

TUBULAR CLUSTER ALGEBRAS II: EXPONENTIAL GROWTH

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ABSTRACT. Among the mutation finite cluster algebras the tubular ones are a particularly interesting class. We show that all tubular (simply laced) cluster algebras are of exponential growth by two different methods: first by studying the automorphism group of the corresponding cluster category and second by giving explicit sequences of mutations.

1. INTRODUCTION

Tubular cluster algebras were introduced in [2] as a proper family of cluster algebras, due to their categorification by tubular cluster categories. These cluster algebras represent three of the 11 exceptional mutation finite cluster algebras with skew symmetric exchange matrix [8] and one is the surface algebra corresponding to the 4-punctured sphere. Figure 1 shows representatives of their exchange matrices in quiver form.

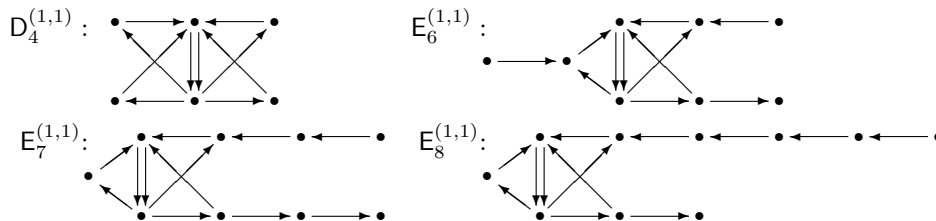


FIGURE 1. Quivers associated to some elliptic root systems

We refer to [2] for more details and context on tubular cluster algebras.

The tubular cluster algebra of type $(2, 2, 2, 2)$ coincides with the surface algebra of the 4-punctured sphere. From a mapping class group argument [14, Sec. 11] it follows that this algebra is of exponential growth. In other words, the number of seeds which can be obtained from a fixed initial seed by at most n mutations is bounded from below by an exponentially growing function of n . Due to the similarity in their categorification one expects this to be true also in the remaining three cases which are not related to surfaces.

Theorem 1.1. *Tubular cluster algebras are of exponential growth.*

We present in this paper two quite different proofs of this result. We think that both proofs are interesting by themselves as they yield two different approaches to this phenomenon.

The first proof, given in Section 3, is based on the result [3, Prop. 7.4] which states that the group of (isomorphism classes of) triangulated self-equivalences of a tubular cluster category contains the group $\mathrm{PSL}_2(\mathbb{Z})$. We show that this descends to an inclusion of $\mathrm{PSL}_2(\mathbb{Z})$ into the corresponding cluster modular group. This is, in some sense, an extension of the above argument which uses the mapping class group.

The second proof, given in Section 4, provides in each of the four cases explicit mutation sequences which directly exhibit the exponential growth. This argument is based on a careful analysis of the lift of mutations (of cluster tilting objects in the tubular cluster category) to Hübner-mutations (of tilting objects in the corresponding category of coherent sheaves over a weighted projective line). Another important ingredient in this approach is the close connection between the exchange graph of Farey triples (a 3-regular tree) and the classification of tilting sheaves over a weighted projective line of tubular type.

We would like to mention that Felikson, Shapiro and Tumarkin recently completed their above mentioned classification of mutation finite cluster algebras by covering also the skew symmetrizable cases [9]. Moreover, they determine in [10] (for the orbifold cases) and in [11] (with H. Thomas for the remaining exceptional cases) the growth rate for all mutation finite cluster algebras. In particular, [11] provides an independent proof for the exponential growth of all tubular cluster algebras which is based on a direct study of the corresponding cluster modular groups. This includes also the non-simply laced cases.

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2. PRELIMINARIES

2.1. Exponential growth of graphs. In our setting a *graph* $G = (G_0, G_1)$ consists of a vertex set G_0 and a edge set G_1 which is a subset of the set of two-element subsets of G_0 . A *path in G of length n* is a sequence of vertices (v_0, v_1, \dots, v_n) such that $\{v_{k-1}, v_k\} \in G_1$ for all $k \in \{1, \dots, n\}$. For a vertex $v \in G_0$ we denote by $v[n] \subset G_0$ the set of vertices which are connected to v by a path of length less than n . Finally, we say that G is of *exponential growth* if for some $v \in G_0$ we can find a function f of exponential growth such that $\#(v[n]) \geq f(n)$ for all $n \in \mathbb{N}$.

For example, we have $\#(v[n]) = 3(2^n - 1)$ for each vertex v of the 3-regular tree \mathbb{T}_3 . Thus \mathbb{T}_3 is of exponential growth.

Let $k \in \mathbb{N}_{\geq 1}$ and G, H be two graphs. By a k -embedding of G into H we mean an injective map $i: G_0 \rightarrow H_0$ such that for each edge $\{v, w\} \in G_1$ there exists a path in H of length at most k , connecting $i(v)$ and $i(w)$. Obviously, H is of exponential growth if for some k , there exists a k -embedding of \mathbb{T}_3 into H .

2.2. Beginning of the proof. By the main result of [2], for a tubular cluster algebra the exchange graph of seeds is isomorphic to the exchange graph \mathcal{G} of cluster tilting objects (in the corresponding tubular cluster category). Thus, by the considerations in Section 2.1, it is sufficient to construct a k -embedding of a tree \mathcal{T} of exponential growth into the exchange graph \mathcal{G} . This will be done in the next two sections by different methods. In the first proof \mathcal{T} is a rooted binary tree and in the second proof it is \mathbb{T}_3 .

3. CLUSTER MODULAR GROUP AND SELF EQUIVALENCES

3.1. Generalities. Let \mathcal{C} be a 2-Calabi-Yau triangulated category, see [20], with split idempotents over some field. We denote the suspension functor of \mathcal{C} by Σ . We suppose that in \mathcal{C} there exists a cluster tilting object T such that there is a cluster structure in the sense of [6] on the cluster tilting objects reachable from T . Without further mentioning all cluster tilting objects will be assumed to be basic. We fix a cluster tilting object with its decomposition into indecomposable direct summands $T = T_1 \oplus \cdots \oplus T_n \in \mathcal{C}$.

Following Keller [21, Sec. 5.5], we consider the groupoid Cl of *cluster tilting sequences* in \mathcal{C} reachable from T . Its objects are the sequences $([T'_1], \dots, [T'_n])$ of isomorphism classes of indecomposable objects such that $T' = \bigoplus_{k=1}^n T'_k$ is a cluster tilting object in \mathcal{C} reachable from T . Note that this implies that the summands T'_i are *rigid*, i.e. $\mathcal{C}(T'_i, \Sigma T'_i) = 0$. Morphisms are formal compositions of (per-)mutations of cluster tilting objects, subject only to the obvious relations: $\mu_k^2 = \text{Id}$ and $\sigma\mu_k = \mu_{\sigma(k)}\sigma$ for $k \in \{1, \dots, n\}$ and each permutation $\sigma \in \mathfrak{S}_n$. For convenience we abbreviate $([T'_1], \dots, [T'_n]) =: [T']$.

We say that a triangulated self-equivalence F of \mathcal{C} is *reachable* if we have $([FT_1], \dots, [FT_n]) \in \text{Cl}$. In this case it is not hard to see that F induces a self-equivalence \overline{F} of Cl which we call *induced*. For a sequence of indices $\mathbf{i} = (i_s, \dots, i_2, i_1)$ with $i_a \in \{1, \dots, n\}$ we define $\mu_{\mathbf{i}} = \mu_{i_s} \cdots \mu_{i_1}$. For a permutation $\sigma \in \mathfrak{S}_n$ we set $\sigma(\mathbf{i}) = (\sigma(i_s), \dots, \sigma(i_1))$.

Proposition 3.1. *Let F and G be two reachable self-equivalences of \mathcal{C} ,*

- (a) $\overline{F} = \overline{G}$ if and only if $([FT_1], \dots, [FT_n]) = ([GT_1], \dots, [GT_n])$.
- (b) Suppose that for two sequences of indices \mathbf{i} and \mathbf{j} and permutations $\sigma, \tau \in \mathfrak{S}_n$ we have $\overline{F}([T]) = \sigma\mu_{\mathbf{i}}([T_1], \dots, [T_n])$ and $\overline{G}([T]) = \tau\mu_{\mathbf{j}}([T_1], \dots, [T_n])$ in Cl . Then $\overline{F} \circ \overline{G}([T]) = \tau\sigma\mu_{\sigma^{-1}(\mathbf{j})}\mu_{\mathbf{i}}([T])$.

Proof. (a) Recall that for each $X \in \mathcal{C}$ there exists a distinguished triangle $T''_X \rightarrow T'_X \rightarrow X \rightarrow \Sigma T''_X$ with $T'_X, T''_X \in \text{add}(T)$. The *index* $\text{ind}_T(X)$

of X with respect to T is $[T'_X] - [T''_X]$ in the split Grothendieck group of $\text{add}(T)$. In [7, Sec. 2] it is shown that in case X is rigid, X is determined up to isomorphism by its index $\text{ind}_T(X)$. Moreover, each triangulated self-equivalence of \mathcal{C} sends cluster tilting objects to cluster tilting objects. Thus our claim follows since the cluster tilting objects reachable from T have by hypothesis a cluster structure.

(b) Since \overline{F} is a self-equivalence of Clt we have

$$\overline{F}(\overline{G}([T])) = \overline{F}(\tau\mu_j([T])) = \tau\mu_j(F([T])) = \tau\mu_j\sigma\mu_i([T]) = \tau\sigma\mu_{\sigma^{-1}(j)}\mu_i([T]).$$

□

By the above proposition, the induced self-equivalences of Clt form a group, which we call the *refined cluster modular group* $\text{Aut}_\iota(\text{Clt})$.

Remark 3.2. The group $\text{Aut}_\iota(\text{Clt})$ seems to be related to the cluster modular group defined by Fock and Goncharov [13, 1.2.5], see also [12, p.28]. Note that the endomorphism ring of a cluster tilting object in \mathcal{C} is in general not determined by its quiver. For example, this occurs for the tubular cluster category of weight type $(2, 2, 2, 2)$, see [5, Expl. 6.12]. This category can be used to categorify the cluster algebra associated to the sphere with four punctures [2, Rem. 1.2].

Corollary 3.3. (a) *We have an injective map*

$$\text{Aut}_\iota(\text{Clt}) \rightarrow \{([T'_1], \dots, [T'_n]) \in \text{Clt} \mid [T'] \equiv [T]\}, \overline{F} \mapsto \overline{F}([T]),$$

where $[T'] \equiv [T]$ means that the assignment $T_i \mapsto T'_i$ for $i = 1, \dots, n$ induces an equivalence of categories between $\text{add}(T)$ and $\text{add}(T')$.

(b) *Suppose, that $\text{Aut}_\iota(\text{Clt})$ contains a free (non-abelian) subgroup in two generators. Then there is a k -embedding of the (rooted) binary tree into the exchange graph of cluster tilting objects.*

Proof. Part (a) follows immediately from Proposition 3.1 (a) whereas part (b) follows from part (a) and Proposition 3.1 (b). □

Remark 3.4. It follows from [22], that in case \mathcal{C} is the generalized cluster category associated to a non-degenerate, Jacobi-finite quiver with potential, the above map is also surjective.

3.2. The tubular case. Let \mathcal{C} be a tubular cluster category. Thus \mathcal{C} is the orbit category of $\mathcal{D} := \mathcal{D}^b(\text{coh } \mathbb{X})$ modulo the self equivalence $\tau^{-1}[1]$, for a weighted projective line \mathbb{X} of tubular type. Here we denoted by τ the Auslander-Reiten automorphism and by $[1]$ the shift automorphism of \mathcal{D} . The canonical projection $\pi: \mathcal{D} \rightarrow \mathcal{C}$ is a triangle functor. In this situation, \mathcal{C} fulfills all the requirements of Section 3.1. Moreover, all cluster tilting sequences are reachable from any given cluster tilting object, see [3, Thm. 8.8].

It is shown in [3, Lemma 6.6] that all triangulated self-equivalences of \mathcal{D} are standard. Thus the isomorphism classes of such self-equivalences form a group which can be identified with the derived Picard group $\text{Aut}_s(\mathcal{D})$ of the corresponding canonical algebra [26]. Furthermore, it is shown in [3, Corollary 6.5] that each triangulated self-equivalence of \mathcal{C} can be lifted along π to a triangulated self-equivalence of \mathcal{D} . In particular, the isomorphism classes of triangulated self-equivalences $\text{Aut}(\mathcal{C})$ of \mathcal{C} form a factor group of $\text{Aut}_s(\mathcal{D})$.

Proposition 3.5. *For a tubular cluster category \mathcal{C} the map*

$$\text{Aut}(\mathcal{C}) \rightarrow \text{Aut}_\iota(\text{Clt}), F \mapsto \overline{F}$$

is an isomorphism of groups.

Proof. By the above observations and the definition, the map $F \mapsto \overline{F}$ is a well-defined, surjective group homomorphism.

Let $F \in \text{Aut}(\mathcal{C})$ be such that $\overline{F} = \text{Id}_{\text{Clt}}$. As explained above, there exists a standard self-equivalence \tilde{F} of \mathcal{D} which lifts F along π . Let $M = M_1 \oplus \cdots \oplus M_n$ be a tilting complex such that $\pi(M_i) = T_i$ for $i = 1, \dots, n$. Since \mathbb{X} is tubular, we may assume that $E := \text{End}_{\mathcal{D}}(M)$ is schurian, i.e. $\dim \mathcal{D}(M_i, M_j) \leq 1$ for all $1 \leq i, j \leq n$. By our hypothesis we may also assume that $\tilde{F}(M_i) \cong M_i$ for all $i = 1, \dots, n$. Since \tilde{F} is standard, it is determined by an element $\omega \in \text{Out}(E)$ (the group of outer automorphisms of E) that fixes the standard primitive idempotents of E , see [26, Prop. 2.3]. Since E is schurian it follows that ω is the identity. Thus \tilde{F} and F are isomorphic to the respective identities. \square

3.3. First proof of Theorem 1.1. By Section 2.2 and Corollary 3.3 (b) it is sufficient to show that in the tubular case $\text{Aut}_\iota(\text{Clt})$ contains a free subgroup in two generators. In fact, by Proposition 3.5 we have $\text{Aut}_\iota(\text{Clt}) \cong \text{Aut}(\mathcal{C})$ and in [3, Prop. 7.4] it is shown, that $\text{Aut}(\mathcal{C})$ is a semidirect product of a finite group by $\text{PSL}_2(\mathbb{Z})$, see also [23, Thm. 6.3]. \square

4. EXPLICIT VERIFICATION USING FAREY TRIPLES

4.1. Hübner mutations. In his Ph.D. thesis [17], Hübner investigated tilting objects in the category $\text{coh } \mathbb{X}$ of coherent sheaves over a weighted projective line \mathbb{X} . In this section, we collect the results from [17] which are relevant in our context.

Let $T = \bigoplus_{i=1}^n T_i$ be a tilting sheaf in $\text{coh } \mathbb{X}$. Let Q_T be the quiver of the endomorphism algebra of T . For simplicity, we shall identify the vertices of Q with the summands T_1, \dots, T_n . For each index $i = 1, \dots, n$ we define the morphism

$$\sigma_i = [\sigma_{i1} \quad \cdots \quad \sigma_{in}] : \bigoplus_{h=1}^n T_h^{r_{ih}} \rightarrow T_i$$

where the entries of $\sigma_{ih} = \begin{bmatrix} \sigma_{ih}^{(1)} & \cdots & \sigma_{ih}^{(r_{ih})} \end{bmatrix}$ constitute a set of morphisms $\sigma_{ih}^{(a)} : T_h \rightarrow T_i$ which are mapped to a basis under the canonical projection

$$\text{rad}(T_h, T_i) \rightarrow \text{rad}(T_h, T_i) / \text{rad}^2(T_h, T_i).$$

Note that r_{ih} is the number of arrows $h \rightarrow i$ in Q_T . Similarly we define a morphism

$$\rho_i = \begin{bmatrix} \rho_{i1} \\ \vdots \\ \rho_{in} \end{bmatrix} : T_i \rightarrow \bigoplus_{h=1}^n T_h^{r_{hi}},$$

where $\rho_{ih} = \sigma_{ih}^\top$.

Proposition 4.1. [17, Prop. 2.6, 2.8] *With the previous notation, we have the following results.*

- (a) *For each index i the morphism σ_i (resp. ρ_i) is either a monomorphism or an epimorphism in $\text{coh } \mathbb{X}$.*
- (b) *For each index i the morphism σ_i is mono (resp. epi) if and only if ρ_i is a mono (resp. epi).*
- (c) *Let $T_k^* = \text{Ker } \sigma_k \oplus \text{Coker } \rho_k$. Then $T_k^* \oplus \bigoplus_{j \neq k} T_j$ is again a tilting object in $\text{coh } \mathbb{X}$.*

In view of Proposition 4.1(b), exactly one of $\text{Ker } \sigma_k$ and $\text{Coker } \rho_k$ is non-zero. This allows us to separate the vertices of Q_T into two classes: T_k is called a *Hübner-source* (resp. a *Hübner-sink*) in case σ_k and ρ_k are mono (resp. epi).

Remark 4.2. The following warning seems in place, see also [17, Bem. 3.3]: it is possible to have two tilting sheaves $T = \bigoplus_{i=1}^n T_i$ and $T' = \bigoplus_{i=1}^n T'_i$ with isomorphic quivers Q_T and $Q_{T'}$, such that some vertex T_i is a Hübner-source of Q_i whereas its corresponding vertex T'_i is a Hübner-sink of $Q_{T'}$.

Definition 4.3. Let $T = \bigoplus_{i=1}^n T_i$ be a tilting sheaf in $\text{coh } \mathbb{X}$. Then for each index $k \in \{1, \dots, n\}$ we define the *mutation of T in direction k* to be the tilting sheaf

$$\mu_k(T) = T_k^* \oplus \bigoplus_{j \neq k} T_j,$$

where $T_k^* = \text{Ker } \sigma_k \oplus \text{Coker } \rho_k$.

Remark 4.4. In [17] the tilting sheaf $\mu_k(T)$ is called *reflection* at the source or sink due to the similarity with the Bernstein-Gelfand-Ponomarev reflections [4]. However, given the role they play as lifts of mutations in the context of cluster algebras we prefer this new terminology.

In view of Remark 4.2, conditions which characterize Hübner-sinks and Hübner-sources are important.

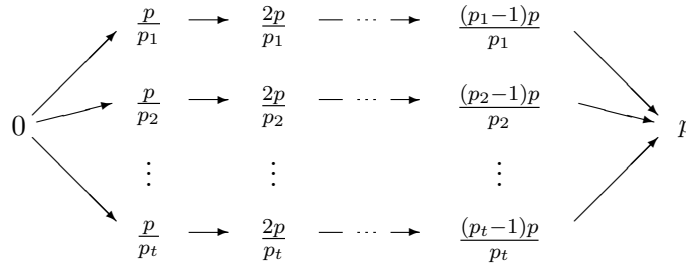
Proposition 4.5. [17, Bem. 2.10, Kor. 3.5] *A sink (resp. source) of a quiver is always a Hübner-sink (resp. Hübner-source). A successor of a Hübner-sink is again a Hübner-sink and a predecessor of a Hübner-source is again a Hübner-source. Furthermore if the endomorphism algebra is given by its quiver and some relations, then any relation starts in a Hübner-source and ends in a Hübner-sink.*

The preceding conditions are not sufficient to decide always whether a given vertex is a Hübner-source or a Hübner-sink. To give a sufficient characterization we need some notions.

We recall that we denote by $\mathcal{D} = \mathcal{D}^b(\text{coh } \mathbb{X})$ the bounded derived category of $\text{coh } \mathbb{X}$. Recall from [16] that there are two \mathbb{Z} -linear forms, rk and deg on $\text{K}_0(\text{coh } \mathbb{X}) = \text{K}_0(\mathcal{D}^b(\text{coh } \mathbb{X}))$, called the *rank* and *degree*. Furthermore the *slope* is defined as $S = \frac{\text{deg}}{\text{rk}}$. Note that each tilting sheaf T in $\text{coh } \mathbb{X}$ gives rise to a triangulated equivalence between the bounded derived categories $\mathcal{D}^b(\text{coh } \mathbb{X})$ and $\mathcal{D}^b(\text{mod } A)$ where $A = \text{End}(T)$. Since such an equivalence induces an isomorphism between the corresponding Grothendieck groups, we can evaluate rank and degree on (classes of) A -modules.

Proposition 4.6. [17, Bem. 3.3] *Let $T = \bigoplus_{i=1}^n T_i$ be a tilting sheaf in $\text{coh } \mathbb{X}$. Further denote by S_i the simple right $\text{End}(T)$ -module associated to T_i . Then T_i is a Hübner-source if and only if $\text{rk}(S_i) > 0$ or $\text{rk}(S_i) = 0$ and $\text{deg}(S_i) > 0$.*

Example 4.7. Let (p_1, p_2, \dots, p_t) be the weight sequence of \mathbb{X} and $p = \text{lcm}(p_1, p_2, \dots, p_t)$. The *canonical configuration* T_{can} , see [16], is a tilting sheaf whose endomorphism algebra is a canonical algebra in the sense of Ringel [25, Sec. 3.7]. The following picture shows its quiver. There are $t - 2$ relations from the unique source to the unique sink of the quiver.



The indecomposable direct summands of T_{can} have rank 1 and degree $j \frac{p}{p_i}$ as shown in the picture above.

The next result, though interesting in its own, permits to calculate the rank in concrete examples.

Proposition 4.8. [17, Thm. 4.6] *For each tilting sheaf T the rank function is an additive function on the quiver Q_T with relations. More precisely, if*

$T = \bigoplus_{i=1}^n T_i$ then for each indecomposable direct summand T_i we have

$$2 \operatorname{rk}(T_i) = \sum_{j \rightarrow i} \operatorname{rk}(T_j) + \sum_{i \rightarrow j} \operatorname{rk}(T_j) - \sum_{j \twoheadrightarrow i} \operatorname{rk}(T_j) - \sum_{i \twoheadrightarrow j} \operatorname{rk}(T_j),$$

where the summation has to be taken over all arrows and relations ending in i .

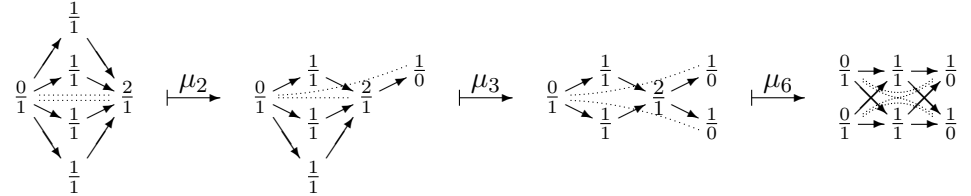
The thesis of Hübner [17] contains also a precise description of the effect on the endomorphism algebra of tilting sheaves under mutation, in terms of arrows and relations. Let $T = \bigoplus_{i=1}^n T_i$ be a tilting sheaf for $\operatorname{coh} \mathbb{X}$ and let $\mu_k(T) = \bigoplus_{i \neq k} T_i \oplus T'_k$ be the mutation in direction k . We state here only the version for the Hübner-source — the one for a Hübner-sink is completely dual and therefore left to the interested reader.

Proposition 4.9. [17, Kor. 4.16] *If T_i is a Hübner-source, then the quiver with relations Q' for $\mu_k(T)$ is obtained from Q as follows.*

- (i) *The quiver Q' has the same vertices as Q .*
- (ii) *For each pair of arrows $i \rightarrow k \rightarrow j$ an arrow $i \rightarrow j$ is added.*
- (iii) *For each pair of an arrow $k \rightarrow i$ and a relation $k \twoheadrightarrow j$ a relation $i \twoheadrightarrow j$ is added.*
- (iv) *Each arrow $i \rightarrow k$ is replaced by a relation $i \twoheadrightarrow k$.*
- (v) *Each arrow $k \rightarrow i$ is replaced by an arrow $i \rightarrow k$.*
- (vi) *Each relation $k \twoheadrightarrow i$ is replaced by an arrow $k \rightarrow i$.*
- (vii) *Pairs of parallel relations and arrows are successively canceled.*
- (viii) *All remaining arrows and relations remain unchanged.*

Remark 4.10. If each relation $i \twoheadrightarrow j$ is replaced by an arrow $j \rightarrow i$ then the corresponding mutation rule is precisely the mutation of diagrams as formulated in [15]. Further we note that this definition is compatible with the mutation of \mathbb{Z}_2 -graded quivers, introduced by Amiot and Oppermann in [1, Def. 6.2].

Example 4.11. We consider the weight sequence $(2, 2, 2, 2)$. Let T_{can} be the canonical configuration, see Example 4.7. We label the vertices of the quiver of $\operatorname{End}(T_{\text{can}})$ by $1, 2, \dots, 6$ such that 1 is the source and 6 is the sink. We consider the mutation sequence $\mu_6 \mu_3 \mu_2$, and indicate the slope $\frac{\deg(T_i)}{\operatorname{rk}(T_i)}$ in the quivers:



4.2. Farey triples. We resume some basic properties of Farey triples, see also [24, Sec. 2]. First, we extend \mathbb{Q} to $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ and observe that

each element $q \in \mathbb{Q}$ defines uniquely two integers $d(q)$ and $r(q)$ which are relatively prime and such that

$$q = \frac{d(q)}{r(q)}, \quad r(q) > 0.$$

Furthermore, we define $d(\infty) = 1$ and $r(\infty) = 0$.

Definition 4.12. For a pair $p, q \in \mathbb{Q}_\infty$ the *Farey distance* is defined as

$$\Delta(p, q) = |d(p)r(q) - d(q)r(p)|.$$

If $\Delta(p, q) = 1$, then p, q are called *Farey neighbours*. A triple $\{q_1, q_2, q_3\}$ of elements of \mathbb{Q}_∞ which are pairwise Farey neighbours is called a *Farey triple*.

Given $p, q \in \mathbb{Q}_\infty$ the *Farey sum* \oplus and *Farey difference* \ominus are defined by

$$p \oplus q = \frac{d(p) + d(q)}{r(p) + r(q)}, \quad p \ominus q = \frac{d(p) - d(q)}{r(p) - r(q)}.$$

If $\bar{q} = \{p, q, r\}$ is a Farey triple then the *mutation* $\mu_p(\bar{q})$ in direction p is defined by

$$\mu_p(\bar{q}) = \begin{cases} \{q \ominus r, q, r\}, & \text{if } q < p < r \text{ or } r < p < q, \\ \{q \oplus r, q, r\}, & \text{if } p < \min(q, r) \text{ or } p > \max(q, r). \end{cases}$$

Lemma 4.13. *To any two Farey neighbours there exist exactly two Farey triples containing them. If \bar{q} is a Farey triple then $\mu_p(\bar{q})$ is again a Farey triple for any $p \in \bar{q}$. Moreover, the mutation of Farey triples is involutive, in the sense that $\mu_{p'}\mu_p(\bar{q}) = \bar{q}$ if $\bar{q} = \{p, q, r\}$ and $\mu_p(\bar{q}) = \{p', q, r\}$.*

Proof. Let $p = \frac{a}{b}$ and $q = \frac{c}{d}$ be Farey neighbours. We may assume that $p > q$, that is $ad - bc = 1$. Now suppose that $\{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\}$ is a Farey triple. Then

$$(1) \quad af - be = \varepsilon \in \{1, -1\}$$

$$(2) \quad cf - de = \varphi \in \{1, -1\}$$

By multiplying (2) by a and using (1) we get $e = \varepsilon a - \varphi c$. Similarly we get $f = \varepsilon b - \varphi d$.

Therefore, the four possible choices for the signs $\varepsilon, \varphi \in \{1, -1\}$ lead to precisely two possible solutions for $\frac{e}{f}$. This proves the first statement and the rest of the proof of the following result is straightforward and left to the interested reader. \square

Remark 4.14. Similarly as in [2] we define the *complexity* $c(q)$ of $q \in \mathbb{Q}_\infty$ to be $|d(q)| + r(q) + |d(q) - r(q)|$. It follows easily that $\{1, 0, \infty\}$ is the unique Farey-triple of minimal sum of the complexities, and that each other Farey triple can be mutated in a unique direction so that its sum of complexities decreases. Consequently, the exchange graph of the Farey triples form a 3-regular tree under mutations.

Lemma 4.15. *If $\{p, q, r\}$ is a Farey triple with $p < q < r$ then*

$$2q \ominus p = q \oplus r, \quad 2q \ominus r = q \oplus p, \quad 2p \ominus q = 2r \ominus q = p \ominus r,$$

where $2\frac{a}{b} \ominus \frac{c}{d} = \frac{2a-c}{2b-d}$.

Proof. Write $p = \frac{a}{b}$, $q = \frac{c}{d}$ and $r = \frac{e}{f}$. The statements follow easily by applying the equations $cb - ad = 1$, $eb - af = 1$ and $ed - cf = 1$. \square

4.3. Tilting sheaves realizing a Farey triple. Given a tilting sheaf $T = \bigoplus_{i=1}^n T_i$ we define its *slope set* to be $\bar{S}(T) = \{S(T_i) \mid i = 1, \dots, n\}$. We say that a tilting sheaf T *realizes* a Farey triple \bar{q} if $\bar{S}(T) = \bar{q}$. In Example 4.11 we have given a tilting sheaf realizing the Farey triple $\{0, 1, \infty\}$ for the weight sequence $(2, 2, 2, 2)$. We now consider the remaining weight sequences $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$. In each case we indicate how a tilting sheaf T realizing the Farey triple $\{0, 1, \infty\}$ may be obtained from the canonical configuration T_{can} by a sequence of mutations. For this, we label the vertices of the quiver of $\text{End}(T_{\text{can}})$ by $1, \dots, n$ in such a way that 1 is the source, n is the sink, and the remaining vertices are labeled $2, \dots, n-1$ from left to right and top to bottom in the picture of the quiver given in Example 4.7.

For the (tubular) weight sequence $(3, 3, 3)$ we use the sequence 7, 5, 3, 8, 1 of mutations (i.e. the composition of mutations $\mu_1\mu_8\mu_3\mu_5\mu_7$) to obtain from T_{can} a tilting sheaf T realizing $\{0, 1, \infty\}$. The quiver of $\text{End}(T)$ is isomorphic to the first quiver in Figure 2. The following table contains the information about degree and rank of the indecomposable direct summands of T .

| i | u_1 | u_2 | u_3 | v_1 | v_2 | w_1 | w_2 | w_3 |
|--|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $\frac{\text{deg}(T_i)}{\text{rk}(T_i)}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{0}{1}$ | $\frac{0}{2}$ | $\frac{1}{0}$ | $\frac{1}{0}$ | $\frac{1}{0}$ |

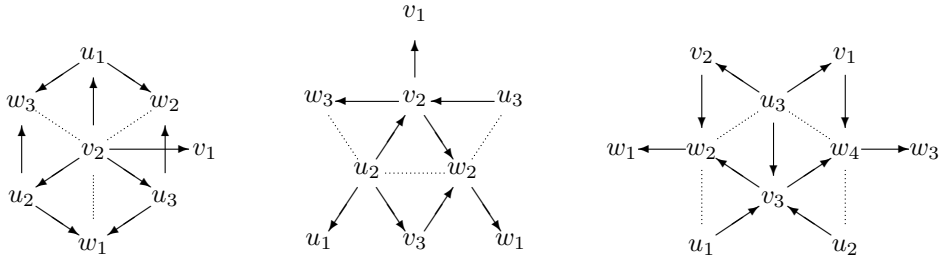


FIGURE 2. Examples of quivers of $\text{End}(T)$ for T realizing a Farey triple. The weight sequences from left to right are $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$.

For the (tubular) weight sequence $(4, 4, 2)$ we use the mutation sequence 3, 6, 9, 4, 7, 9, 7, 8, 1, 3 to obtain from T_{can} a tilting sheaf T realizing $\{0, 1, \infty\}$. The quiver of $\text{End}(T)$ is isomorphic to the second quiver in Figure 2. The

following table contains the information about degree and rank of the indecomposable direct summands of T .

| i | u_1 | u_2 | u_3 | v_1 | v_2 | v_3 | w_1 | w_2 | w_3 |
|------------------------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $\frac{\deg(T_i)}{\text{rk}(T_i)}$ | $\frac{0}{1}$ | $\frac{0}{2}$ | $\frac{0}{1}$ | $\frac{1}{1}$ | $\frac{2}{2}$ | $\frac{1}{1}$ | $\frac{1}{0}$ | $\frac{2}{0}$ | $\frac{1}{0}$ |

For the (tubular) weight sequence $(6, 3, 2)$ we use the mutation sequence $5, 10, 8, 6, 10, 3, 4, 8, 9, 7, 2, 4, 6, 5$ to obtain from T_{can} a tilting sheaf T realizing $\{0, 1, \infty\}$. The quiver of $\text{End}(T)$ is isomorphic to the third quiver in Figure 2. The following table contains the information about degree and rank of the indecomposable direct summands of T .

| i | u_1 | u_2 | u_3 | v_1 | v_2 | v_3 | v_4 | w_1 | w_2 | w_3 |
|------------------------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $\frac{\deg(T_i)}{\text{rk}(T_i)}$ | $\frac{0}{1}$ | $\frac{0}{1}$ | $\frac{0}{2}$ | $\frac{1}{1}$ | $\frac{2}{2}$ | $\frac{1}{1}$ | $\frac{2}{2}$ | $\frac{1}{0}$ | $\frac{1}{0}$ | $\frac{2}{0}$ |

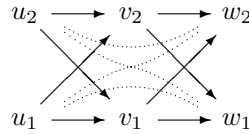
4.4. Explicit mutation sequences. For each of the tubular weight sequences $(2, 2, 2, 2)$, $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$ we give explicit mutation sequences which transform a given tilting sheaf realizing a Farey triple into another one realizing a mutated Farey triple.

We start by considering the case where the weight sequence of \mathbb{X} is $(2, 2, 2, 2)$. Let $\mathcal{T}_{2,2,2,2}$ be the set of (isomorphism classes of) tilting sheaves

$$T = \bigoplus_{i \in I} T_i \text{ with } I = \{u_1, u_2, v_1, v_2, w_1, w_2\},$$

such that

- $S(T_{x_1}) = S(T_{x_2})$ for all $x \in \{u, v, w\}$,
- $\bar{S}(T)$ is a Farey triple,
- the quiver Q with relations of $\text{End}(T)$ looks as follows:



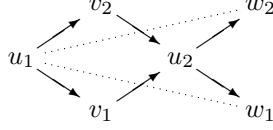
Note that $\mathcal{T}_{2,2,2,2}$ is not empty as shown in Section 4.3. Define the mutation sequences $\mu_u = \mu_{u_1}\mu_{u_2}$, $\mu_v = \mu_{v_1}\mu_{v_2}$ and $\mu_w = \mu_{w_1}\mu_{w_2}$.

Proposition 4.16. *Let \mathbb{X} be a weighted projective line with weight sequence $(2, 2, 2, 2)$. Then for any $T \in \mathcal{T}_{2,2,2,2}$ and $x \in \{u, v, w\}$ we have that $\mu_x(T) \in \mathcal{T}$ and $\bar{S}(\mu_x(T)) = \mu_{q_x}(\bar{S}(T))$, where $q_x = S(T_{x_1})$.*

Proof. Let $\bar{S}(T) = \{q_u, q_v, q_w\}$ with $q_u < q_v < q_w$. Since $S(T_{u_i}) < S(T_{v_i}) < S(T_{w_i})$ for each $i = 1, 2$ it follows that $S(T_{u_i}) = q_u$, $S(T_{v_i}) = q_v$ and $S(T_{w_i}) = q_w$ for $i = 1, 2$.

If $x = u$ (resp. $x = w$) then the mutation takes place in two Hübner-sources (resp. Hübner-sinks). It follows from Proposition 4.8 and the symmetry of the quiver that $\text{rk}(T_{u_1}) = \text{rk}(T_{u_2})$ and similarly $\text{rk}(T_{v_1}) = \text{rk}(T_{v_2})$ and $\text{rk}(T_{w_1}) = \text{rk}(T_{w_2})$. Therefore the corresponding equalities hold also for the degree. We denote $r_u = \text{rk}(T_{u_i})$, $d_f = \text{rk}(T_{u_i})$ and similarly define r_v , d_v , r_w and d_w .

Now assume that $x = u$. By mutation in u_2 we obtain the following quiver.



Thus by Proposition 4.5 the vertex T_{u_1} is a Hübner source of the quiver of $\mu_{u_2}(T)$. Hence the quiver of $\mu_u(T)$ is isomorphic to Q . Using that the rank and the degree are additive on exact sequences we get easily that $\text{rk}(T'_{u_i}) = 2r_v - r_u$ and $\text{deg}(T'_{u_i}) = 2d_v - d_u$ for $i = 1, 2$, where T'_{u_1} and T'_{u_2} denote the two summands of $\mu_u(T)$ which are obtained instead of T_{u_1} and T_{u_2} by mutation. Using Lemma 4.15 we get

$$S(T'_{u_i}) = \frac{2d_v - d_u}{2r_v - r_u} = \frac{d_v + d_w}{r_v + r_w} = q_v \oplus q_w.$$

This shows that $\bar{S}(\mu_f(T)) = \mu_{q_u}(\bar{S}(T))$ and hence the result in case $x = u$.

The case where $x = w$ is completely similar and the case $x = v$ is also similar with the unique difference that it is possible for the vertices T_{v_1} , T_{v_2} to be a Hübner-source or a Hübner-sink (both of the same kind). \square

We now focus on the case where the weight sequence is $(3, 3, 3)$. Here we look at three possible quivers with relations.

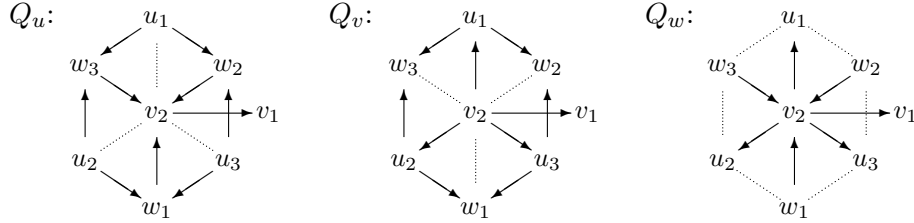


FIGURE 3. Quivers with relations for the weight sequence $(3, 3, 3)$.

Let $\mathcal{T}_{(3,3,3)}$ be the set of (isomorphism classes of) tilting sheaves

$$T = \bigoplus_{i \in I} T_i \text{ with } I = \{u_1, u_2, u_3, v_1, v_2, w_1, w_2, w_3\},$$

such that

- the rank and degree of the indecomposable summands of T are related as shown in the following table:

| i | u_1 | u_2 | u_3 | v_1 | v_2 | w_1 | w_2 | w_3 |
|------------------------------------|---------------|---------------|---------------|---------------|-----------------|---------------|---------------|---------------|
| $\frac{\deg(T_i)}{\text{rk}(T_i)}$ | $\frac{a}{b}$ | $\frac{a}{b}$ | $\frac{a}{b}$ | $\frac{c}{d}$ | $\frac{2c}{2d}$ | $\frac{e}{f}$ | $\frac{e}{f}$ | $\frac{e}{f}$ |

- $\bar{S}(T) = \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\}$ is a Farey triple,
- the quiver with relations of $\text{End}(T)$ is one of the three quivers of Figure 3.

Note that $\mathcal{T}_{3,3,3}$ is not empty by the construction in Section 4.3.

We define the mutation sequences

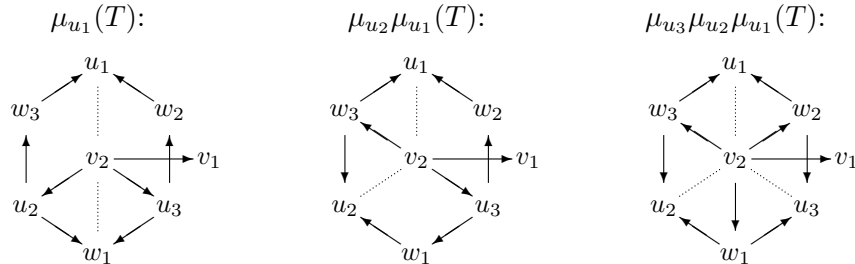
$$\begin{aligned}\mu_u &= \mu_{u_3}\mu_{u_2}\mu_{u_1}, \\ \mu_v &= \mu_{u_3}\mu_{v_2}\mu_{v_1}\mu_{u_2}\mu_{w_3}\mu_{u_1}\mu_{v_2}, \\ \mu_w &= \mu_{w_1}\mu_{w_2}\mu_{w_3}.\end{aligned}$$

Proposition 4.17. *Let \mathbb{X} be a weighted projective line with weight sequence $(3, 3, 3)$. Then for any $T \in \mathcal{T}_{(3,3,3)}$ and $x \in \{u, v, w\}$ we have $\mu_x(T) \in \mathcal{T}_{(3,3,3)}$ and $\bar{S}(\mu_x(T)) = \mu_{q_x}(\bar{S}(T))$, where $q_x = S(T_{x_1})$.*

Proof. Let $T \in \mathcal{T}_{(3,3,3)}$ and denote $T' = \mu_x(T)$. We start with the case $x = u$. Let us first assume that the quiver of $\text{End}(T)$ is Q_c . Then $\frac{c}{d} < \frac{a}{b} < \frac{e}{f}$ and therefore $\frac{a}{b} = \frac{c}{d} \oplus \frac{e}{f}$. Using Proposition 4.8 we get that $b = d + f$ and then $a = c + e$. We will distinguish whether T_{u_1} is a Hübner-source or Hübner-sink using Proposition 4.6 and observe that if $b = 2d$ then it follows that $a > 2c$. Hence we get the two cases:

- T_{f_i} is a Hübner-source, which is equivalent to $b \geq 2d$, or
- T_{f_i} is a Hübner-sink, which is equivalent to $b < 2d$.

In case (a), the effect of the mutation sequence μ_u on the quiver of $\text{End}(T)$ is as depicted in the next illustration. We observe that by Proposition 4.6 also T_{u_2} is a Hübner-source in $\mu_{u_1}(T)$ and similarly T_{u_3} is a Hübner-source of $\mu_{u_2}\mu_{u_1}(T)$.

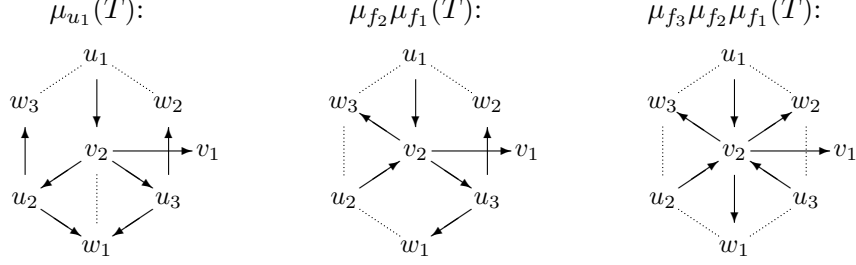


The new summands T'_{u_i} are obtained as cokernels:

$$T_{u_i} \rightarrow T_{w_j} \oplus T_{w_h} \rightarrow T'_{u_i}$$

where j, h are such that $\{h, i, j\} = \{1, 2, 3\}$. Consequently, for each $i = 1, 2, 3$ the rank and the degree of T'_{u_i} equals $2f - b = f - d$ and $2e - a = e - c$ respectively. Hence $\mu(T'_{u_i}) = \frac{e}{f} \ominus \frac{c}{d}$. This shows that $\overline{\mathcal{S}}(\mu_u(T)) = \mu_{\frac{a}{b}}(\overline{\mathcal{S}}(T))$.

In case (b) we obtain the following sequence where we now observe that T_{u_2} is a Hübner-sink of $\mu_{u_1}(T)$ and T_{u_3} is a Hübner-sink of $\mu_{u_2}\mu_{u_1}(T)$.



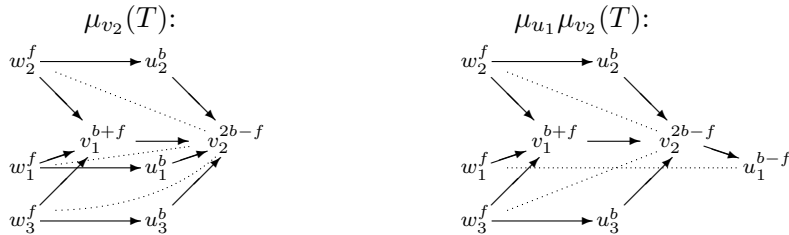
For each $i = 1, 2, 3$ the rank and the degree of T'_{u_i} equal $2d - b = d - f$ and $2c - a = c - e$ respectively. Hence $\mu(T'_{u_i}) = \frac{c}{d} \ominus \frac{e}{f}$ and again $\overline{\mathcal{S}}(\mu_u(T)) = \mu_{\frac{a}{b}}(\overline{\mathcal{S}}(T))$ follows.

Still having $x = u$ we now look at the case where $\text{End}(T)$ has as quiver Q_u (resp. Q_w). Then it is easily observed that μ_u yields the inverse process as in case (a) (resp. in case (b)) above. This concludes the assertion if $x = u$. The case $x = w$ is handled completely similar. We now assume that $x = v$. Since the arguments are the same as used for $x = u$ except that the mutation sequence is substantially longer we will use certain abbreviations.

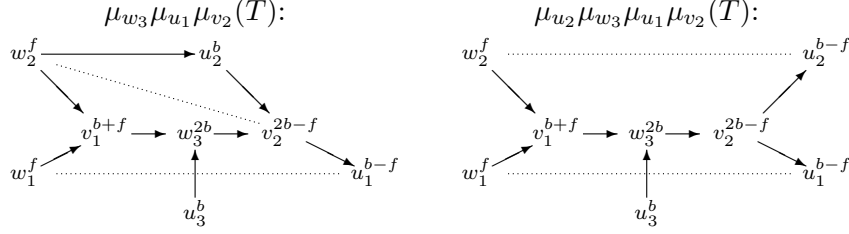
We start considering first the case where $\text{End}(T)$ has quiver Q_w . Then $\frac{e}{f} < \frac{c}{d} < \frac{a}{b}$ and therefore we conclude as before that $c = a + e$ and $d = b + f$. We shall distinguish the following three cases:

- (a) $f \leq b$, (b) $\frac{1}{2}f \leq b < f$, (c) $\frac{1}{2}f < b$.

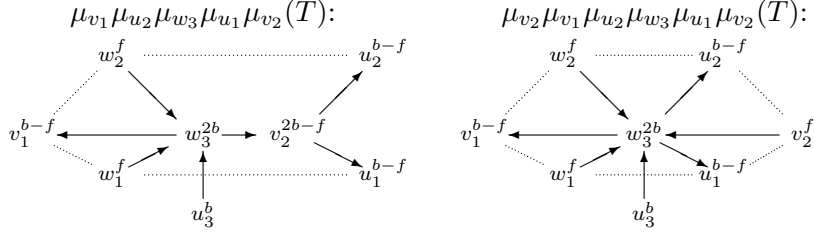
In case (a) the summand T_{v_2} is a Hübner-source and T_{u_1} a Hübner source of $\mu_{v_2}(T)$. The first two steps in the mutation sequence look as follows. We have indicated the rank of each summand as superscript. The degree is obtained by replacing b by a and f by e .



Now T_{w_3} is a Hübner-source of $\mu_{u_1}\mu_{v_2}(T)$ since a relation starts in w_3 . The resulting quiver after mutation is shown in the next illustration on the left. Then T_{u_2} is a Hübner-source of $\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$ since $f \geq b$.

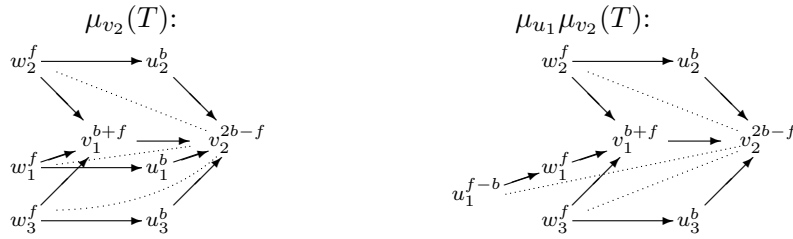


Proceeding further this way, we see that T_{v_1} is a Hübner-source of $\mu_{u_2}\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$, again since $b \geq f$. The resulting quiver after mutation is shown below on the left. Then T_{v_2} is a Hübner-sink of $\mu_{v_1}\mu_{u_2}\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$ since $2f - b - 2f = -b < 0$ since $b = 0$ is not possible ($b = 0$ would imply $f = 0$ and then $\frac{a}{b} = \frac{e}{f} = \infty$).

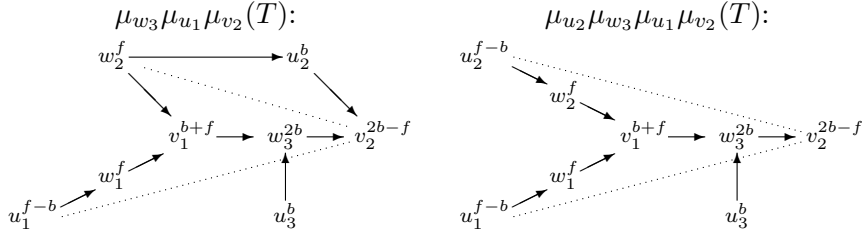


The final step just changes the direction of the arrow $u_3 \rightarrow w_3$ but not the slope. We therefore see that in case (a) we have $\mu_v(T) \in \mathcal{T}$ and $\bar{S}(\mu_v(T)) = \mu_{\frac{c}{a}}(\bar{S}(T))$.

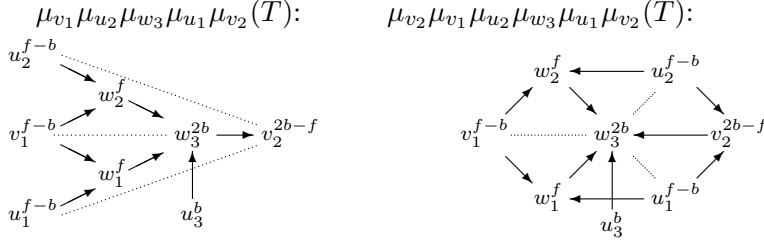
We now consider case (b), where we can copy the first step and start with $\mu_{v_2}(T)$ already calculated as in case (a) since again T_{v_2} is a Hübner-source of T . Now T_{u_1} is a Hübner-sink of $\mu_{v_2}(T)$.



Clearly T_{w_3} is a Hübner-source of $\mu_{u_1}\mu_{v_2}(T)$ and then T_{u_2} is a Hübner-sink of $\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$ since $b < f$.

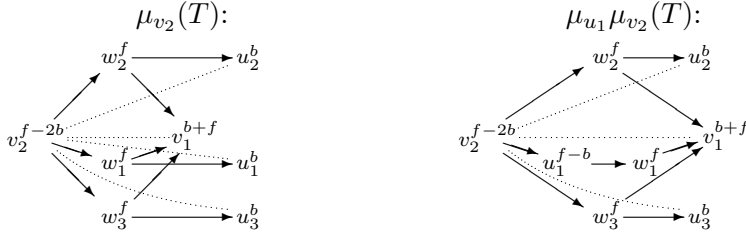


Now, since $b - f < 0$ we have that T_{v_1} is a Hübner-sink of $\mu_{u_2} \mu_{w_3} \mu_{u_1} \mu_{v_2}(T)$. The resulting quiver is shown on the left hand side in the next picture. Clearly T_{v_2} is a Hübner-sink in $\mu_{u_2} \mu_{w_3} \mu_{u_1} \mu_{v_2}(T)$.

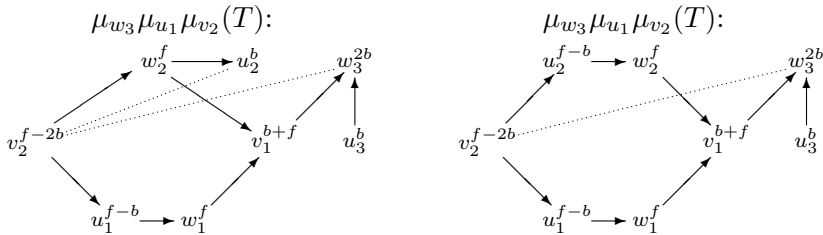


Again, the last step only changes the direction of the arrow $u_3 \rightarrow w_3$ but no slope. Hence we have shown that also in case (b) we have $\mu_v(T) \in \mathcal{T}$ and that $\bar{S}(\mu_v(T)) = \mu_{\bar{d}}(\bar{S}(T))$.

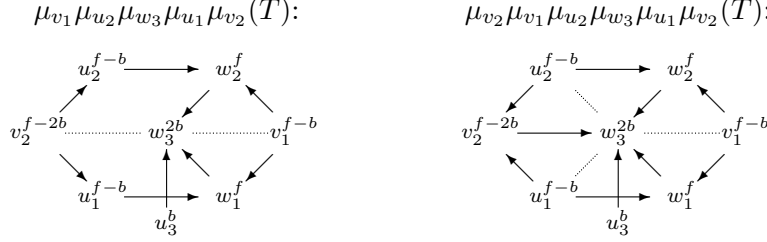
We consider now the case (c) where T_{v_2} is a Hübner-sink of T . The resulting situation is shown in the next picture on the left. Clearly T_{u_1} is a sink of $\mu_{v_2}(T)$.



Since $b \geq 0$ we have that T_{w_3} is Hübner-source of $\mu_{u_1} \mu_{v_2}(T)$. Also T_{u_2} is a Hübner-sink of the resulting $\mu_{w_3} \mu_{u_1} \mu_{v_2}(T)$.



Again, since $b - f < 0$ we have that T_{v_1} is a Hübner-sink of $\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$. Clearly T_{v_2} is a Hübner-source of $\mu_{v_1}\mu_{u_2}\mu_{w_3}\mu_{u_1}\mu_{v_2}(T)$.



And again the last mutation in the sequence does only revert the direction of the arrow $u_3 \rightarrow w_3$ without changing any slope. Hence we have proved the assertion in case (c) and therefore in all possible cases where the quiver of $\text{End}(T)$ is Q_w .

A similar calculation deals with the cases where $\text{End}(T)$ has as quiver Q_u or Q_v . However these cases need distinction since in no step there is a possible choice for a vertex to be a Hübner-source or a Hübner-sink. These cases are therefore left to the interested reader. This concludes the proof of the statement. \square

Since the remaining two tubular weight sequences require no new type of argument we restrict to give the proper definition and statement and leave the verification to the interested reader.

If the weight sequence is $(4, 4, 2)$, let $\mathcal{T}_{(4,4,2)}$ be the set of (isomorphism classes of) tilting sheaves

$$T = \oplus_{i \in I} T_i \text{ with } I = \{u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3\},$$

such that

- the rank and degree of the indecomposable summands of T are related as shown in the following table:

| i | u_1 | u_2 | u_3 | v_1 | v_2 | v_3 | w_1 | w_2 | w_3 |
|--|---------------|-----------------|---------------|---------------|-----------------|---------------|---------------|-----------------|---------------|
| $\frac{\text{deg}(T_i)}{\text{rk}(T_i)}$ | $\frac{a}{b}$ | $\frac{2a}{2b}$ | $\frac{a}{b}$ | $\frac{c}{d}$ | $\frac{2c}{2d}$ | $\frac{c}{d}$ | $\frac{e}{f}$ | $\frac{2e}{2f}$ | $\frac{e}{f}$ |

- $\bar{S}(T) = \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\}$ is a Farey triple,
- the quiver with relations of $\text{End}(T)$ is one shown in the middle of Figure 2.

Note that $\mathcal{T}_{4,4,2}$ is not empty by the construction in Section 4.3.

The mutation sequences in this case are then defined to be

$$\begin{aligned}\mu_u &= \mu_{w_1}\mu_{u_3}\mu_{v_3}\mu_{v_1}\mu_{u_2}\mu_{v_2}\mu_{u_2}\mu_{w_2}\mu_{u_2}\mu_{v_3}, \\ \mu_v &= \mu_{u_1}\mu_{v_3}\mu_{w_3}\mu_{w_1}\mu_{v_2}\mu_{w_2}\mu_{v_2}\mu_{u_2}\mu_{v_2}\mu_{w_3}, \\ \mu_w &= \mu_{v_1}\mu_{w_3}\mu_{u_3}\mu_{u_1}\mu_{c_2}\mu_{u_2}\mu_{w_2}\mu_{v_2}\mu_{w_2}\mu_{u_3}.\end{aligned}$$

Proposition 4.18. *Let \mathbb{X} be a weighted projective line with weight sequence $(4, 4, 2)$. Then for any $T \in \mathcal{T}_{(4,4,2)}$ and $x \in u, v, w$ we have $\mu_x(T) \in \mathcal{T}_{(4,4,2)}$ and $\bar{S}(\mu_x(T)) = \mu_{q_x}(\bar{S}(T))$, where $q_x = S(T_{x_1})$.*

In case the weight sequence is $(6, 3, 2)$ we need to define three quivers Q_x for $x \in \{u, v, w\}$ as follows:

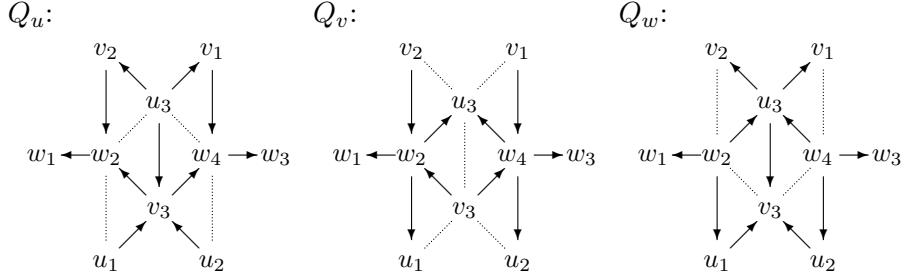


FIGURE 4. Quivers with relations for the weight sequence $(6, 3, 2)$.

Let $\mathcal{T}_{(6,3,2)}$ to be the set of (isomorphism classes of) all tilting sheaves

$$T = \bigoplus_{i \in I} T_i \text{ with } I = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4, w_1, w_2, w_3\},$$

such that

- the rank and degree of the indecomposable summands of T are related as shown in the following table:

| i | u_1 | u_2 | u_3 | v_1 | v_2 | v_3 | v_4 | w_1 | w_2 | w_3 |
|------------------------------------|---------------|---------------|-----------------|---------------|-----------------|---------------|-----------------|---------------|---------------|-----------------|
| $\frac{\deg(T_i)}{\text{rk}(T_i)}$ | $\frac{a}{b}$ | $\frac{a}{b}$ | $\frac{2a}{2b}$ | $\frac{c}{d}$ | $\frac{2c}{2d}$ | $\frac{c}{d}$ | $\frac{2c}{2d}$ | $\frac{e}{f}$ | $\frac{e}{f}$ | $\frac{2e}{2f}$ |

- $\bar{S}(T) = \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\}$ is a Farey triple,
- the quiver with relations of $\text{End}(T)$ is one of the three quivers of Figure 4.

Note that $\mathcal{T}_{6,3,2}$ is not empty by the construction in Section 4.3.

The mutations sequences are defined as

$$\begin{aligned}\mu_u &= \mu_{u_3}\mu_{u_2}\mu_{u_1}, \\ \mu_v &= \mu_{v_3}\mu_{v_2}\mu_{v_1}, \\ \mu_w &= \mu_{w_3}\mu_{w_1}\mu_{v_3}\mu_{w_4}\mu_{u_2}\mu_{w_3}\mu_{w_4}\mu_{w_2}\mu_{u_1}\mu_{w_1}\mu_{w_2}.\end{aligned}$$

Proposition 4.19. *Let \mathbb{X} be a weighted projective line with weight sequence $(6, 3, 2)$. Then for any $T \in \mathcal{T}_{(6,3,2)}$ and $x \in \{u, v, w\}$ we have $\mu_x(T) \in \mathcal{T}_{(6,3,2)}$ and $\overline{S}(\mu_x(T)) = \mu_{q_x}(\overline{S}(T))$, where $q_x = S(T_{x_1})$.*

Remark 4.20. Another warning seems in place: Although the mutation sequences μ_u , μ_v and μ_w are involutive on Farey triples in the sense of Lemma 4.13, they are not involutive on the isomorphism classes of tilting sheaves in the four tubular cases.

4.5. Second proof of Theorem 1.1. For the four tubular types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$ the Propositions 4.16, 4.17, 4.18 and 4.19 respectively provide a recursive procedure to construct a k -embedding of the 3-regular tree \mathbb{T}_3 (identified with the exchange graph of Farey-triples) into the exchange graph of tilting sheaves over the weighted projective line of the corresponding tubular weight type. Now, this exchange graph can be identified by [3] with the exchange graph of cluster tilting objects in the respective tubular cluster category. Thus, in view of Section 2.2 we are done. \square

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