_{h \in I} v^u_h \leq 6 \cdot |I|$ by Corollary 2.7, this sequence must stop when we reach a form q^u without positive (off-diagonal) coefficients. This completes the proof of the statement. □

Dynkin diagram Δ	immediate predecessors Γ of Δ with $\|\Gamma\| < \|\Delta\|$	immediate predecessors Γ of Δ with $\|\Gamma\| = \|\Delta\|$
$\mathbb{A}_n$	$\mathbb{A}_{n-1}$ $\mathbb{A}_i \coprod \mathbb{A}_{n-i-1}$ $(1 \leq i \leq n-2)$	
$\mathbb{D}_4$	$\mathbb{A}_3$	$\mathbb{A}_1^4$
$\mathbb{D}_5$	$\mathbb{A}_4$ $\mathbb{D}_4$	$\mathbb{A}_1^2 \coprod \mathbb{A}_3$
$\mathbb{D}_6$	$\mathbb{A}_5$ $\mathbb{D}_5$	$\mathbb{A}_1^2 \coprod \mathbb{D}_4$ $\mathbb{A}_3^2$
$\mathbb{D}_n$ $(7 \leq n)$	$\mathbb{A}_{n-1}$ $\mathbb{D}_{n-1}$	$\mathbb{A}_1^2 \coprod \mathbb{D}_{n-2}$ $\mathbb{A}_3 \coprod \mathbb{D}_{n-3}$ $\mathbb{D}_i \coprod \mathbb{D}_{n-i}$ $(4 \leq i \leq n-4)$
$\mathbb{E}_6$	$\mathbb{D}_5$	$\mathbb{A}_1 \coprod \mathbb{A}_5$ $\mathbb{A}_2^3$
$\mathbb{E}_7$	$\mathbb{E}_6$	$\mathbb{A}_7$ $\mathbb{A}_1 \coprod \mathbb{D}_6$ $\mathbb{A}_2 \coprod \mathbb{A}_5$
$\mathbb{E}_8$		$\mathbb{A}_8$ $\mathbb{D}_8$ $\mathbb{A}_1 \coprod \mathbb{E}_7$ $\mathbb{A}_2 \coprod \mathbb{E}_6$ $\mathbb{A}_3 \coprod \mathbb{D}_5$ $\mathbb{A}_4^2$

Table 1: Immediate predecessors of Dynkin diagrams

Table 1 shows the *immediate* predecessors of all Dynkin diagrams, separated into two columns according to the two cases described in the Theorem above.

5.3. **Remark.** We note briefly the relationship of the above defined order with Lie theory. The semisimple Lie algebras (over $\mathbb{C}$) are up to isomorphism determined by the Dynkin types, and for each Dynkin type Δ, denote by $\mathfrak{g}(\Delta)$ some fixed representative. We recall that $\mathfrak{g}(\Delta)$ is graded by $\Phi \cup \{0\}$, where Φ is a root system. Now, $\Gamma \leq \Delta$ precisely when there is a injective homomorphism of graded Lie algebras $f : \mathfrak{g}(\Gamma) \to \mathfrak{g}(\Delta)$, in the sense that $f(\mathfrak{g}(\Gamma)_r) \subseteq \mathfrak{g}(\Delta)_{\hat{f}(r)}$ for some linear map $\hat{f}$.

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