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Generalized Serre relations for Lie algebras associated with positive unit forms

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Abstract

Every semisimple Lie algebra defines a root system on the dual space of a Cartan subalgebra and a Cartan matrix, which expresses the dual of the Killing form on a root base. Serre's Theorem [J.-P. Serre, Complex Semisimple Lie Algebras (G.A. Jones, Trans.), Springer-Verlag, New York, 1987] gives then a representation of the given Lie algebra in generators and relations in terms of the Cartan matrix.

In this work, we generalize Serre's Theorem to give an explicit representation in generators and relations for any simply laced semisimple Lie algebra in terms of a positive quasi-Cartan matrix. Such a quasi-Cartan matrix expresses the dual of the Killing form for a $\mathbb{Z}$-base of roots. Here, by a $\mathbb{Z}$-base of roots, we mean a set of linearly independent roots which generate all roots as linear combinations with integral coefficients.

MSC: 17B20

1. Introduction and main result

A square matrix with integer coefficients A is called a *quasi-Cartan matrix*, see [1], if it is symmetrizable (that is, there exists a diagonal matrix D with positive diagonal entries such that DA is symmetric) and $A_{ii} = 2$ for all i. A quasi-Cartan matrix is called a *Cartan matrix*, see [3], if it is positive definite, that is, all principal minors are positive, and $A_{ij} \leq 0$ for all $i \neq j$.

A *unit form* is a quadratic form $q : \mathbb{Z}^N \to \mathbb{Z}$, $q(x) = \sum_{i=1}^N x_i^2 + \sum_{i<j} q_{ij} x_i x_j$, with integer coefficients $q_{ij} \in \mathbb{Z}$. Any unit form $q : \mathbb{Z}^N \to \mathbb{Z}$ has an associated symmetric quasi-Cartan matrix $A = A(q)$ given by $A_{ij} = q(c_i + c_j) - q(c_i) - q(c_j)$, where $c_1, \ldots, c_n$ is the canonical basis of $\mathbb{Z}^N$. To simplify notation, set $q_{ij} = q_{ji}$ for $i > j$. It will be convenient to switch sometimes to a more graphical language and associate with any unit form $q : \mathbb{Z}^N \to \mathbb{Z}$ a bigraph $B(q)$ with vertices $1, \ldots, N$ and edges as follows. Two different vertices i and j are joined by $|q_{ij}|$ *full* edges if $q_{ij} < 0$ and by q_{ij} *broken* edges if $q_{ij} > 0$. If $A(q)$ is a Cartan matrix then $B(q)$ is a graph (there are

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no broken edges) Δ, which by the Cartan–Killing classification is a disjoint union of Dynkin diagrams $\mathbb{A}_m$ ($m \geq 1$), $\mathbb{D}_m$ ($m \geq 4$) and $\mathbb{E}_m$ ($m = 6, 7, 8$). In that case, we write $q = q_\Delta$ and call Δ the *Dynkin type* of q (or of A).

Given a unit form q, set $A = A(q)$ and let $\mathfrak{g}_4(q)$ be the Lie algebra defined by the generators e_i, e_{-i}, h_i ($1 \leq i \leq N$) and the relations

$R_1(q)$ $[h_i, h_j] = 0$ for all i, j,
$R_2(q)$ $[h_i, e_{\varepsilon j}] = -\varepsilon A_{ij} e_{\varepsilon j}$, for all i, j and $\varepsilon \in \{1, -1\}$,
$R_3(q)$ $[e_{\varepsilon i}, e_{-\varepsilon i}] = \varepsilon h_i$ for all i and $\varepsilon \in \{1, -1\}$,
$R_4(q)$ $(\mathbf{ad}\, e_{\varepsilon i})^{1+n}(e_{\delta j}) = 0$, where $n = \max\{0, -\varepsilon\delta A_{ij}\}$, for $\varepsilon, \delta \in \{1, -1\}$ and $1 \leq i, j \leq N$.

Theorem 1.1 (*[5]*). *If q is positive definite unit form such that its quasi-Cartan matrix is a Cartan matrix then $\mathfrak{g}_4(q)$ is a semisimple (and finite dimensional) Lie algebra.*

Notice that in general, when A is not necessarily a Cartan matrix, the relations $R_4(q)$ are a subset of the relations

$R_\infty(q)$ $[e_{\varepsilon_1 i_1}, \ldots, e_{\epsilon_t i_t}] = 0$ if $q(\sum_{j=1}^t \varepsilon_j c_{i_j}) > 1$ and $\varepsilon_j \in \{1, -1\}$,

where we used multibrackets, defined inductively by

$$[x_1, x_2, \ldots, x_t] = [x_1, [x_2, \ldots, x_t]].$$

Let $\mathfrak{g}_\infty(q)$ be the Lie algebra defined by the generators e_i, e_{-i}, h_i ($1 \leq i \leq N$) and by the relations $R_1(q)$, $R_2(q)$, $R_3(q)$ and $R_\infty(q)$. We recall that any positive definite unit form has a unique associated *Dynkin type* Δ such that q is *equivalent* to q_Δ, that is $q = q_\Delta \circ T$ for some $\mathbb{Z}$-invertible integer matrix T; see also the proof of Proposition 2.1. The fact that two unit forms q and q' are equivalent will be denoted by $q \sim q'$.

Theorem 1.2 (*[2]*). *If q is positive definite of Dynkin type Δ then $\mathfrak{g}_\infty(q)$ is isomorphic to $\mathfrak{g}_4(q_\Delta)$.*

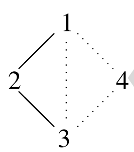
Notice that the set of relations $R_\infty(q)$ is infinite and although it has been shown in [2, Proposition 6.6] that there exists a finite subset S of $R_\infty(q)$ which suffices to define $\mathfrak{g}_\infty(q)$, it remains unsatisfactory, because S is usually very large and its definition depends heavily on a factorization of the matrix T, for which $q = q_\Delta \circ T$, into certain elementary transformations.

The main result of this paper is to give an explicit and finite set of relations for which the defined Lie algebra is isomorphic to $\mathfrak{g}_4(q_\Delta)$. This set includes $R_1(q)$, $R_2(q)$, $R_3(q)$ and $R_4(q)$ as above and additionally some relations $R_5(q)$ depending on the set of *chordless cycles* in q: a *chordless cycle* is a tuple of indices $(i_1, \ldots, i_t)$ such that $q_{i_a i_b} \neq 0$ if and only if $a - b \equiv \pm 1 \mod t$. Clearly the chordless cycles in q correspond to the chordless cycles in $B(q)$ (in graph theory a *cycle* is a closed path $(i_1, \ldots, i_t)$ and a *chord* is an edge $\{i_a, i_b\}$ for which $a - b \not\equiv \pm 1 \mod t$). The importance of chordless cycles for the classification of cluster algebras [1] of finite type should be mentioned at this point.

Let

$R_5(q)$ $[e_{\varepsilon_1 i_1}, \ldots, e_{\varepsilon_t i_t}] = 0$, where $(i_1, \ldots, i_t)$ is a chordless cycle in q and $\varepsilon_t \in \{1, -1\}$, $\varepsilon_l = -q_{i_l, i_{l+1}} \varepsilon_{l+1}$ for $1 \leq l \leq t - 1$.

Example 1.3. Let $q : \mathbb{Z}^4 \to \mathbb{Z}$, $q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 + x_1x_4 - x_2x_3 + x_3x_4$. The bigraph $B(q)$ of q looks as follows:



There are two subsets, namely $\{1, 2, 3\}$ and $\{1, 3, 4\}$, each of which gives rise to six chordless cycles, which in turn define two relations each. Therefore $R_5(q)$ consists of the following relations:

$$[e_1, e_2, e_3] = 0, \quad [e_2, e_3, e_{-1}] = 0, \quad [e_3, e_{-1}, e_{-2}] = 0,$$
$$[e_1, e_{-3}, e_{-2}] = 0, \quad [e_2, e_1, e_{-3}] = 0, \quad [e_3, e_2, e_1] = 0,$$
$$[e_{-1}, e_{-2}, e_{-3}] = 0, \quad [e_{-2}, e_{-3}, e_1] = 0, \quad [e_{-3}, e_1, e_2] = 0,$$

$$
\begin{array}{lll}
[e_{-1}, e_3, e_2] = 0, & [e_{-2}, e_{-1}, e_3] = 0, & [e_{-3}, e_{-2}, e_{-1}] = 0,\\
[e_1, e_{-4}, e_3] = 0, & [e_4, e_{-3}, e_1] = 0, & [e_3, e_{-1}, e_4] = 0,\\
[e_1, e_{-3}, e_4] = 0, & [e_4, e_{-1}, e_3] = 0, & [e_3, e_{-4}, e_1] = 0,\\
[e_{-1}, e_4, e_{-3}] = 0, & [e_{-4}, e_3, e_{-1}] = 0, & [e_{-3}, e_1, e_{-4}] = 0,\\
[e_{-1}, e_3, e_{-4}] = 0, & [e_{-4}, e_1, e_{-3}] = 0, & [e_{-3}, e_4, e_{-1}] = 0.
\end{array}
$$

Let $\mathfrak{g}_5(q)$ be the Lie algebra defined by the generators e_i, e_{-i}, h_i $(1 \le i \le N)$ and by the relations $R_1(q)$, $R_2(q)$, $R_3(q)$, $R_4(q)$ and $R_5(q)$. Observe that all these sets are finite and given in a very combinatorial way.

The following is the main result of this paper.

Theorem 1.4. *Let q and q' be positive definite unit forms. Then*

(i) *$q \sim q'$ if and only if $\mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q')$,*
(ii) *$\mathfrak{g}_5(q) \simeq \mathfrak{g}_4(q_\Delta)$, where Δ is the Dynkin type of q.*

Remark 1.5. By Theorem 1.2, it follows that $\mathfrak{g}_5(q) \simeq \mathfrak{g}_\infty(q)$ for any positive definite unit form q.

Remark 1.6. In order to prove Theorem 1.4 it is sufficient to show the implication $q \sim q' \Rightarrow \mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q')$ for any two positive definite unit forms.

Proof. Indeed, the rest then follows easily: to see (ii), let $q = q_\Delta$. Then $\mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q_\Delta)$, but $\mathfrak{g}_5(q_\Delta) = \mathfrak{g}_4(q_\Delta)$, since there is no chordless cycle for q_Δ and consequently $R_5(q)$ is empty.

Now, suppose that $\mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q')$. If Δ is the Dynkin type of q and Δ' is the Dynkin type of q' then it follows from (ii) that $\mathfrak{g}_4(q_\Delta) \simeq \mathfrak{g}_4(q_{\Delta'})$ and therefore $\Delta = \Delta'$; see [5]. Consequently $q \sim q_\Delta = q_{\Delta'} \sim q'$. ■

If q is a positive definite unit form then $q(c_i \pm c_j) \ge 1$ and hence $|q_{ij}| \le 1$. If a unit form q satisfies $(-q_{i_1 i_2})(-q_{i_2 i_3}) \dots (-q_{i_{t-1} i_t})(-q_{i_t i_1}) = -1$ for any chordless cycle $(i_1, \dots, i_t)$ in q, we say that q *satisfies the cycle condition*. For instance, if q is a positive definite unit form then q satisfies the cycle condition.

Remark 1.7. If q is positive definite then the set of relations $R_5(q)$ is a subset of $R_\infty(q)$.

Proof. Let $\gamma = (i_1, \dots, i_t)$ be a chordless cycle and $\varepsilon_1, \dots, \varepsilon_t$ be defined as in $R_5(q)$. Then $\varepsilon_1 = \prod_{l=1}^{t-1}(-q_{i_l i_{l+1}})\varepsilon_t = q_{i_t i_1}\varepsilon_t$ since q satisfies the cycle condition and hence $q(\sum_{l=1}^{t} \varepsilon_l c_{i_l}) = t + q_{i_1 i_t}\varepsilon_1\varepsilon_t + \sum_{l=1}^{t-1} q_{i_l i_{l+1}}\varepsilon_l\varepsilon_{l+1} = t + \varepsilon_1^2 + \sum_{l=1}^{t-1} q_{i_l i_{l+1}}(-q_{i_l i_{l+1}}\varepsilon_{l+1})\varepsilon_{l+1} = t + 1 - (t-1) = 2$. ■

The article is structured as an iterated reduction to more and more special situations, where the main steps and the implications are as follows

Theorem 1.4 $\Leftarrow$ Proposition 2.8 $\Leftarrow$ Lemma 3.2 $\Leftarrow$ Lemma 4.3.

We show each of the above implications in a separate section and use the last two sections to prove Lemma 4.3 itself.

2. Reduction to elementary transformations

Given a unit form $q : \mathbb{Z}^N \to \mathbb{Z}$, we define a linear transformation I_r, given by $I_r(c_i) = c_i$ for any $i \neq r$ and $I_r(c_r) = -c_r$. We say that q' is obtained from q by a sign inversion if $q' = q \circ I_r$ for some r.

Let $q : \mathbb{Z}^N \to \mathbb{Z}$ be a unit form. For any $r \neq s$ and $\sigma \in \{1, -1\}$ we define a linear transformation T_{sr}^{σ} by $T_{sr}^{\sigma} c_i = c_i$ for any $i \neq r$ and $T_{sr}^{\sigma} c_r = c_r + \sigma c_s$.

Note that if $\sigma := -q_{rs} \in \{1, -1\}$, the form $q' = q \circ T_{sr}^{\sigma}$ is again a unit form and T_{sr}^{σ} is called a *Gabrielov transformation* for q. Let q and q' be two unit forms. If $q' = q \circ P$ where P is a permutation matrix or $q' = q \circ T_{sr}^{-q_{rs}}$ or $q' = q \circ I_r$ then we write $q \sim_G q'$. Closing by transitivity, we get an equivalence relation $\sim_G$ on the unit forms and call two unit forms in the same equivalence class *Gabrielov-equivalent*, or just G-*equivalent*, for short.

Although the proof of the following result is well known to specialists it is rather hard to find an explicit reference for it and therefore we include a proof of it.

Proposition 2.1. *If q and q' are positive definite unit forms then $q \sim q'$ if and only if q and q' have the same Dynkin type if and only if $q \sim_G q'$.*

Proof. For any positive definite unit form q there exists a Dynkin type Δ such that $q \sim_G q_\Delta$; see for example [4, Theorem 6.2]. This Dynkin type is uniquely determined by q: Define the graph $G(q)$ to have as vertices the elements of $q^{-1}(1)$ and edges $\{x, y\}$ for every two vertices x, y for which $q(x - y) \in \{0, 1\}$. Observe that $G(q) \simeq G(q')$ if q and q' are equivalent. Hence the components of $G(q)$ correspond to the components of Δ and for each component G of $G(q)$, the number of vertices of G together with the number of indices i such that $\pm c_i \in G$ determine the corresponding Dynkin diagram uniquely.

Hence, if Δ and Δ' denote the Dynkin types of the positive definite unit forms q and q' respectively, we have $q \sim_G q_\Delta$ and $q' \sim_G q_{\Delta'}$. Hence $q \sim q' \Rightarrow \Delta = \Delta' \Rightarrow q \sim_G q_\Delta = q_{\Delta'} \sim_G q' \Rightarrow q \sim q'$. ■

In order to prove Theorem 1.4 it is enough to show the following result.

Theorem 2.2. *If q and q' are positive definite unit forms then $q \sim_G q'$ if and only if $\mathfrak{g}_5(q)$ is isomorphic to $\mathfrak{g}_5(q')$.*

Remark 2.3. Again, we only have to show the implication $q \sim_G q' \Rightarrow \mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q')$.

Proof. Indeed assume this is shown; then by Proposition 2.1 we get $q \sim q' \Rightarrow q \sim_G q' \Rightarrow \mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q')$ and hence by Remark 1.6, Theorem 1.4 holds. Therefore $\mathfrak{g}_5(q) \simeq \mathfrak{g}_5(q') \Rightarrow q \sim q' \Rightarrow q \sim_G q'$, the latter again by Proposition 2.1. ■

The following result is useful for reducing to special situations.

Lemma 2.4. *Let q be a unit form, $s \neq r$, $\sigma = -q_{sr}$ and $1 \le i_1, \dots, i_t \le n$. Then $q \circ T_{sr}^{\sigma} \circ I_{i_1} \circ \cdots \circ I_{i_t} = q \circ I_{i_1} \circ \cdots \circ I_{i_t} \circ T_{sr}^{\sigma'}$, where $\sigma' = (-1)^{\varepsilon}\sigma$ and ε is the number of indices a with $1 \le a \le t$ and $i_a \in \{r, s\}$.*

Proof. This follows directly from the fact that $T_{sr}^{\sigma} \circ I_i = I_i \circ T_{sr}^{-\sigma}$ for $i = r, s$ and $T_{sr}^{\sigma} \circ I_i = I_i \circ T_{sr}^{\sigma}$ else. ■

Remark 2.5. In order to show Theorem 2.2, it is enough to consider the two cases $q' = q \circ I_r$ and $q' = q \circ T_{sr}^{+1}$ if $q_{rs} = -1$.

Proof. By the definition of G-equivalence, it is enough to consider the cases $q' = q \circ P$ where P is a permutation matrix, $q' = q \circ I_r$ and $q' = q \circ T_{sr}^{-q_{rs}}$ if $q_{rs} \in \{1, -1\}$. However, the first case, that is, $q' = q \circ P$, is straightforward and if $q_{sr} = 1$ then $q \circ T_{sr}^{-1} = q \circ I_r \circ T_{sr}^{+1} \circ I_r$, by Lemma 2.4. ■

Proposition 2.6. *Let q be a positive unit form. If $q' = q \circ I_r$ then the Lie algebras $\mathfrak{g}_5(q)$ and $\mathfrak{g}_5(q')$ are isomorphic.*

Proof. Denote by e_i, e_{-i}, h_i and by e'_i, e'_{-i}, h'_i the generators of $\mathfrak{g}_5(q)$ and $\mathfrak{g}_5(q')$ respectively. Let $A = A(q)$ and $A' = A(q')$. Further we set

$$\widetilde{e}_{\varepsilon i} = \begin{cases} e_{\varepsilon i}, & \text{if } i \neq r \\ e_{-\varepsilon r}, & \text{if } i = r \end{cases} \quad \text{and} \quad \widetilde{h}_i = \begin{cases} h_i, & \text{if } i \neq r \\ -h_r, & \text{if } i = r. \end{cases} \tag{2.1}$$

The verification that these elements satisfy the relations $\mathrm{R}_1(q')$ to $\mathrm{R}_4(q')$ is easy (it was also stated in [2]) and we leave it to the interested reader.

To verify $\mathrm{R}_5(q')$ let $\gamma = (i_1, \dots, i_t)$ be any chordless cycle for q'. Observe first that γ is also a chordless cycle for q. If $r \notin \{i_1, \dots, i_t\}$ the verification is straightforward. If $r \in \{i_1, \dots, i_t\}$, say $r = i_a$, then let $\varepsilon'_t \in \{1, -1\}$ and inductively $\varepsilon'_l = -q'_{i_l i_{l+1}} \varepsilon'_{l+1}$ for $1 \le l \le t - 1$. Then

$$\begin{aligned}[\widetilde{e}_{\varepsilon'_1 i_1}, \dots, \widetilde{e}_{\varepsilon'_{a-1} i_{a-1}}, \widetilde{e}_{\varepsilon'_a i_a}, \widetilde{e}_{\varepsilon'_{a+1} i_{a+1}}, \dots, \widetilde{e}_{\varepsilon'_t i_t}] &= [e_{\varepsilon'_1 i_1}, \dots, e_{\varepsilon'_{a-1} i_{a-1}}, e_{-\varepsilon'_a i_a}, e_{\varepsilon'_{a+1} i_{a+1}}, \dots, e_{\varepsilon_t i_t}] \\ &= [e_{\varepsilon_1 i_1}, \dots, e_{\varepsilon_{a-1} i_{a-1}}, e_{\varepsilon_a i_a}, e_{\varepsilon_{a+1} i_{a+1}}, \dots, e_{\varepsilon_t i_t}] =: x,\end{aligned}$$

where $\varepsilon_a = -\varepsilon'_a$ and $\varepsilon_j = \varepsilon'_j$ for all $j \neq a$. In order to see that $x = 0$, we will use $\mathrm{R}_5(q)$. To do so we have to ensure that $\varepsilon_l = -q_{i_l i_{l+1}} \varepsilon_{l+1}$, for $1 \le l \le t - 1$. This follows easily from $q'_{i_l i_{l+1}} = q_{i_l i_{l+1}}$ for $l \neq a, a-1$ and from $q'_{i_l i_{l+1}} = -q_{i_l i_{l+1}}$ in the case where $l = a$ or $l = a - 1$.

The Lie subalgebra of $\mathfrak{g}_5(q)$ generated by $\widetilde{e}_{\varepsilon i}$ and $\widetilde{h}_i$ for ($i = 1, \dots, N$, and $\varepsilon \in \{1, -1\}$) is clearly $\mathfrak{g}_5(q)$. Therefore we obtain a homomorphism of Lie algebras $\varphi : \mathfrak{g}_5(q') \to \mathfrak{g}_5(q)$ which maps $e'_{\varepsilon i}$ to $\widetilde{e}_{\varepsilon i}$ and h'_i to $\widetilde{h}_i$. Similarly, we obtain a homomorphism of Lie algebras $\psi : \mathfrak{g}_5(q) \to \mathfrak{g}_5(q')$. It is straightforward to check that φ and ψ are inverse to each other. This finishes the proof. ■

Corollary 2.7. *There is an automorphism Φ of $\mathfrak{g}_5(q)$ which sends $e_{\varepsilon i}$ to $e_{-\varepsilon i}$ and h_i to $-h_i$ for any $1 \le i \le N$.*

Proof. Denote by $\varphi_r : \mathfrak{g}_5(q) \to \mathfrak{g}_5(q)$ the isomorphism which maps $e_{\varepsilon i}$ to $\widetilde{e}_{\varepsilon i}$ and h_i to $\widetilde{h}_i$, where $\widetilde{e}_{\varepsilon i}$ and $\widetilde{h}_i$ are defined as in (2.1). Then the isomorphism $\Phi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_N$ maps $e_{\varepsilon i}$ to $e_{-\varepsilon i}$ and h_i to $-h_i$ for any $1 \le i \le N$, whereas the effect on the unit form is the identity since $q(-x) = q(x)$. ∎

The remainder of the article is divided into several steps in order to prove the following result, which by Remark 2.5 is enough to show Theorem 2.2 (and hence Theorem 1.4).

Proposition 2.8. *Assume that q is a positive definite unit form, $q_{rs} = -1$ and $q' = q \circ T_{sr}^{+1}$. Then $\mathfrak{g}_5(q)$ and $\mathfrak{g}_5(q')$ are isomorphic Lie algebras.*

3. Reduction to special chordless cycles

In this section, we will reduce the proof of Proposition 2.8 to the verification that certain monomials are zero in $\mathfrak{g}_5(q)$.

Assume that q is positive definite, $q_{rs} = -1$ and $q' = q \circ T_{sr}^{+1}$. Once again, denote the generators of $\mathfrak{g}_5(q)$ by e_i, e_{-i}, h_i and the generators of $\mathfrak{g}_5(q')$ by e'_i, e'_{-i}, h'_i.

Then define the following elements in $\mathfrak{g}_5(q)$:

$$\widetilde{e}_{\varepsilon i} = \begin{cases} [e_{\varepsilon r}, e_{\varepsilon s}], & \text{if } i = r \\ e_{\varepsilon i}, & \text{if } i \ne r, \end{cases} \qquad \widetilde{h}_i = \begin{cases} h_r + h_s, & \text{if } i = r \\ h_i, & \text{if } i \ne r. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $q_{sr} = -1$ and $q' = q \circ T_{sr}^{+1}$. The elements $\widetilde{e}_{\varepsilon i}$ and $\widetilde{h}_i$ satisfy the relations* $\mathrm{R}_1(q')$, $\mathrm{R}_2(q')$, $\mathrm{R}_3(q')$ *and* $\mathrm{R}_4(q')$.

Proof. It has been shown in [2] that these elements satisfy the relations $\mathrm{R}_1(q')$, $\mathrm{R}_2(q')$ and $\mathrm{R}_3(q')$ and it only remains to show $\mathrm{R}_4(q')$, that is we have to show that $(\mathbf{ad}\, \widetilde{e}_{\varepsilon i})^{1+m'}(\widetilde{e}_{\delta j}) = 0$ where $m' = \max\{0, -\varepsilon\delta A'_{ij}\}$, for any $\varepsilon, \delta \in \{1, -1\}$ and any $i, j = 1, \dots, n$. For $i = j$ the case $\varepsilon = \delta$ is obvious and the case $\varepsilon = -\delta$ easy: we have $m' = 2$ and $(\mathbf{ad}\, \widetilde{e}_{\varepsilon i})(\widetilde{e}_{-\varepsilon i}) = \varepsilon \widetilde{h}_i$ by $\mathrm{R}_3(q')$. Therefore $(\mathbf{ad}\, \widetilde{e}_{\varepsilon i})^2(\widetilde{e}_{-\varepsilon i})$ is a multiple of $\widetilde{e}_{\varepsilon i}$ by $\mathrm{R}_2(q')$ which implies $(\mathbf{ad}\, \widetilde{e}_{\varepsilon i})^3(\widetilde{e}_{-\varepsilon i}) = 0$. For $i \ne j$ we distinguish several cases.

Case $i \ne r$, $j \ne r$: Then $\widetilde{e}_{\varepsilon i} = e_{\varepsilon i}$, $\widetilde{e}_{\delta j} = e_{\delta j}$ and $A'_{ij} = A_{ij}$, $m' = \max\{0, -\varepsilon\delta A_{ij}\}$ and therefore $(\mathbf{ad}\, \widetilde{e}_{\varepsilon i})^{1+m'}(\widetilde{e}_{\delta j}) = (\mathbf{ad}\, e_{\varepsilon i})^{1+m'}(e_{\delta j}) = 0$ by $\mathrm{R}_4(q)$.

Case $i = r$, $j \ne r, s$: Then $A'_{rj} = A_{rj} + A_{sj}$. Suppose first that $m' = 0$. Then either $A'_{rj} = 0$ or $\varepsilon\delta A'_{rj} > 0$. In the first case, we have $A_{rj} = -A_{sj}$ so either both are zero (and then $[e_{\varepsilon r}, e_{\delta j}] = 0$, $[e_{\varepsilon s}, e_{\delta j}] = 0$ and consequently $(\mathbf{ad}\, \widetilde{e}_{\varepsilon r})(\widetilde{e}_{\delta j}) = 0$ by $\mathrm{R}_4(q)$) or both are non-zero and then (j, r, s) is a chordless cycle in q and we get $(\mathbf{ad}\, \widetilde{e}_{\varepsilon r})^{1+m'}(\widetilde{e}_{\delta j}) = [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}] = -[e_{\delta j}, [e_{\varepsilon r}, e_{\varepsilon s}]] = 0$ by $\mathrm{R}_5(q)$. In the second case, where $\varepsilon\delta A'_{rj} > 0$, we have $\varepsilon\delta A_{rj} \ge 0$ and $\varepsilon\delta A_{sj} \ge 0$ since $|A_{ij}| \le 1$ for all $i \ne j$. Therefore, we have $[e_{\varepsilon r}, e_{\delta j}] = 0 = [e_{\varepsilon s}, e_{\delta j}]$ by $\mathrm{R}_4(q)$. Thus, using the Jacobi identity, we see that $0 = [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}] = (\mathbf{ad}\, \widetilde{e}_{\varepsilon r})^{1+m'}(\widetilde{e}_{\delta j})$.

Suppose now that $m' > 0$; then $m' = 1$ and $A'_{rj} = A_{rj} + A_{sj} = -\varepsilon\delta$ and either $A_{sj} = 0$ or $A_{rj} = 0$. In the case where $A_{sj} = 0$, we have

$$[e_{\varepsilon s}, e_{\delta j}] = 0, \tag{3.2}$$

$$[e_{\varepsilon r}, e_{\varepsilon r}, e_{\delta j}] = 0, \tag{3.3}$$

$$[e_{\varepsilon s}, e_{\varepsilon s}, e_{\varepsilon r}] = 0. \tag{3.4}$$

Using the general fact (valid in any Lie algebra $\mathfrak{g}$ and for any $x, y, z \in \mathfrak{g}$) that

$$\begin{aligned} [x, y] = 0 \;\Rightarrow\; & [x, y, z] = [y, x, z], \qquad [x, z, y] = [y, z, x], \quad \text{and} \\ & [[x, z], y] = [[y, z], x], \qquad [[z, x], y] = [[z, y], x], \end{aligned} \tag{3.5}$$

we get

$$[e_{\varepsilon s}, [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}]] = 0. \tag{3.6}$$

Using (3.5) repeatedly and the above equations as indicated, we can calculate

$$\begin{aligned}
(\mathbf{ad}\,\widetilde{e}_{\varepsilon r})^2(\widetilde{e}_{\delta j}) &= [[e_{\varepsilon r}, e_{\varepsilon s}], [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}]] \\
&\overset{(3.2)}{=} [[e_{\varepsilon r}, e_{\varepsilon s}], [[e_{\varepsilon r}, e_{\delta j}], e_{\varepsilon s}]] \\
&\overset{(3.4)}{=} [e_{\varepsilon s}, [[e_{\varepsilon r}, e_{\delta j}], [e_{\varepsilon r}, e_{\varepsilon s}]]] \\
&\overset{(3.3)}{=} [e_{\varepsilon s}, [e_{\varepsilon r}, [[e_{\varepsilon r}, e_{\delta j}], e_{\varepsilon s}]]] \\
&\overset{(3.2)}{=} [e_{\varepsilon s}, [e_{\varepsilon r}, [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}]]] \\
&\overset{(3.6)}{=} [[[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}], [e_{\varepsilon r}, e_{\varepsilon s}]] \\
&= -[[e_{\varepsilon r}, e_{\varepsilon s}], [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}]] \\
&= -(\mathbf{ad}\,\widetilde{e}_{\varepsilon r})^2(\widetilde{e}_{\delta j}).
\end{aligned}$$

Hence $(\mathbf{ad}\,\widetilde{e}_{\varepsilon r})^2(\widetilde{e}_{\delta j}) = 0$. In the second case, where $A_{rj} = 0$, notice that $[[e_{\varepsilon r}, e_{\varepsilon s}], [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\delta j}]] = [[e_{\varepsilon s}, e_{\varepsilon r}], [[e_{\varepsilon s}, e_{\varepsilon r}], e_{\delta j}]]$ and proceed similarly, interchanging the roles of r and s.

Case $i = r$, $j = s$: Observe that $A'_{rs} = -A_{rs} = 1$ and therefore $(\mathbf{ad}\,\widetilde{e}_{\varepsilon r})^2(\widetilde{e}_{-\varepsilon s}) = [\widetilde{e}_{\varepsilon r}, [e_{\varepsilon r}, e_{\varepsilon s}], e_{-\varepsilon s}] \overset{R_4(q)}{=} [\widetilde{e}_{\varepsilon r}, [e_{-\varepsilon s}, e_{\varepsilon s}], e_{\varepsilon r}] \overset{R_3(q)}{=} [\widetilde{e}_{\varepsilon r}, (-\varepsilon)h_s, e_{\varepsilon r}] \overset{R_2(q)}{=} [\widetilde{e}_{\varepsilon r}, \varepsilon^2 A_{rs} e_{\varepsilon r}] = A_{rs}[[e_{\varepsilon r}, e_{\varepsilon s}], e_{\varepsilon r}] = -A_{rs}(\mathbf{ad}\, e_{\varepsilon r})^2(e_{\varepsilon s}) = 0$, which is zero by $R_4(q)$ since $1+\max\{0, -\varepsilon^2 A_{rs}\} = 2$. On the other hand, we have $(\mathbf{ad}\,\widetilde{e}_{\varepsilon r})(\widetilde{e}_{\varepsilon s}) = [[e_{\varepsilon r}, e_{\varepsilon s}], e_{\varepsilon s}] = (\mathbf{ad}\, e_{\varepsilon s})^2(e_{\varepsilon r}) = 0$ again by $R_4(q)$.

Case $i \neq r, s$, $j = r$: If $A_{ir} = 0 = A_{is}$ then it is straightforward to check that $(\mathbf{ad}\,\widetilde{e}_{\varepsilon i})(\widetilde{e}_{\delta r}) = 0$. Otherwise we must have $A_{ir} \neq A_{is}$ (since $|A'_{ir}| = |A_{ir} + A_{is}| < 2$) and therefore we have $[e_{\varepsilon i}, e_{\delta r}] = 0$ or $[e_{\varepsilon i}, e_{\delta s}] = 0$ by $R_4(q)$. Assume $[e_{\varepsilon i}, e_{\delta r}] = 0$ (the case where $[e_{\varepsilon i}, e_{\delta s}] = 0$ is completely similar). Then we obtain from (3.5) that $(\mathbf{ad}\, e_{\varepsilon i})([e_{\delta r}, (\mathbf{ad}\, e_{\varepsilon i})^a(e_{\delta s})]) = [e_{\delta r}, (\mathbf{ad}\, e_{\varepsilon i})^{a+1}(e_{\delta s})]$ for any $a \geq 0$ and therefore get $(\mathbf{ad}\,\widetilde{e}_{\varepsilon i})^{1+m'}(\widetilde{e}_{\delta r}) = (\mathbf{ad}\, e_{\varepsilon i})^{1+m'}([e_{\delta r}, e_{\delta s}]) = [e_{\delta r}, (\mathbf{ad}\, e_{\varepsilon i})^{1+m'}(e_{\delta s})]$, which is zero if $m' \geq \max\{0, -\varepsilon\delta A_{is}\}$, in particular if $A_{is} = 0$ or $\varepsilon\delta A_{is} > 0$. So it remains to consider the case where $\varepsilon\delta A_{is} = -1$ and $m' = 0$, that is $\varepsilon\delta A'_{ir} \geq 0$. Since $0 \leq \varepsilon\delta A'_{ir} = \varepsilon\delta A_{ir} + \varepsilon\delta A_{is} = \varepsilon\delta A_{ir} - 1$ we must have $\varepsilon\delta A_{ir} = 1$, but then (i, r, s) is a chordless cycle in q and therefore $(\mathbf{ad}\,\widetilde{e}_{\varepsilon i})^{1+m'}(\widetilde{e}_{\delta r}) = [e_{\varepsilon i}, e_{\delta r}, e_{\delta s}]$ is zero by $R_5(q)$.

Case $i = s$, $j = r$: Since $A'_{rs} = -A_{rs} = 1$, we get for $\varepsilon = -\delta$ that $m' = 1$ and calculate $(\mathbf{ad}\,\widetilde{e}_{\varepsilon s})^2(\widetilde{e}_{-\varepsilon r}) = [e_{\varepsilon s}, e_{\varepsilon s}, e_{-\varepsilon s}, e_{-\varepsilon r}] = [e_{\varepsilon s}, e_{-\varepsilon r}, e_{-\varepsilon s}, e_{\varepsilon s}]$, where the last equation is due to (3.5). Hence $(\mathbf{ad}\,\widetilde{e}_{\varepsilon s})^2(\widetilde{e}_{-\varepsilon r}) = [e_{\varepsilon s}, e_{-\varepsilon r}, -\varepsilon h_s] = [e_{\varepsilon s}, A_{rs} e_{-\varepsilon r}] = 0$ by $R_4(q)$. For $\varepsilon = \delta$, we have $m' = 0$ and get $(\mathbf{ad}\,\widetilde{e}_{\varepsilon s})(\widetilde{e}_{\varepsilon r}) = -(\mathbf{ad}\, e_{\varepsilon s})^2(e_{\varepsilon r}) = 0$ by $R_4(q)$ since $\max\{0, -\varepsilon^2 A_{sr}\} = 1$. ■

Lemma 3.2. *With the above notation, for any chordless cycle* $\gamma = (r, i_1, i_2, \ldots, i_t)$ *in* q' *with* $q'_{ri_1} = -1$, $q'_{i_1 i_2} = -1, \ldots, q'_{i_{t-1} i_t} = -1$ *and* $q'_{i_t r} = 1$, *the following monomials are zero in* $\mathfrak{g}_5(q)$*:*

$$\begin{aligned}
E^{+}_{\gamma,0} &:= [\widetilde{e}_r, \widetilde{e}_{i_1}, \widetilde{e}_{i_2}, \ldots, \widetilde{e}_{i_t}] \\
E^{+}_{\gamma,u} &:= [\widetilde{e}_{-i_u}, \widetilde{e}_{-i_{u+1}}, \ldots, \widetilde{e}_{-i_t}, \widetilde{e}_r, \widetilde{e}_{i_1}, \ldots, \widetilde{e}_{i_{u-1}}] \quad (1 \leq u \leq t) \\
E^{-}_{\gamma,0} &:= [\widetilde{e}_{-r}, \widetilde{e}_{i_t}, \widetilde{e}_{i_{t-1}}, \ldots, \widetilde{e}_{i_1}] \\
E^{-}_{\gamma,u} &:= [\widetilde{e}_{-i_u}, \widetilde{e}_{-i_{u-1}}, \ldots, \widetilde{e}_{-i_1}, \widetilde{e}_{-r}, \widetilde{e}_{i_t}, \ldots, \widetilde{e}_{i_{u+1}}] \quad (1 \leq u \leq t).
\end{aligned} \tag{3.7}$$

We will prove Lemma 3.2 in the next section; however we show at once its importance, as it allows us to prove Proposition 2.8:

Proof of Proposition 2.8. By Corollary 2.7 it follows from Lemma 3.2 that for $\varepsilon \in \{1, -1\}$, we have

$$\begin{aligned}
E^{+,\varepsilon}_{\gamma,0} &:= [\widetilde{e}_{\varepsilon r}, \widetilde{e}_{\varepsilon i_1}, \widetilde{e}_{\varepsilon i_2}, \ldots, \widetilde{e}_{\varepsilon i_t}] = 0, \\
E^{+,\varepsilon}_{\gamma,u} &:= [\widetilde{e}_{-\varepsilon i_u}, \widetilde{e}_{-\varepsilon i_{u+1}}, \ldots, \widetilde{e}_{-\varepsilon i_t}, \widetilde{e}_{\varepsilon r}, \widetilde{e}_{\varepsilon i_1}, \ldots, \widetilde{e}_{\varepsilon i_{u-1}}] = 0, \\
E^{-,\varepsilon}_{\gamma,0} &:= [\widetilde{e}_{-\varepsilon r}, \widetilde{e}_{\varepsilon i_t}, \widetilde{e}_{\varepsilon i_{t-1}}, \ldots, \widetilde{e}_{\varepsilon i_1}] = 0, \\
E^{-,\varepsilon}_{\gamma,u} &:= [\widetilde{e}_{-\varepsilon i_u}, \widetilde{e}_{-\varepsilon i_{u-1}}, \ldots, \widetilde{e}_{-\varepsilon i_1}, \widetilde{e}_{-\varepsilon r}, \widetilde{e}_{\varepsilon i_t}, \ldots, \widetilde{e}_{\varepsilon i_{u+1}}] = 0.
\end{aligned}$$

We now will show that the elements $\widetilde{e}_{\varepsilon i}$ and $\widetilde{h}_i$ defined in (3.1) also satisfy the relations $\mathrm{R}_5(q')$. Therefore, we have to show that for any chordless cycle $\gamma = (j_1, \ldots, j_t)$ in q' the element

$$F_\gamma^{\varepsilon_t} = [\widetilde{e}_{\varepsilon_1 j_1}, \widetilde{e}_{\varepsilon_2 j_2}, \ldots, \widetilde{e}_{\varepsilon_t j_t}] \in \mathfrak{g}_5(q)$$

is zero, where $\varepsilon_t \in \{1, -1\}$ and $\varepsilon_l = -q'_{j_l j_{l+1}} \varepsilon_{l+1}$ for $l = t-1, t-2, \ldots, 1$.

Now, if r does not belong to γ then $\widetilde{e}_{\varepsilon_a j_a} = e_{\varepsilon_a j_a}$ for $1 \le a \le t$ and γ is also a chordless cycle in q. Consequently $F_{\gamma}^{\varepsilon_t} = 0$ by $\mathrm{R}_5(q)$.

Thus, it remains to consider the case where r belongs to γ, say $j_a = r$. If $s \neq j_{a-1}$ (where $j_0 := j_t$), we can assume (using Lemma 2.4 and Proposition 2.6) that $q'_{r j_{a+1}} = -1, q'_{j_{a+1} j_{a+2}} = -1, \ldots, q'_{j_t j_1} = -1, \ldots, q'_{j_{a-2} j_{a-1}} = -1$ and then $q'_{j_{a-1} r} = 1$ since q' satisfies the cycle condition. Then let $\gamma' = (r, j_{a+1}, \ldots, j_t, j_1, \ldots, j_{a-1})$ and observe that we can apply Lemma 3.2 and conclude that $F_\gamma^{\varepsilon_t} = E^{+,\varepsilon_t}_{\gamma', t-a+1} = 0$.

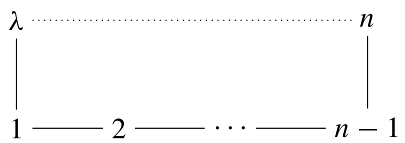
If $s = j_{a-1}$ then $q'_{j_{a-1} r} = 1$ (since $q_{sr} = -1$) and we can assume (again using Lemma 2.4 and Proposition 2.6) that $q'_{r j_{a+1}} = -1, q'_{j_{a+1} j_{a+2}} = -1, \ldots, q'_{j_t j_1} = -1, \ldots, q'_{j_{a-2} j_{a-1}} = -1$. Then let $\gamma' = (r, j_{a+1}, \ldots, j_t, j_1, \ldots, j_{a-1})$ and observe that again we can apply Lemma 3.2 and conclude that $F_\gamma^{\varepsilon_t} = E^{+,\varepsilon_t}_{\gamma', t-a+1} = 0$. This shows that the elements $\widetilde{e}_{\varepsilon i}$ and $\widetilde{h}_i$ satisfy the relations $\mathrm{R}_5(q')$ and therefore it follows now from Lemma 3.1 that there exists a homomorphism $\varphi : \mathfrak{g}_5(q') \to \mathfrak{g}_5(q)$ which maps $e'_{\varepsilon i}$ to $\widetilde{e}_{\varepsilon i}$ and h'_i to $\widetilde{h}_i$.

Similarly, there exists a homomorphism $\psi : \mathfrak{g}_5(q) \to \mathfrak{g}_5(q')$ and it is easily verified that they are inverse to each other; see [2] for details. ∎

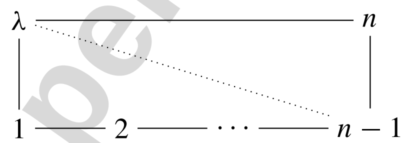
4. Reduction to special monomials

In this section, we describe how chordless cycles can occur in $q' = q \circ T_{sr}$ which are not necessarily chordless cycles in q. Therefore we switch to the more combinatorial language of bigraphs.

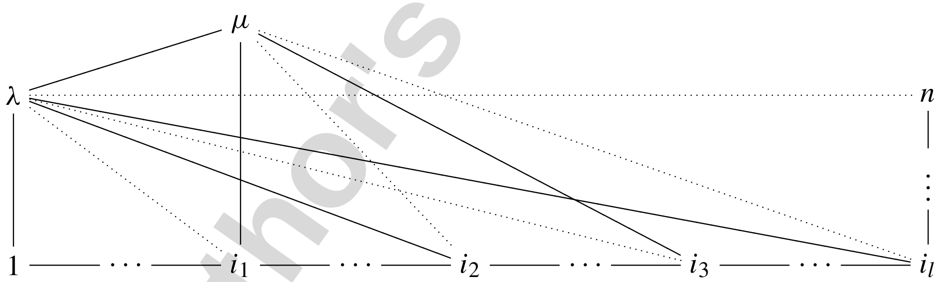
We introduce four types of bigraphs, where the first is just a chordless cycle $\mathbf{c}_n^0(\lambda) = (\lambda, 1, \ldots, n)$, where $\{\lambda, n\}$ is the only broken edge:



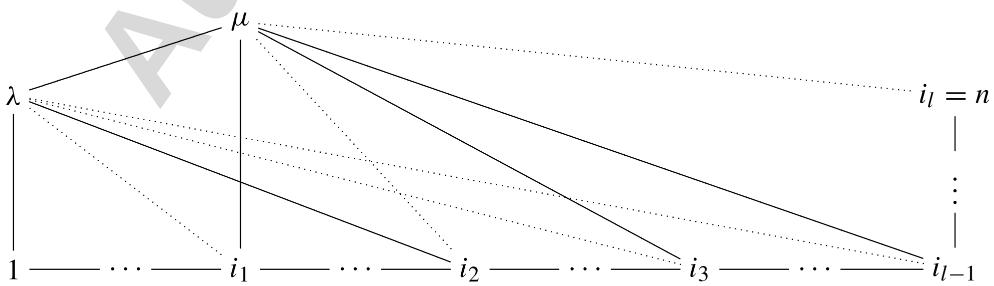
$\mathbf{c}_n^0(\lambda)$



$\mathbf{g}_n^0(\lambda)$



$\mathbf{g}_n^1(\lambda, \mu; i_1, \ldots, i_l)$, $\quad l$ is even,



$\mathbf{g}_n^2(\lambda, \mu; i_1, \ldots, i_l)$, $\quad l > 0$ is even.

By definition, the edges of $\mathbf{g}_n^0(\lambda)$ are precisely those lying in the two chordless cycles $(\lambda, 1, 2, \dots, n-1)$ and $(\lambda, n-1, n)$ where $\{\lambda, n-1\}$ is the only broken edge.

For any even l and any sequence of indices $1 < i_1 < i_2 < \cdots < i_l \le n$ we define two more bigraphs as follows. The bigraph $\mathbf{g}_n^1(\lambda, \mu, i_1, \dots, i_l)$ is obtained by adding a vertex μ to the chordless cycle $\mathbf{c}_n^0(\lambda)$, a full edge $\{\lambda, \mu\}$, and adding for each odd $a = 1, 3, \dots, l-1$ a broken edge $\{\lambda, i_a\}$ and a full edge $\{\mu, i_a\}$ and adding for each even $a = 2, 4, \dots, l$ a full edge $\{\lambda, i_a\}$ and a broken edge $\{\mu, i_a\}$. The definition of $\mathbf{g}_n^2(\lambda, \mu, i_1, \dots, i_l)$ is quite similar: start with a chordless cycle $(\mu, \lambda, 1, \dots, n)$ where $\{n, \mu\}$ is the only broken edge and add more edges as follows: for each odd $a = 1, 3, \dots, l-1$ a broken edge $\{\lambda, i_a\}$ and a full edge $\{\mu, i_a\}$ and for each even $a = 2, 4, \dots, l-2$ a full edge $\{\lambda, i_a\}$ and a broken edge $\{\mu, i_a\}$.

Recall that it only remains to prove Lemma 3.2 and that we therefore can restrict our attention to the case where $\gamma \subseteq B(q')$ is a chordless cycle with $\gamma = \mathbf{c}_n^0(r)$. A subbigraph Γ' of a bigraph Γ is called *induced* if any edge $\{x, y\}$ (full or broken) of Γ with vertices x, y of Γ' is contained in Γ'.

Lemma 4.1. *Suppose that q is positive definite, $q_{rs} = -1$ and $q' = q \circ T_{rs}^{+1}$. Furthermore let $\gamma = \mathbf{c}_n^0(r)$ be a chordless cycle in $B(q')$ and denote by Γ the induced subbigraph of $B(q)$ given by the vertices $r, s, 1, \dots, n$.*

If $s \in \gamma$, then $\Gamma = \mathbf{g}_n^0(r)$ and $n = s$. If $s \notin \gamma$, then let $i_1, \dots, i_l$ (with $1 \le i_1 < i_2 < \cdots < i_l \le n$) be the vertices i_a such that $q_{si_a} \neq 0$. Then either $l = 0$ and $\Gamma = \mathbf{g}_n^1(r, s;)$ or $l > 0$ is even and the following hold:

(a) *If $i_1 \neq 1$ and $i_l \neq n$, then $\Gamma = \mathbf{g}_n^1(r, s; i_1, \dots, i_l)$.*
(b) *If $i_1 \neq 1$ and $i_l = n$, then $\Gamma = \mathbf{g}_n^2(r, s; i_1, \dots, i_l)$.*
(c) *If $i_1 = 1$ and $i_l \neq n$, then $\Gamma = \mathbf{g}_n^2(s, r; i_2, \dots, i_l, n)$.*
(d) *If $i_1 = 1$ and $i_l = n$, then $\Gamma = \mathbf{g}_n^1(s, r; i_2, \dots, i_{l-1})$.*

Proof. If s is also a vertex of γ then, since $q'_{rs} = 1$, we must have $s = n$ and then the induced subbigraph Γ of $B(q)$ has the form $\Gamma = \mathbf{g}_n^0(r)$.

Now, assume that s does not lie in γ. If $l = 0$, that is $q_{si_a} = 0$ for $1 \le a \le n$, then $\Gamma = \mathbf{g}_n^1(r, s;)$ and we are done. So, suppose now that $l > 0$.

Since $(s, r, 1, \dots, i_1)$ is a chordless cycle in $B(q')$, we must have $q'_{si_1} = -1$ by the cycle condition. Inductively for $1 < a \le l$ we have that $(s, i_{a-1}, i_{a-1}+1, \dots, i_a)$ is a chordless cycle in $B(q')$ and we infer again by the cycle condition that $q_{s,i_a} = (-1)^a$.

If $i_l = n$ then (r, i_l, s) is a chordless cycle in $B(q')$ and since $q'_{s,r} = 1 = q'_{r,i_l}$ we get that $(-1)^l = q'_{s,i_l} = 1$. Therefore l is even.

If $i_l \neq n$ then $(r, s, i_l, \dots, n)$ is a chordless cycle in $B(q')$. Again by the cycle condition, we get $(-1)^l = q_{si_l} = 1$ and infer again that l is even.

The rest of the verification is now straightforward using that $q_{ri} = q'_{ri} - q'_{si}$ for any $i \neq r, s$. ■

The following simple result will help to reduce our calculations by half.

Lemma 4.2. *If $A = [A_1, A_2, \dots, A_{n+1}]$ is a monomial which satisfies $[A_i, A_j] = 0$ whenever $|i - j| \neq 1$ then*

$$A^{\leftarrow} := [A_{n+1}, A_n, \dots, A_2, A_1] = (-1)^n [A_1, A_2, \dots, A_{n+1}]. \tag{4.1}$$

Proof. This is easily seen by induction and (3.5). ■

Now, we formulate a result which only involves knowledge about $B(q)$ and monomials in $\mathfrak{g}_5(q)$, but which will imply the Lemma 3.2, as shown below.

Lemma 4.3. *Suppose that q is positive definite and that Γ is an induced subbigraph of $B(q)$ which is of the form $\mathbf{g}_n^0(\lambda)$, $\mathbf{g}_n^1(\lambda, \mu; i_1, \dots, i_l)$ (for some even l) or $\mathbf{g}_n^2(\lambda, \mu; i_1, \dots, i_l)$ (for some even $l > 0$).*

Then the following monomials are zero in $\mathfrak{g}_5(q)$ (where in the case where $\Gamma = \mathbf{g}_n^0(\lambda)$ we assume that $\mu = n$):

$$F_{n,u}(\lambda, \mu) := [e_u, e_{u-1}, \dots, e_1, [e_\lambda, e_\mu], e_{-n}, \dots, e_{-(u+1)}], \tag{4.2}$$

for $u = 0, 1, \dots, n$, where $F_{n,0}(\lambda, \mu) = [[e_\lambda, e_\mu], e_{-n}, e_{-(n-1)}, \dots, e_{-1}]$.

Proof of Lemma 3.2. Define the following monomials in $\mathfrak{g}_5(q)$:

$$\begin{aligned} G^+_{n,0}(\lambda,\mu) &:= [[e_\lambda, e_\mu], e_1, \ldots, e_n], \\ G^+_{n,u}(\lambda,\mu) &:= [e_{-u}, e_{-(u+1)}, \ldots, e_{-n}, [e_\lambda, e_\mu], e_1, \ldots, e_{u-1}], \\ G^-_{n,0}(\lambda,\mu) &:= [[e_{-\lambda}, e_{-\mu}], e_n, e_{n-1}, \ldots, e_1], \\ G^-_{n,u}(\lambda,\mu) &:= [e_{-u}, e_{-(u-1)}, \ldots, e_{-1}, [e_{-\lambda}, e_{-\mu}], e_n, \ldots, e_{u+1}]. \end{aligned} \tag{4.3}$$

Suppose that $\gamma = \mathbf{c}^0_n(r)$ is a chordless cycle in $B(q')$ and denote by Γ the induced subbigraph of $B(q)$ given by the vertices $r, s, 1, \ldots, n$. If $s \in \gamma$, we have by Lemma 4.1 that $\Gamma = \mathbf{g}^0_n(r)$ and the monomials from (3.7) translate directly into the monomials of (4.3) by $E^\varepsilon_{\gamma,v} = G^\varepsilon_{n,v}(r, n)$.

If $s \notin \gamma$, we define $i_1, \ldots, i_l$ as in Lemma 4.1. Assume first $l = 0$. Then $\Gamma = \mathbf{g}^1_n(r, s;)$ and $E^\varepsilon_{\gamma,v} = G^\varepsilon_{n,v}(r, s)$. In the case where $l > 0$, by Lemma 4.1, l is even and there are four cases to be considered: in the two cases (a) and (b) we have $E^\varepsilon_{\gamma,v} = G^\varepsilon_{n,v}(r, s)$ and in the remaining two cases (c) and (d) we have $E^\varepsilon_{\gamma,v} = G^\varepsilon_{n,v}(s, r) = -G^\varepsilon_{n,v}(r, s)$.

So, in order to prove Lemma 3.2, it remains to show that the monomials $G^\varepsilon_{n,v}(r, s)$ are zero. Since Γ is an induced subbigraph of $B(q)$ which is of the form $\mathbf{g}^0_n(\lambda)$, $\mathbf{g}^1_n(\lambda, \mu; i_1, \ldots, i_l)$ or $\mathbf{g}^2_n(\lambda, \mu; i_1, \ldots, i_l)$, we can apply Lemma 4.3.

Thus $G^-_{n,v}(\lambda, \mu) = \Phi(F_{n,v}(\lambda, \mu)) = 0$ for $1 \le v \le n$ and $G^-_{n,0}(\lambda, \mu) = \Phi(F_{n,0}(\lambda, \mu)) = 0$, where Φ is the automorphism of Corollary 2.7. Observe that $G^\varepsilon_{n,v}(\lambda, \mu)$ is a monomial which satisfies the hypothesis of Lemma 4.2. Hence for $1 \le u \le n$, we get $G^+_{n,u}(\lambda, \mu) = \pm\Phi(G^-_{n,u-1}(\lambda, \mu)^{\leftarrow}) = 0$ and $G^+_{n,0}(\lambda, \mu) = \pm\Phi(G^-_{n,n}(\lambda, \mu)^{\leftarrow}) = 0$. Therefore the result. ■

5. Preparatory results on monomials

In this section we shall provide some necessary tools for handling complicated monomials.

5.1. Monomials in the free magma

Let $X = \{e_i, e_{-i}, h_i \mid 1 \le i \le N\}$ and denote by $\mathcal{M}(X)$ the free magma on X; see [3]. The binary operation in the magma is denoted by parentheses: (A, B). For $n > 2$ and $A_1, \ldots, A_n \in \mathcal{M}(X)$, we define inductively

$$(A_1, \ldots, A_n) := (A_1, (A_2, \ldots, A_n)).$$

Often, the elements that we consider are of the form $A = (A_1, \ldots, A_n)$ with $A_1, \ldots, A_n \in X$; we shall write then $A_i \in A$.

Let $\pi : \mathcal{M}(X) \to \mathfrak{g}_5(q)$ be the projection defined by $\pi(A) = A$ if $A \in X$ and $\pi((A_1, A_2)) = [\pi(A_1), \pi(A_2)]$. It is easy to see that for $A_1, \ldots, A_n, B \in \mathcal{M}(X)$ we have

$$[\pi(B), \pi(A_i)] = 0, \quad \text{for } i = 1, \ldots, n \ \Rightarrow [\pi(B), \pi((A_1, \ldots, A_n))] = 0. \tag{5.1}$$

Now we define a new binary operation $\bullet : \mathcal{M}(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$A \bullet B = \begin{cases} (A, B), & \text{if } A \in X; \\ (A_1, A_2 \bullet B), & \text{if } A = (A_1, A_2). \end{cases}$$

It is easy to see that this new operation is associative. As a consequence of (3.5), we have for $a_1, \ldots, a_t \in X$ and $A = (a_1, \ldots, a_t)$, $B, C \in \mathcal{M}(X)$ that

$$\forall i, \quad [a_i, \pi(B)] = 0 \ \Rightarrow \ \pi(A \bullet (B, C)) = [\pi(B), \pi(A \bullet C)], \tag{5.2}$$

$$\forall i, \quad [a_i, \pi(C)] = 0 \ \Rightarrow \ \pi(A \bullet (B, C)) = [\pi(A \bullet B), \pi(C)]. \tag{5.3}$$

Lemma 5.1. *Let $a_1, \ldots, a_t \in X$ and $A = (a_1, \ldots, a_t)$, $B \in \mathcal{M}(X)$ be such that $[a_i, \pi(B)] = 0$ for all $i < t$. Then $\pi(A \bullet B) = [\pi(A), \pi(B)]$.*

Proof. For $t = 1$ this follows directly from the definitions. For $t > 1$ we have $\pi(A \bullet B) = [a_1, \pi(A' \bullet B)]$, where $A' = (a_2, \ldots, a_t)$. By induction and (3.5), we get $\pi(A \bullet B) = [a_1, \pi(A'), \pi(B)] = [\pi(B), \pi(A'), a_1]$ and the result follows now by antisymmetry. ■

Lemma 5.2. *Let $a_1, \dots, a_s, b_1, \dots, b_t \in X$ be such that $[a_i, b_j] = 0$ and $A = (a_1, \dots, a_s)$, $B = (b_1, \dots, b_t)$, $C \in \mathcal{M}(X)$. Then $\pi(A \bullet B \bullet C) = \pi(B \bullet A \bullet C)$.*

Proof. By definition we have $\pi(A \bullet B \bullet C) = [a_1, \dots, a_s, b_1, \dots, b_t, \pi(C)] = [a_1, \dots, a_{s-1}, b_1, \dots, b_t, a_s, \pi(C)]$ by applying t times (3.5) and then inductively $[a_1, \dots, a_{s-1}, b_1, \dots, b_t, a_s, \pi(C)] = [b_1, \dots, b_t, a_1, \dots, a_s, \pi(C)] = \pi(B \bullet A \bullet C)$. ■

5.2. Full chains

A tuple of vertices $(p_1, \dots, p_n)$ of a bigraph Γ is a *chain* if there exists an edge between the vertices p_i and p_j if only if $|i - j| = 1$. The chain is said to be *full* if all edges are full. If $\Delta = (p_1, \dots, p_n)$ is a chain in Γ and a is a vertex not belonging to Δ then we say that $i \in \Delta$ is *linked* to a if $q_{ai} \neq 0$ and we denote by $L_a(\Delta)$ the set of vertices of Δ which are linked to a.

The following lemma shows how the monomial along a full chain can be broken down to subchains.

Lemma 5.3. *Let q be a positive definite unit form and $\Delta = (1, \dots, u)$ be a full chain in $B(q)$. Then for any $1 < i_1 < i_2 < \cdots < i_l = u$ we have that*

$$\pi(D_l \bullet \cdots \bullet D_1) = [\pi(D_l), \dots, \pi(D_1)],$$

where $D_m = (e_{\varepsilon i_m}, \dots, e_{\varepsilon(i_{m-1}+1)})$ for $1 \leq m \leq l$ and $i_0 = 0$.

Proof. For $l = 2$ we have $D_1 = (e_{\varepsilon i_1}, \dots, e_{\varepsilon 1})$ and $D_2 = (e_{\varepsilon i_2}, \dots, e_{\varepsilon(i_1+1)})$ and $[e_{\varepsilon j}, \pi(D_1)] = 0$ for all $j > i_1 + 1$, and hence by Lemma 5.1, we get $\pi(D_2 \bullet D_1) = [\pi(D_2), \pi(D_1)]$. The general case follows by induction. ■

Lemma 5.4. *Let q be a positive definite unit form, let $\Delta = (1, \dots, u)$ be a full chain in $B(q)$ and let a be a vertex not belonging to Δ. Then there is an even number of vertices of Δ which are linked to a if and only if $\sum_{j=1}^{u} q_{aj} = 0$ and in that case*

(a) $[e_{\sigma a}, e_{\varepsilon u}, \dots, e_{\varepsilon 1}] = 0$ and
(b) $[e_{\varepsilon 1}, \dots, e_{\varepsilon u}, e_{\sigma a}] = 0$,

for any $\varepsilon, \sigma \in \{1, -1\}$.

Proof. Since q is a positive definite unit form, we have $-1 \leq q_{ij} \leq 1$. Let $L_a(\Delta) = \{i_1, \dots, i_l\}$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq u$. Since q satisfies the cycle condition, we have $q_{ai_s} = -q_{ai_{s+1}}$ for $1 \leq s < l$. Therefore we have that l is even if and only if $\sum_{j=1}^{u} q_{aj} = 0$.

Now suppose that l is even. Then it is possible to divide the full chain Δ into k subchains $\Delta_m = (v_m, \dots, w_m)$ for $1 \leq m \leq k$, with $v_1 = 1$, $w_k = u$, $v_m < w_m$ and $w_m + 1 = v_{m+1}$ $(1 \leq m < k)$ such that for each subchain Δ_m either no vertex is linked to a or $L_a(\Delta_m) = \{v_m, w_m\}$. By Lemma 5.3, we have $[e_{\sigma a}, e_{\varepsilon u}, \dots, e_{\varepsilon 1}] = [e_{\sigma a}, \pi(D_k), \dots, \pi(D_1)]$, where $D_m = (e_{\varepsilon w_m}, \dots, e_{\varepsilon v_m})$ for $1 \leq m \leq k$.

By (5.1) it is enough to show that $[e_{\sigma a}, \pi(D_m)] = 0$ for any $1 \leq m \leq k$. This is clear if no vertex of the subchain Δ_m is linked to a and follows otherwise by $R_5(q)$. This shows (a).

Clearly, (b) is trivially true if $[e_{\varepsilon u}, e_{\sigma a}] = 0$, so we shall assume that $[e_{\varepsilon u}, e_{\sigma a}] \neq 0$. Then let $j < u$ be maximal with $q_{ja} \neq 0$. We have then $\sum_{i=j}^{u} q_{ia} = 0$ and $[e_{\varepsilon i}, e_{\sigma a}] = 0$ for $j \leq i < u$ and $[e_{\varepsilon j}, \dots, e_{\varepsilon u}, e_{\sigma a}]$ satisfies the hypothesis of Lemma 4.2 and consequently $[e_{\varepsilon j}, \dots, e_{\varepsilon u}, e_{\sigma a}] = \pm[e_{\sigma a}, e_{\varepsilon u}, \dots, e_{\varepsilon 1}]$, which is zero by (a). ■

5.3. Zero monomials associated with full chains with links

Lemma 5.5. *Let q be a positive definite unit form, let $\Delta = (1, \dots, u)$ be a full chain in $B(q)$ and let a be a vertex not belonging to Δ. Suppose that $\sum_{j=1}^{u} q_{aj} = 0$ and that $B \in \mathcal{M}(X)$ is such that $[e_{\varepsilon j}, \pi(B)] = 0$ for $1 \leq j \leq u$. Then $\pi(D_\varepsilon \bullet (e_{\sigma a}, B)) = 0$ holds for any $\sigma, \varepsilon \in \{1, -1\}$ where $D_\varepsilon = (e_{\varepsilon u}, \dots, e_{\varepsilon 1}) \in \mathcal{M}(X)$.*

Proof. It follows from (5.3) that $\pi(D_\varepsilon \bullet (e_{\sigma a}, B)) = [\pi(D_\varepsilon \bullet e_{\sigma a}), \pi(B)]$. Now $\pi(D_\varepsilon \bullet e_{\sigma a}) = [e_{\varepsilon u}, \dots, e_{\varepsilon 1}, e_{\sigma a}]$ which is zero if $q_{a1} = 0$. Otherwise let $i > 1$ be minimal with $q_{ai} \neq 0$ (such an i exists since $\sum_{j=1}^{n} q_{aj} = 0$ and $q_{a1} \neq 0$). Then we have $[e_{\varepsilon i}, \dots, e_{\varepsilon 1}, e_{\sigma a}] = 0$ by $R_5(q)$. ■

For the rest of this section we shall assume the following situation.

Hypothesis 5.6. Let q be a positive definite unit form, $\Delta = (1, \dots, u)$ a full chain in $B(q)$, $k \geq 0$ and let $1 < i_1 < i_2 < \cdots < i_k \leq u$ be fixed indices. Furthermore, if $k = 0$ let $D_1 = (e_u, \dots, e_1)$ and if $k > 0$ then let

$$D_1 = (e_{i_1-1}, \dots, e_1)$$
$$D_m = (e_{i_m-1}, \dots, e_{i_{m-1}}) \quad (1 < m \leq k)$$
$$D_{k+1} = (e_u, \dots, e_{i_k}).$$

Lemma 5.7. *Assume Hypothesis 5.6. Then for $A \in \mathcal{M}(X)$ and any m with $3 \leq m \leq k+1$ we have*

$$\pi(D_m \bullet D_{m-2} \bullet \cdots \bullet D_1 \bullet A) = \pi(D_{m-2} \bullet \cdots \bullet D_1 \bullet D_m \bullet A), \quad \text{if } m \text{ is odd},$$
$$\pi(D_m \bullet D_{m-2} \bullet \cdots \bullet D_2 \bullet A) = \pi(D_{m-2} \bullet \cdots \bullet D_2 \bullet D_m \bullet A), \quad \text{if } m \text{ is even}.$$

Proof. Let $D = D_{m-2} \bullet \cdots \bullet D_1$ if m is odd ($D = D_{m-2} \bullet \cdots \bullet D_2$ if m is even). Then we have $[e_i, e_j] = 0$ for any $e_i \in D_m$ and any $e_j \in D$. Therefore by Lemma 5.2 we get the result. ■

Hypothesis 5.8. Let $B, C \in \mathcal{M}(X)$ be such that

$$[e_j, \pi(C)] = 0, \quad \text{for all } e_j \in D_1,$$
$$[e_j, \pi(D_{m-1} \bullet C)] = 0, \quad \text{for all } e_j \in D_m, m \text{ odd with } 1 < m \leq k+1,$$
$$[e_j, \pi(D_{m-1} \bullet B)] = 0, \quad \text{for all } e_j \in D_m, m \text{ even with } 1 < m \leq k+1.$$

Lemma 5.9. *Assume Hypotheses 5.6 and 5.8. Then for any m with $1 < m \leq k+1$, and each $e_j \in D_m$ we have*

$$\pi(e_j \bullet D_{m-1} \bullet D_{m-3} \bullet \cdots \bullet D_3 \bullet D_1 \bullet B) = 0 \quad \text{if } m \text{ is even},$$
$$\pi(e_j \bullet D_{m-1} \bullet D_{m-3} \bullet \cdots \bullet D_4 \bullet D_2 \bullet C) = 0, \quad \text{if } m \text{ is odd}.$$

Proof. Suppose that m is even (the case where m is odd is completely similar) and let $E = \pi(e_j \bullet D_{m-1} \bullet D_{m-3} \bullet \cdots \bullet D_3 \bullet D_1 \bullet B)$. Set $D' = (e_j, D_{m-1})$ and observe that for any $x \in D'$ and any $y \in A = D_{m-3} \bullet \cdots \bullet D_1$ we have $[x, y] = 0$. Hence by Lemma 5.2, we have $E = \pi(D' \bullet A \bullet B) = \pi(A \bullet D' \bullet B)$. The result follows now from the fact that $\pi(D' \bullet B) = [e_j, \pi(D_{m-1} \bullet B)] = 0$ by Hypothesis 5.8. ■

Lemma 5.10. *Again, assume Hypotheses 5.6 and 5.8.*

(i) *For any even m, we have that $\pi(D_{m+1} \bullet \cdots \bullet D_2 \bullet D_1 \bullet (B, C))$ is equal to $[\pi(D_{m+1} \bullet \cdots \bullet D_1 \bullet B), \ \pi(D_m \bullet D_{m-2} \bullet \cdots \bullet D_2 \bullet C)]$.*

(ii) *For any odd m, we have that $\pi(D_{m+1} \bullet \cdots \bullet D_2 \bullet D_1 \bullet (B, C))$ is equal to $[\pi(D_m \bullet D_{m-2} \bullet \cdots \bullet D_1 \bullet B), \ \pi(D_{m+1} \bullet D_{m-1} \bullet \cdots \bullet D_2 \bullet C)]$.*

Proof. The proof is by induction on m. Let $E_m = \pi(D_{m+1} \bullet \cdots \bullet D_2 \bullet D_1 \bullet (B, C))$.

If $m = 1$ then $E_1 = \pi(D_2 \bullet D_1 \bullet (B, C)) = [e_{i_2-1}, \dots, e_{i_1}, \pi(D_1 \bullet (B, C))]$ and by Hypothesis 5.8 we have $[e_j, \pi(C)] = 0$ for all $e_j \in D_1$. Therefore we can apply (5.3) and get that $\pi(D_1 \bullet (B, C)) = [\pi(D_1 \bullet B), C]$. Again by Hypothesis 5.8, we have $[e_i, \pi(D_1 \bullet B)] = 0$ for any $e_i \in D_2$, and therefore get $E_1 = \pi(D_2 \bullet (D_1 \bullet B, C)) = [\pi(D_1 \bullet B), \pi(D_2 \bullet C)]$ by (5.2).

Assume now that $m > 1$ and that m is even (the odd case is very similar). Then we have $E_m = [e_{i_{m+1}-1}, \dots, e_{i_m}, \pi(D_m \bullet \cdots \bullet D_1 \bullet (B, C))]$ and get by induction

$$E_m = [e_{i_{m+1}-1}, \dots, e_{i_m}, \pi(D_{m-1} \bullet \cdots \bullet D_1 \bullet B), \pi(D_m \bullet \cdots \bullet D_2 \bullet C)].$$

Since by Lemma 5.9, we have $[e_j, \pi(D_m \bullet \cdots \bullet D_2 \bullet C)] = 0$ for all $e_j \in D_{m+1}$, we get by (5.3) that

$$E_m = [\pi(D_{m+1} \bullet D_{m-1} \bullet \cdots \bullet D_1 \bullet B), \pi(D_m \bullet \cdots \bullet D_2 \bullet C)],$$

which is what we had to show. ■

Hypothesis 5.11. If k is odd then $\pi(D_{k+1} \bullet C) = 0$ and if k is even then $\pi(D_{k+1} \bullet B) = 0$.

The following technical result is an important tool in our proof of Lemma 4.3.

Lemma 5.12. *Assume Hypotheses 5.6, 5.8 and 5.11. Then*

$$\pi(D_{k+1} \bullet \cdots \bullet D_1 \bullet (B, C)) = 0.$$

Proof. Let $E = \pi(D_{k+1} \bullet \cdots \bullet D_1 \bullet (B, C))$. If $k = 1$ then, by Lemma 5.10, we have that $E = [\pi(D_1 \bullet B), \pi(D_2 \bullet C)]$ and the result follows directly from Hypothesis 5.11.

If $k > 1$ (say k is even; the odd case is similar) then it follows from Lemma 5.10 that

$$E = [\pi(D_{k+1} \bullet D_{k-1} \bullet \cdots \bullet D_1 \bullet B), \pi(D_k \bullet D_{k-2} \bullet \cdots \bullet D_2 \bullet C)].$$

Using Lemma 5.7 we get $\pi(D_{k+1} \bullet \cdots \bullet D_1 \bullet B) = \pi(D_{k-1} \bullet \cdots \bullet D_1 \bullet D_{k+1} \bullet B)$ and the result follows from $\pi(D_{k+1} \bullet B) = 0$. ∎

6. Proof of Lemma 4.3

Recall that q is a positive definite unit form which contains $\mathbf{g}_n^0(\lambda)$, $\mathbf{g}_n^1(\lambda, \mu)$ or $\mathbf{g}_n^2(\lambda, \mu)$. In particular $(1, \ldots, n)$ is a full chain in $B(q)$ and we denote by $1 < i_1 < i_2 < \cdots < i_l \le n$ the vertices of $(1, \ldots, n)$ which are linked to μ.

The proof of Lemma 4.3 is quite different for $\mathbf{g}_n^0(\lambda)$ to that for the two cases $\mathbf{g}_n^1(\lambda, \mu)$ and $\mathbf{g}_n^2(\lambda, \mu)$, which can be considered simultaneously and are rather more difficult, so we start with them.

6.1. Strategy for $\mathbf{g}_n^1(\lambda, \mu)$ and $\mathbf{g}_n^2(\lambda, \mu)$

Again denote by Γ the induced subbigraph of $B(q)$ given by the vertices $\lambda, \mu, 1, \ldots, n$. We have $\Gamma = \mathbf{g}_n^1(\lambda, \mu, i_1, \ldots, i_l) =: \mathbf{g}^1$ or $\Gamma = \mathbf{g}_n^2(\lambda, \mu, i_1, \ldots, i_l) =: \mathbf{g}^2$ and want to show that $F = F_{n,u}(\lambda, \mu)$, as defined in Lemma 4.3, is zero in $\mathfrak{g}_5(q)$ for $0 \le u \le n$.

In the proof the following cases are distinguished.

I: $\Gamma = \mathbf{g}^1, \mathbf{g}^2$ and $i_k \le u < i_{k+1}$ for some *even* $k < l$.
II: $\Gamma = \mathbf{g}^1, \mathbf{g}^2$ and $i_k \le u < i_{k+1}$ for some *odd* $k < l - 1$.
III: $\Gamma = \mathbf{g}^1$ and $i_l \le u < n$.
IV: $\Gamma = \mathbf{g}^2$ and $i_{l-1} \le u < i_l = n$.
V: $\Gamma = \mathbf{g}^1, \mathbf{g}^2$ and $1 \le u < i_1$.
VI: $\Gamma = \mathbf{g}^1, \mathbf{g}^2$ and $u = n$.
VII: $\Gamma = \mathbf{g}^1, \mathbf{g}^2$ and $u = 0$.

The last case is the easiest: since there are an even number of vertices of $\Delta = (1, 2, \ldots, n)$ linked to λ and μ, we have $[e_{-\alpha}, e_n, e_{n-1}, \ldots, e_1] = 0$ by Lemma 5.4 for $\alpha = \lambda, \mu$ and therefore $F = F_{n,0} = 0$.

The remaining cases are more difficult. However, since in each case the procedure is quite similar, we follow a common scheme of argument in three steps as follows.
First step: In each case certain definitions are given and a couple of equalities are proved:

I: Define $B_1 = (e_{-i_{k+1}}, \ldots, e_{-(u+1)})$ and $C_1 = (e_{-n}, \ldots, e_{-(i_{k+1}+1)})$. Furthermore, let $B = (e_\lambda, B_1)$ and $C = (e_\mu, C_1)$. We show that (i) $[e_\mu, \pi(C_1 \bullet B_1)] = 0$, (ii) $[e_\lambda, \pi(C_1)] = 0$ and (iii) $[e_\mu, \pi(B)] = 0$.
II: Define $B_1 = (e_{-n}, \ldots, e_{-(i_{k+1}+1)})$ and $C_1 = (e_{-i_{k+1}}, \ldots, e_{-(u+1)})$ and, furthermore, $B = (e_\lambda, B_1)$ and $C = (e_\mu, C_1)$. We show that (i) $[e_\lambda, \pi(B_1 \bullet C_1)] = 0$, (ii) $[e_\mu, \pi(B_1)] = 0$ and (iii) $[e_\lambda, \pi(C)] = 0$.
III: Define $k = l$, $B_1 = (e_{-n}, \ldots, e_{-(u+1)})$, $B = (e_\mu, B_1)$ and $C = e_\lambda$. Then show that (i) $[e_\mu, \pi(B_1)] = 0$.
IV: Define $k = l - 1$, $B = e_\lambda$, $C_1 = (e_{-n}, \ldots, e_{-(u+1)})$ and $C = (e_\mu, C_1)$. Show that (i) $[e_\lambda, \pi(C_1)] = 0$.
V: Define $k = 0$, $B_1 = (e_{-i_1}, \ldots, e_{-(u+1)})$ and $C_1 = (e_{-n}, \ldots, e_{-(i_1+1)})$ and then $B = (e_\lambda, B_1)$, $C = (e_\mu, C_1)$. Then show that (i) $[e_\mu, \pi(C_1 \bullet B_1)] = 0$, (ii) $[e_\lambda, \pi(C_1)] = 0$ and (iii) $[e_\mu, \pi(B)] = 0$.
VI: Define $u = n$, $k = l$, $B = e_\lambda$, $C = e_\mu$ and there will be nothing to prove in this step.

Second step: In all the cases prove that

$$F = \pm\pi(D \bullet (B, C))$$

where $D = D_{k+1} \bullet D_k \bullet \cdots \bullet D_1$ with $D_i \in \mathcal{M}(X)$ as in Hypothesis 5.6.
Third step: Show that in all the cases the Hypotheses 5.8 and 5.11 are satisfied. By Lemma 5.12 we get then $F = 0$, which is what we had to prove.

6.2. *Proof of the first step*

We start with case I. To show (i), let $\Delta = (u+1, \ldots, n)$. The vertices of the full chain Δ which are linked to μ are $i_{k+1}, \ldots, i_l$, and hence $L_\mu(\Delta)$ has even cardinality and $[e_\mu, \pi(C_1 \bullet B_1)] = 0$, by Lemma 5.4.

To see property (ii), let $\Delta = (i_{k+1}+1, \ldots, n)$. Then, if $\Gamma = \mathbf{g}^1$, we have that $L_\lambda(\Delta) = \{i_{k+2}, \ldots, i_l, n\}$ has even cardinality and if $\Gamma = \mathbf{g}^2$, then $L_\lambda(\Delta) = \{i_{k+2}, \ldots, i_{l-1}\}$ also has even cardinality. In any case, (ii) follows from Lemma 5.4.

For (iii), observe that $[e_\lambda, e_j] = 0$ and $[e_\mu, e_j] = 0$ (for $u+1 \le j < i_{k+1}$) and therefore $[e_\lambda, \pi(A)] = [e_\mu, \pi(A)] = 0$ where $A = (e_{-i_{k+1}-1}, \ldots, e_{-(u+1)})$. Hence we have $[e_\mu, \pi(B)] = [e_\mu, e_\lambda, e_{-i_{k+1}}, \pi(A)] = [e_\mu, \pi(A), e_{-i_{k+1}}, e_\lambda] = [\pi(A), e_\mu, e_{-i_{k+1}}, e_\lambda]$, which is zero since $[e_\mu, e_{-i_{k+1}}, e_\lambda] = 0$ by $R_5(q)$.

To see property (i) in case II, let $\Delta = (u+1, \ldots, n)$. If $\Gamma = \mathbf{g}^1$, we have $L_\lambda(\Delta) = \{i_{k+1}, \ldots, i_l, n\}$ and if $\Gamma = \mathbf{g}^2$, we have $L_\lambda(\Delta) = \{i_{k+1}, \ldots, i_{l-1}\}$. In any case $L_\lambda(\Delta)$ has even cardinality, and by Lemma 5.4 we get (i).

For (ii), observe that for $\Delta = (i_{k+1}+1, \ldots, n)$ the set $L_\mu(\Delta) = \{i_{k+2}, \ldots, i_l\}$ has even cardinality, since k is odd, and (ii) follows again by Lemma 5.4. Property (iii) in case II follows like in case I.

The property (i) in case III (respectively in case IV) is trivial since there is no vertex of $\Delta = (u+1, \ldots, n)$ linked to μ (respectively to λ).

In case V, let $\Delta = (u+1, \ldots, n)$ and observe that $L_\mu(\Delta) = \{i_1, \ldots, i_l\}$ has even cardinality. Hence (i) holds by Lemma 5.4. Similarly, if $\Delta = (i_1+1, \ldots, n)$ then, in case $\Gamma = \mathbf{g}^1$, we have $L_\lambda(\Delta) = \{i_2, \ldots, i_l, n\}$ and in case $\Gamma = \mathbf{g}^2$, we have $L_\lambda(\Delta) = \{i_2, \ldots, i_{l-1}\}$. In both cases $L_\lambda(\Delta)$ has even cardinality and once again we can use Lemma 5.4 to deduce (ii). Property (iii) in case V follows like property (iii) in case I. ■

6.3. *Proof of the second step*

Let $D = D_{k+1} \bullet \cdots \bullet D_2 \bullet D_1$, where the D_i are as in Hypothesis 5.6.
Cases I and V: By definition, we have $F = [e_u, \ldots, e_1, G]$ where $G = [[e_\lambda, e_\mu], \pi(C_1 \bullet B_1)]$. By (3.5) and antisymmetry we deduce from property (i) that $G = -[e_\mu, e_\lambda, \pi(C_1 \bullet B_1)]$. And now, it follows from Lemma 5.3 that $G = -[e_\mu, e_\lambda, \pi(C_1), \pi(B_1)]$. Again by (3.5), we deduce from property (ii) that $G = -[e_\mu, \pi(C_1), e_\lambda, \pi(B_1)] = -[e_\mu, \pi(C_1), \pi(B)]$. Hence, we obtain from property (iii) and (3.5) that $G = [\pi(B), e_\mu, \pi(C_1)] = [\pi(B), \pi(C)]$. Finally, substitute G in F to get $F = [e_u, \ldots, e_1, \pi(B), \pi(C)] = \pm\pi(D \bullet (B, C))$.

The proof in case II is identical, after interchanging B_1 with C_1 and B with C.

In case III, we have by definition $F = [e_u, \ldots, e_1, G]$, where $G = [[e_\lambda, e_\mu], \pi(B_1)]$. Now, by antisymmetry, property (i) and (3.5), we have $G = -[\pi(B_1), e_\lambda, e_\mu] = -[e_\mu, e_\lambda, \pi(B_1)] = -[\pi(C), \pi(B)] = [\pi(B), \pi(C)]$. Therefore $F = \pm\pi(D \bullet (B, C))$.

The proof in case IV is almost identical to that in case III after interchanging C_1 with B_1 and C with B.

Case VI is trivial since $F = \pi(D \bullet (C, B)) = -\pi(D \bullet (B, C))$. ■

6.4. *Proof of the third step*

Case I. For each $e_j \in D_1$ (that is $1 \le j < i_1$), we have $[e_j, \pi(C_1)] = 0$ and $[e_j, e_\mu] = 0$ since $q_{j\mu} = 0$. Therefore $[e_j, \pi(C)] = 0$. For any odd m with $1 < m < k$, we have for $e_j \in D_m$, that $[e_j, \pi(C_1)] = 0$ and $[e_j, e_\mu] = 0$ since $q_{j\mu} \ge 0$ and conclude that $[e_j, \pi(C)] = 0$. If $j > i_{m-1}$ then $[e_j, \pi(D_{m-1})] = 0$ and therefore $[e_j, \pi(D_{m-1} \bullet C)] = 0$. If $j = i_{m-1}$ then let $D' = D_{m-1} \bullet e_\mu$ and we obtain $[e_j, \pi(D_{m-1} \bullet C)] = [e_j, \pi(D' \bullet C_1)] = [\pi(e_j \bullet D'), \pi(C_1)]$ by (5.3). By Lemma 5.4(b), we get the second assertion of Hypothesis 5.8.

For any even m with $1 < m \le k$, we have for $e_j \in D_m$ that $[e_j, \pi(B)] = 0$ since $[e_j, e_\lambda] = 0$ and $[e_j, \pi(B_1)] = 0$. Like in the case when m is odd, we conclude that $[e_j, \pi(D_{m-1} \bullet B)] = 0$. This shows that Hypothesis 5.8 holds. To

see Hypothesis 5.11, we use that $(u, \dots, i_k, \lambda, i_{k+1}, \dots, u+1)$ is a chordless cycle and therefore $\pi(D_{k+1} \bullet B) = 0$ by $R_5(q)$.
Case II. The argument is completely similar to that for case I.
Case III. For $e_j \in D_1$, we have $[e_j, \pi(C)] = 0$ since $q_{j\mu} = 0$. If m is odd and $e_j \in D_m$, then again $[e_j, \pi(C)] = 0$ since $q_{j\mu} \geq 0$ and for $j > i_{m-1}$, we get directly $[e_j, \pi(D_{m-1} \bullet C)] = 0$, whereas if $j = i_{m-1}$ then $[e_{i_{m-1}}, \pi(D_{m-1} \bullet C)] = 0$ follows from $R_5(q)$ since $(i_{m-1}, i_{m-1} - 1, \dots, i_{m-2}, \mu)$ is a chordless cycle in q. Similarly one can argue for even m. Hypothesis 5.11 follows again by $R_5(q)$ since $(u, \dots, i_l, \lambda, n, \dots, u+1)$ is a chordless cycle.

The case IV again follows similarly to III after interchanging C_1 with B_1 and C with B.
Case V. Here we have $D = D_1 = [e_u, \dots, e_1]$ and therefore for each $e_j \in D_1$, we get $[e_j, e_\mu] = 0$ since $q_{j\mu} = 0$. Consequently $[e_j, \pi(C)] = 0$ and there is nothing left to prove for Hypotheses 5.8 and 5.11 follows by $R_5(q)$ since $(e_u, \dots, e_1, e_\lambda, e_{-i_1}, \dots, e_{-(u+1)})$ is a chordless cycle in q.
Case VI. We have $k = l$ and $D_{k+1} = (e_n, \dots, e_{i_l})$, which reduces to $D_{k+1} = e_n$ in the case where $\Gamma = \mathbf{g}^2$. Hypothesis 5.8 follows very similarly to in case I. Hypothesis 5.11 follows, in the case where $\Gamma = \mathbf{g}^2$ from $[e_n, e_\lambda] = 0$ and in the case where $\Gamma = \mathbf{g}^1$ from $R_5(q)$ since $(n, \dots, i_l, \lambda)$ is a chordless cycle in q. ■

6.5. *Zero monomials in the bigraphs* $\mathbf{g}_n^0(\lambda)$

Recall that here we have $\Gamma = \mathbf{g}_n^0(\lambda)$ and that we have to show that the monomials

$$F_{n,u}(\lambda, n) = [e_u, \dots, e_1, [e_\lambda, e_n], e_{-n}, \dots, e_{-(u+1)}]$$

for $0 \leq u \leq n$ are zero in $\mathfrak{g}_5(q)$.

We start with the case where $0 \leq u < n$: Then we have $F_{n,u}(\lambda, n) = [e_u, \dots, e_1, G]$, where $G = [[e_\lambda, e_n], e_{-n}, \dots, e_{-(u+1)}]$ and since Lemma 4.2 can be applied, we get $G = \pm[[e_\lambda, e_n], e_{-n}, \dots, e_{-(u+1)}]^{\leftarrow} = \pm[e_{-(u+1)}, \dots, e_{-n}, e_\lambda, e_n]$. Make the replacement $[e_{-n}, e_\lambda, e_n] = [e_\lambda, e_{-n}, e_n] = [e_\lambda, h_n] = e_\lambda$ in the former expression to get $G = [e_{-(u+1)}, \dots, e_{-(n-1)}, e_\lambda]$ which is zero because $[e_{-(n-1)}, e_\lambda] = 0$ by $R_4(q)$.

It remains to consider the case where $u = n$. Then we first observe that $G = [e_n, e_{n-1}, e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda]$ equals zero in $\mathfrak{g}_5(q)$. Indeed, $G = [e_n, [e_{n-1}, e_n], e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda]$ since $[e_{n-1}, e_{n-2}, \dots, e_1, e_\lambda] = 0$ by $R_5(q)$. Therefore, $G = [[e_{n-1}, e_n], e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda]$ since $[e_n, e_{n-1}, e_n] = 0$ by $R_4(q)$. Finally, we have $G = -[[e_n, e_{n-1}], e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda]$ by antisymmetry and then $G = -[e_n, e_{n-1}, e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda] = -G$ since $[e_n, e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda] = 0$ by $R_4(q)$. That is, we have $G = -G$ and therefore $G = 0$.

Now, $F_{n,n}(\lambda, n) = [e_n, \dots, e_1, [e_\lambda, e_n]] = -[e_n, \dots, e_1, e_n, e_\lambda]$ by antisymmetry and since $[e_i, e_n] = 0$ (for $i < n-1$), we get finally $F_{n,n}(\lambda, n) = -[e_n, e_{n-1}, e_n, e_{n-2}, e_{n-3}, \dots, e_1, e_\lambda] = G = 0$. ■

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