TUBULAR CLUSTER ALGEBRAS I: CATEGORIFICATION

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Abstract. We present a categorification of four mutation finite cluster algebras by the cluster category of the category of coherent sheaves over a weighted projective line of tubular weight type. Each of these cluster algebras which we call tubular is associated to an elliptic root system. We show that via a cluster character the cluster variables are in bijection with the positive real Schur roots associated to the weighted projective line. In one of the four cases this is achieved by the approach to cluster algebras of Fomin-Shapiro-Thurston using a 2-sphere with 4 marked points whereas in the remaining cases it is done by the approach of Geiss-Leclerc-Schröer using preprojective algebras.

1. Introduction

Cluster algebras were introduced around 2001 by Fomin and Zelevinsky [12] as a tool to study questions concerning dual canonical bases and total positivity. Though mainly combinatorial in their conception, many important questions about cluster algebras were answered by introducing some extra structure like for example a "categorification" [14],[9] or some Poisson geometric context [18].

Somehow simplifying, a cluster algebra $\mathcal{A}(B)$ of rank n with trivial coefficients is determined by the skew symmetrizable exchange matrix $B \in \mathbb{Z}^{n\times n}$. If B' is mutation equivalent [12, Def. 4.2] to B, the cluster algebra $\mathcal{A}(B')$ is isomorphic to $\mathcal{A}(B)$.

The cluster algebras we will study in this article are those which are given by a matrix B whose associated diagram is one in Figure 1 (the symbol close to the diagram denotes the associated elliptic root system, see Point (c) below). We call these four cluster algebras *tubular*, due to their categorifaction by a tubular cluster category which we discuss in the present paper.

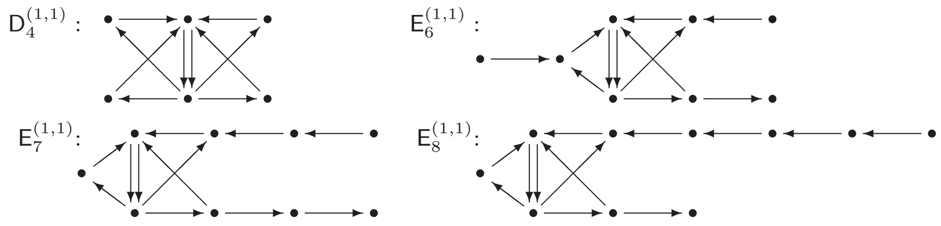

Figure 1. Quivers associated to some elliptic root systems

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In [13] Fomin and Zelevinsky showed that a cluster algebra $\mathcal{A}(B)$ is of finite type if and only if B is mutation equivalent to the skew-symmetrization of a Cartan matrix of finite type. Moreover, they obtained in this case a natural bijection between the cluster variables and almost positive roots in the corresponding root system. A broader class of cluster algebras are those with a mutation finite exchange matrix. Each cluster algebras associated to the signed adjacency matrix of arcs of a triangulation on a (possibly bordered) two-dimensional surface, studied by Fomin, Shapiro and Thurston [11], belongs to this class. Note that in this case cluster variables are parametrized by tagged arcs on the corresponding surface. In a recent paper by Felikson, Shapiro and Tumarkin [10] it is shown that there are besides the above mentioned family of mutation finite exchange matrices only 11 further (exceptional) mutation classes of skew-symmetric exchange matrices of rank ≥ 3, namely those given by:

(a) Cartan matrices of type E_6, E_7, E_8. The corresponding cluster algebras are of finite type.
(b) generalized Cartan matrices of type $\tilde{\mathsf{E}}_6$, $\tilde{\mathsf{E}}_7$, $\tilde{\mathsf{E}}_8$. The corresponding cluster algebras are categorified by the cluster category of a tame hereditary algebra of the corresponding type, and cluster variables correspond naturally to almost positive Schur roots.
(c) certain orientations of the diagrams describing elliptic root systems of type $\mathsf{E}_6^{(1,1)}, \mathsf{E}_7^{(1,1)}$ resp. $\mathsf{E}_8^{(1,1)}$, see Figure 1 above.
(d) the Diagrams X_6 and X_7 recently discovered by Derksen and Owen [8].

Note that also the exchange matrix associated to the quiver of type $\mathsf{D}_4^{(1,1)}$ in Figure 1 is mutation finite. In fact, it corresponds in the setup of [11] to the 2-sphere with 4 punctures.

We give in this paper a uniform categorification of the cluster algebras associated to the Diagrams in Figure 1. To this end, we denote by $\operatorname{coh}\mathbb{X}$ the category of coherent sheaves over a weighted projective line $\mathbb{X}$ in the sense of Geigle-Lenzing [15], see also Section 2. The orbit category $\mathcal{C}_{\mathbb{X}} := \mathcal{D}^b(\operatorname{coh}\mathbb{X})/\langle\tau^{-1}[1]\rangle$, called the *cluster category* associated to $\mathbb{X}$, of the derived category $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$ is a triangulated 2-Calabi-Yau category [22] admitting a cluster structure in the sense of [6]. Moreover, it is easy to see that $\operatorname{coh}\mathbb{X}$ and $\mathcal{C}_{\mathbb{X}}$ have the same Auslander-Reiten quiver. An object X in $\mathcal{C}_{\mathbb{X}}$ is called *rigid* if $\operatorname{Ext}^1_{\mathcal{C}}(X, X) = 0$.

The weighted projective line $\mathbb{X}$ comes with a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$. The complexity of $\operatorname{coh}\mathbb{X}$ depends essentially on the value of the virtual genus $g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^t \frac{p}{p_i}\right)$, where $p = \operatorname{lcm}(p_1, \ldots, p_t)$. More precisely, $\operatorname{coh}\mathbb{X}$ is tame if and only if $g_{\mathbb{X}} \leq 1$. If $g_{\mathbb{X}} = 1$ the weighted projective line $\mathbb{X}$ is said to be of *tubular type* and we call the category $\mathcal{C}_{\mathbb{X}}$ *tubular*. In this case each connected component of the Auslander-Reiten quiver of $\mathcal{C}_{\mathbb{X}}$ is a tube. The following is the main result of this paper.

Theorem 1.1. *Each cluster algebra $\mathcal{A}$ of type $\mathsf{D}_4^{(1,1)}, \mathsf{E}_6^{(1,1)}, \mathsf{E}_7^{(1,1)}$ resp. $\mathsf{E}_8^{(1,1)}$ can be categorified by $\mathcal{C}_{\mathbb{X}}$, where $\mathbb{X}$ is of tubular type with weight sequence $\mathbf{p} = (2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$ respectively. More precisely, there exists a cluster character in the sense of Palu [25], which induces a bijective map from the set of (isomorphism classes of) indecomposable objects E of $\mathcal{C}_{\mathbb{X}}$ which are rigid to the* *set of cluster variables of $\mathcal{A}$ and a bijection between the (isomorphism classes of) cluster-tilting objects and the clusters.*

Remark 1.2. **(a)** The parametrization of the indecomposable rigid objects in $\mathcal{C}_{\mathbb{X}}$ reduces to the description of positive real Schur roots in the Grothendieck group $K_0(\mathbb{X})$ of $\operatorname{coh}\mathbb{X}$. In the tubular case this can be done in a closed form. The condition for two indecomposable rigid objects of being Ext-orthogonal can also be translated into a handy criterion in terms of Schur roots. This facilitates a combinatorial description of the exchange graph and the cluster complex. We shall carry out these description in a forthcoming paper [4].

(b) For the proof in the D-case we produce an explicit bijection between positive real Schur roots and tagged arcs on the 2-sphere with 4 punctures, which respects the respective notions of compatibility. See Proposition 5.11 for the precise statement. We believe that this bijection is of independent interest.

The following result is an immediate consequence of Theorem 1.1.

Corollary 1.3. *The cluster complex of each tubular cluster algebra $\mathcal{A}$ is the clique complex of the compatibility relation.*

In the E-cases we obtain some addtional information from the proof of our result.

Corollary 1.4. *Each tubular cluster algebra $\mathcal{A}$ of type $\mathsf{E}_6^{(1,1)}, \mathsf{E}_7^{(1,1)}$ or $\mathsf{E}_8^{(1,1)}$ is finitely generated and the cluster monomials form a linearly independent family.*

In fact, these cluster algebras are mutation equivalent to the certain cluster algebras which are categorified in [17], see Section 3.2. All cluster algebras of this kind have the above mentioned properties, see Section 3.1.

2. Coherent sheaves over weighted projective lines

2.1. **Basic Notions.** Let k be an algebraically closed field and denote by $\mathbb{P}^1$ the projective line over k. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t)$ be a collection of pairwise different points of $\mathbb{P}^1$ and $\mathbf{p} = (p_1, \dots, p_t)$ a weight sequence with $p_i \in \mathbb{N}_{\geq 2}$. Then the tuple $\mathbb{X} = (\mathbb{P}^1, \boldsymbol{\lambda}, \mathbf{p})$ is called a weighted projective line. The weight function is defined by

$$p_\lambda \colon \mathbb{P}^1 \to \mathbb{N}, \mu \mapsto \begin{cases} p_i & \text{if } \mu = \lambda_i \text{ for some } i \\ 1 & \text{else.} \end{cases}$$

For later use we set $p := \operatorname{lcm}(p_1, \dots, p_t)$.

Geigle and Lenzing [15] associate to $\mathbb{X}$ a category of coherent sheaves as follows: Let $L(\mathbf{p})$ the rank 1 additive group

$$L(\mathbf{p}) := \langle \vec{x}_1, \dots, \vec{x}_t \mid p_1\vec{x}_1 = \cdots = p_t\vec{x}_t = \vec{c} \rangle$$

and $S(\mathbf{p}, \boldsymbol{\lambda})$ the $L(\mathbf{p})$-graded commutative algebra

$$S(\mathbf{p}, \boldsymbol{\lambda}) = k[u, v, x_1, \dots, x_t]/(x_i^{p_i} - \lambda_i' u - \lambda_i'' v \mid i = 1, \dots, t)$$

where $\deg x_i = \vec{x}_i$ and $\lambda_i = [\lambda_i' : \lambda_i''] \in \mathbb{P}^1$. Now, $\operatorname{coh}\mathbb{X}$ is the quotient category of finitely generated $L(\mathbf{p})$-graded $S(\mathbf{p}, \boldsymbol{\lambda})$-modules by the Serre subcategory of finite length modules.

Geigle and Lenzing showed that $\operatorname{coh}\mathbb{X}$ is a hereditary abelian category with finite dimensional Hom and Ext spaces. The free module gives a structure sheaf $\mathcal{O}$, and shifting the grading gives twists $E(\vec{x})$ for any sheaf $E \in \operatorname{coh}\mathbb{X}$ and $\vec{x} \in L(\mathbf{p})$. Moreover, they showed that with $\vec{w} := \sum_{i=1}^{t}(\vec{c} - \vec{x}_i) - 2\vec{c}$ the following version of Serre duality

$$D\operatorname{Ext}^1_{\mathbb{X}}(E, F) \cong \operatorname{Hom}_{\mathbb{X}}(F, E(\vec{w}))$$

holds. As a consequence, $\operatorname{coh}\mathbb{X}$ has Auslander-Reiten sequences with the shift $E \mapsto E(\vec{w})$ acting as Auslander-Reiten translation.

Every sheaf is a direct sum of a 'vector bundle' which has a filtration with factors of the form $\mathcal{O}(\vec{x})$, and a sheaf of finite length. The indecomposable sheaves of finite length are readily described: For each $\mu \in \mathbb{P}^1 \setminus \boldsymbol{\lambda}$ there is a unique simple sheaf S_μ 'concentrated at μ' and for $\lambda_i \in \mathbb{P}^1$ there are simple sheaves $S_{i,1}, \ldots, S_{i,p_i}$ 'concentrated at λ_i' and the only non-trivial extension between them are

$$\operatorname{Ext}^1_{\mathbb{X}}(S_\mu, S_\mu) \cong k \text{ and } \operatorname{Ext}^1_{\mathbb{X}}(S_{i,j}, S_{i,j'}) \cong k \text{ if } j - j' \equiv 1 \pmod{p_i}.$$

As a consequence, for each simple sheaf S and $l \in \mathbb{N}$ there exists a unique indecomposable sheaf $S^{(l)}$ of length l and socle S, and up to isomorphism all indecomposable sheaves of finite length are of this form. Resuming:

Proposition 2.1. *The category $\operatorname{coh}_0\mathbb{X}$ of sheaves of finite length is an exact abelian, uniserial subcategory of $\operatorname{coh}\mathbb{X}$ which is stable under Auslander-Reiten translation. The components of the Auslander-Reiten quiver of $\operatorname{coh}_0\mathbb{X}$ form a family of standard tubes $(\mathcal{T}_\mu)_{\mu \in \mathbb{P}^1}$ with rank $\operatorname{rk}\mathcal{T}_\mu = p_\lambda(\mu)$, see* [26] *for definitions.*

2.2. **Discrete invariants.** The Grothendieck group $K_0(\mathbb{X})$ of $\operatorname{coh}\mathbb{X}$ is a free abelian group of rank $n = 2 + \sum_{i=1}^{t}(p_i - 1)$. Since $\operatorname{coh}\mathbb{X}$ is hereditary, the homological form

$$\langle E, F\rangle = \dim \operatorname{Hom}_{\mathbb{X}}(E, F) - \dim \operatorname{Ext}^1_{\mathbb{X}}(E, F)$$

descends to an integral bilinear form on $K_0(\mathbb{X})$. For the same reason, the Auslander-Reiten translate induces a linear transformation τ on $K_0(\mathbb{X})$ such that $[E(\vec{w})] = \tau[E]$ for all $E \in \operatorname{coh}\mathbb{X}$. From Serre duality follows

$$\langle \tau\mathbf{e}, \tau\mathbf{f}\rangle = \langle \mathbf{e}, \mathbf{f}\rangle = -\langle \mathbf{f}, \tau\mathbf{e}\rangle$$

for all $\mathbf{e}, \mathbf{f} \in K_0(\mathbb{X})$. Thus $\tau\mathbf{e} = \mathbf{e}$ implies $\langle \mathbf{e}, \mathbf{e}\rangle = 0$.

There exists $\mathbf{h}_\infty \in K_0(\mathbb{X})$ such that $\mathbf{h}_\infty = [S_\mu]$ for all $\mu \in \mathbb{P}^1 \setminus \boldsymbol{\lambda}$. We define the *rank* of a sheaf E by

$$\operatorname{rk} E = \langle [E], \mathbf{h}_\infty\rangle.$$

It is easy to see that $\operatorname{rk}\mathcal{O}(\vec{x}) = 1$ for all $\vec{x} \in L(\mathbf{p})$ and $\operatorname{rk} S = 0$ for all simple sheaves.

Next, with $\mathbf{h}_0 := \sum_{k=0}^{p-1}[\mathcal{O}(k\vec{w})]$ we may define the *degree* of E as

$$\deg E = \langle \mathbf{h}_0, [E]\rangle - (\operatorname{rk} E)\langle \mathbf{h}_0, [\mathcal{O}]\rangle,$$

and it turns out that $\deg S = p/p_\lambda(\mu)$ if S is a simple concentrated at $\mu \in \mathbb{P}^1$, and $\deg \mathcal{O}(\vec{x}) = \delta(\vec{x})$ with $\delta\colon L(\mathbf{p}) \to \mathbb{Z}$ defined by $\delta(\vec{x}_i) = p/p_i$.

Clearly, rank and degree are linear functionals on $K_0(\mathbb{X})$. Thus, we may define for $\mathbf{e} \in K_0(\mathbb{X})$

$$0 \prec \mathbf{e} :\Leftrightarrow \operatorname{rk}\mathbf{e} > 0 \text{ or } (\operatorname{rk}\mathbf{e} = 0 \text{ and } \deg\mathbf{e} > 0).$$

This converts $K_0(\mathbb{X})$ in an ordered group and we say $\mathbf{e}$ is *positive* if $0 \prec \mathbf{e}$. Note that for each $E \in \operatorname{coh}\mathbb{X}$ we have $0 \prec [E]$.

Definition 2.2. Let $\mathbb{X}$ be a weighted projective line. We say that $\mathbf{e} \in K_0(\mathbb{X})$ is a *positive root* if there exists an *indecomposable* $E \in \operatorname{coh}\mathbb{X}$ with $[E] = \mathbf{e}$. Such a root is called a *Schur root* if E can be chosen such that $\operatorname{End}_{\mathbb{X}}(E) \cong k$, it is called *real* if $\langle \mathbf{e}, \mathbf{e} \rangle = 1$ and *isotropic* if $\langle \mathbf{e}, \mathbf{e} \rangle = 0$.

Recall that a sheaf $E \in \operatorname{coh}\mathbb{X}$ is called *rigid* if $\operatorname{Ext}^1_{\mathbb{X}}(E, E) = 0$. Remarkably, we have the following alternative characterization of Schur roots [24, Prop. 4.4.1].

Proposition 2.3. *The map $E \mapsto [E]$ induces a bijection between the isomorphism classes of indecomposable rigid sheaves and the real positive Schur roots.*

2.3. **Stability.** For each positive $\mathbf{e} \in K_0(\mathbb{X})$ we define its *slope* by $\operatorname{slope}(\mathbf{e}) = \frac{\deg \mathbf{e}}{\operatorname{rk} \mathbf{e}} \in \mathbb{Q}_\infty$. If $E \in \operatorname{coh}\mathbb{X}$ we write $\operatorname{slope}(E) = \operatorname{slope}([E])$.

We say that E es *stable* (resp. *semistable*) if for each non-trivial subbundle E' of E we have $\operatorname{slope}(E') < \operatorname{slope}(E)$ (resp. $\operatorname{slope}(E') \leq \operatorname{slope}(E)$). We have the following result [15, Prop. 5.2]:

Proposition 2.4. *For each $q \in \mathbb{Q}_\infty$ let $\mathcal{C}_q$ be the subcategory of $\operatorname{coh}\mathbb{X}$ consisting of all semistable coherent sheaves of slope q. In particular, $\mathcal{C}_\infty$ is the subcategory of finite length sheaves. Then the following holds:*

(a) *$\mathcal{C}_q$ is an extension closed exact abelian finite length subcategory of $\operatorname{coh}\mathbb{X}$ with the simple objects being precisely the stable vector bundles.*
(b) $\operatorname{Hom}_{\mathbb{X}}(\mathcal{C}_q, \mathcal{C}_{q'}) = 0$ *if $q' < q$.*

2.4. **Tubular weighted projective lines.** The complexity of the classification of indecomposables in $\operatorname{coh}\mathbb{X}$ depends essentially on the value of the virtual genus $g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right)$, If $g_{\mathbb{X}} \leq 1$ the category $\operatorname{coh}\mathbb{X}$ is derived equivalent to the module category of a tame hereditary algebra. For $g_{\mathbb{X}} > 1$ the classification problem is known to be wild. It is elementary to see that $g_{\mathbb{X}} = 1$ if and only if

$$\mathbf{p} \in \{(6,3,2),\ (4,4,2),\ (3,3,3),\ (2,2,2,2)\}.$$

In this case $\mathbb{X}$ is called *tubular*. We have then $p\vec{\omega} = 0$ so that $\langle \mathbf{h}_0, [\mathcal{O}] \rangle = 0$ and the degree simplifies to $\deg \mathbf{e} = \langle \mathbf{h}_0, \mathbf{e} \rangle$. The type $(2,2,2,2)$ (resp. $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$) is closely related to the Dynkin diagram D_4 (resp. E_6, E_7 and E_8), see Remark 4.4, and therefore we shall say that $\mathbb{X}$ is of D-*type* (resp. of E-*type*).

Lenzing and his collaborators [15, Thm. 5.6] and [23, Thm. 4.6], showed the following fundamental result:

Theorem 2.5. *Let $\mathbb{X}$ be a tubular weighted projective line. Then:*

(a) *For any $q \in \mathbb{Q}_\infty$ the subcategory $\mathcal{C}_q \subset \operatorname{coh}\mathbb{X}$ (see Proposition 2.4) is stable under Auslander-Reiten translation and it is equivalent to $\mathcal{C}_\infty$.*
(b) *If $E \in \operatorname{coh}\mathbb{X}$ is indecomposable then $E \in \mathcal{C}_q$ for some $q \in \mathbb{Q}_\infty$ and $[E] \in K_0(\mathbb{X})$ is a positive real or isotropic positive root.*
(c) *If $0 \prec \mathbf{e} \in K_0(\mathbb{X})$ and $\langle \mathbf{e}, \mathbf{e} \rangle \in \{0, 1\}$ then $\mathbf{e}$ is a real or isotropic root.*
(d) *If $\mathbf{e} \in K_0(\mathbb{X})$ is a positive real root there exists up to isomorphism a unique indecomposable $E \in \operatorname{coh}\mathbb{X}$ with $[E] = \mathbf{e}$. If $\mathbf{f} \in K_0(\mathbb{X})$ is a positive*

isotropic root, there exists a $\mathbb{P}^1$*-family of indecomposable sheaves* $(F_\mu)_{\mu\in\mathbb{P}^1}$ *with* $[F_\mu] = \mathbf{f}$.

Remark 2.6. If $g_{\mathbb{X}} < 1$ it is well-known that similar statements to (b), (c) and (d) hold true. For $g_{\mathbb{X}} > 1$ a description of the positive roots, paralleling Kac's theorem for representations of quivers, was found recently by Crawley-Boevey [7].

2.5. **Positive Schur roots in the tubular case.** Let $\mathbb{X}$ be a tubular weighted projective line.

Definition 2.7. For $q \in \mathbb{Q}_\infty$ we define $a(q) \in \mathbb{Z}$ and $b(q) \in \mathbb{Z}_{\geq 0}$ such that $\gcd(a(q), b(q)) = 1$ and $q = \frac{a(q)}{b(q)}$. Furthermore we define $\mathbf{h}_q = b(q)\mathbf{h}_0 + a(q)\mathbf{h}_\infty$.

Note that $\mathbf{h}_q$ is a positive isotropic root with $\operatorname{slope}(\mathbf{h}_q) = q$.

For $\mathbf{e} \in K_0(\mathbb{X})$ we write $\tau^{\mathbb{Z}}\mathbf{e} = \{\tau^i\mathbf{e} \mid i \in \mathbb{Z}\}$. Since $\mathbb{X}$ is tubular $\tau^{\mathbb{Z}}\mathbf{e}$ is a finite set of cardinality $|\tau^{\mathbb{Z}}\mathbf{e}|$ which is a divisor of p (recall $p\vec{\omega} = 0$), and $\Sigma\tau^{\mathbb{Z}}\mathbf{e} := \sum_{\mathbf{e}'\in\tau^{\mathbb{Z}}\mathbf{e}} \mathbf{e}'$ is well defined.

Lemma 2.8. *For* $0 \prec \mathbf{e} \in K_0(\mathbb{X})$ *with* $\operatorname{slope}(\mathbf{e}) = q \in \mathbb{Q}_\infty$ *there exists a positive integer* $\operatorname{ql}(\mathbf{e})$ *such that* $\Sigma\tau^{\mathbb{Z}}\mathbf{e} = \operatorname{ql}(\mathbf{e})\mathbf{h}_q$.

Proof. Since $\mathbb{X}$ is tubular, $\operatorname{slope}(\mathbf{e}) = \operatorname{slope}(\tau\mathbf{e}) = q$, thus $\operatorname{slope}(\Sigma\tau^{\mathbb{Z}}\mathbf{e}) = q$. Moreover $\Sigma\tau^{\mathbb{Z}}\mathbf{e}$ is positive and $\tau(\Sigma\tau^{\mathbb{Z}}\mathbf{e}) = \Sigma\tau^{\mathbb{Z}}\mathbf{e}$, so $\langle\Sigma\tau^{\mathbb{Z}}\mathbf{e}, \Sigma\tau^{\mathbb{Z}}\mathbf{e}\rangle = 0$. Our claim follows now from [23, Lemma 2.6]. □

Proposition 2.9. *Let* $\mathbb{X}$ *be a tubular weighted projective line. The positive isotropic Schur roots are precisly of the form* $\mathbf{h}_q$ *with* $q \in \mathbb{Q}_\infty$. *Moreover, for* $0 \prec \mathbf{e} \in K_0(\mathbb{X})$ *the following are equivalent:*

(a) $\mathbf{e}$ *is a real positive Schur root,*
(b) *there exists an indecomposable* $E \in \operatorname{coh}\mathbb{X}$ *with* $\operatorname{Ext}^1_{\mathbb{X}}(E, E) = 0$ *and* $[E] = \mathbf{e}$,
(c) $\langle\mathbf{e}, \mathbf{e}\rangle = 1$ *and* $\operatorname{ql}(\mathbf{e}) < |\tau^{\mathbb{Z}}\mathbf{e}|$,

Proof. The equivalence of (a) and (b) was stated in Proposition 2.3. By Proposition 2.1 and Theorem 2.5(a) the Auslander-Reiten quiver of $\operatorname{coh}\mathbb{X}$ consists only of standard tubes. It is well known, that any indecomposable E in a standard tube of rank r has trivial endomorphism rings if and only if the quasi-length of E is less or equal to r.

Now, $\mathbf{h}_q$ is by [23, Lemma 2.6] and Theorem 2.5(c) the smallest positive isotropic root of slope q. By the above observation it is a Schur root and all other isotropic roots of slope q are not Schur.

It remains to show that in case $[E]$ is a real root $\operatorname{ql}([E])$ is the quasi-length of E. This follows from [23, Thm. 4.6(iv)] applied to the family representing $\mathbf{h}_q$. □

2.6. **Ext-orthogonality in the tubular case.** Let $\mathbb{X}$ be a tubular weighted projective line and E, F be two indecomposable rigid sheaves. We call E and F *Ext-orthogonal* or *compatible* if $E \oplus F$ is rigid. We denote by $\mathbf{e} = [E]$ and $\mathbf{f} = [F]$ their classes in the Grothendieck group. We will show that the condition for E and F to be Ext-orthogonal can be checked using only $\mathbf{e}$ and $\mathbf{f}$ with simple arguments and the homological form.

First of all, slope($\mathbf{e}$) and slope($\mathbf{f}$) are calculated using the homological form. If slope($\mathbf{e}$) $<$ slope($\mathbf{f}$) then $E \oplus F$ is rigid if and only if $\langle \mathbf{f}, \mathbf{e} \rangle = 0$. It remains to consider the case where $q :=$ slope($\mathbf{e}$) $=$ slope($\mathbf{f}$). Since the indecomposable sheaves with slope q form a family of pairwise orthogonal tubes $E \oplus F$ is rigid whenever E and F belong to different tubes. The latter happens if and only if $\langle \mathbf{e}, \tau^i \mathbf{f} \rangle = 0$ for all $i = 0, \ldots, r-1$ where $r = |\tau^{\mathbb{Z}} \mathbf{f}|$.

It remains to characterize when two indecomposable rigid sheaves of the same tube are Ext-orthogonal. For this we recall that each tube is an abelian category whose quasi-simples form an orbit $\tau^{\mathbb{Z}} S$ under the Auslander-Reiten translation τ of a single quasi-simple object S. Now, each indecomopsable rigid sheaf E defines $\mathrm{B}(E)$, the set of quasi-simples which occur as composition factor of E, more precisely, if i is minimal with $\tau^i S \in \mathrm{B}(E)$ then $\mathrm{B}(E) = \{\tau^{i+j} S \mid j = 0, \ldots, \mathrm{ql}(\mathbf{e}) - 1\}$. It is not hard to see that E and F are Ext-orthogonal if and only if one of the following three conditions is satisfied: (i) $\mathrm{B}(E) \subseteq \mathrm{B}(F)$, (ii) $\mathrm{B}(F) \subseteq \mathrm{B}(E)$ or (iii) $\mathrm{B}(E) \cap \tau^h \mathrm{B}(F) = 0$ for $h = -1, 0, 1$.

In the following we give equivalent conditions on the vectors $\mathbf{e}$ and $\mathbf{f}$ for each of these cases. We first show how the values of $\langle \mathbf{f}, - \rangle$ and $\langle -, \mathbf{f} \rangle$ vary in the tube. We have shown this in Figure 2 where the bottom dotted line indicates the τ-orbit of the quasi-simples and the dotted line on the top the τ-orbit of indecomposable rigid objects of maximal quasi-length. We have indicated the values as vectors $\begin{bmatrix} \langle -, \mathbf{f} \rangle \\ \langle \mathbf{f}, - \rangle \end{bmatrix}$. The region of indecomposable rigid objects which are Ext-orthogonal to F is shown gray.

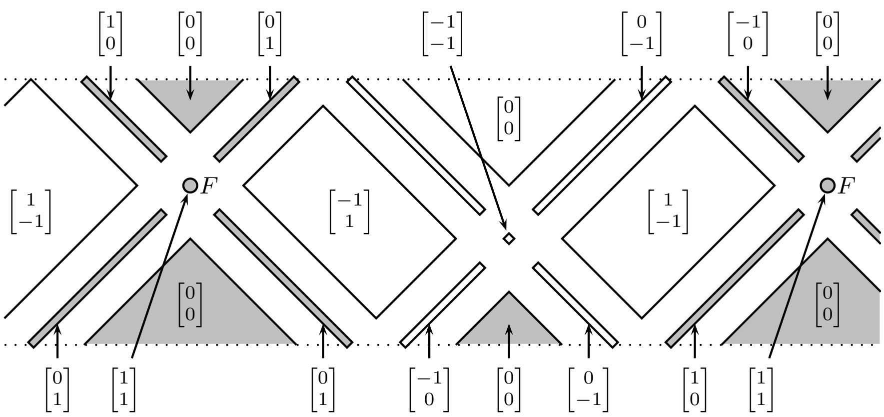


FIGURE 2. Ext-orthogonality in a tube

Note that in the cases (i) and (ii) we have $\langle \mathbf{e}, \mathbf{f} \rangle \geq 0$ and $\langle \mathbf{f}, \mathbf{e} \rangle \geq 0$ and in case (iii) we have $\langle \mathbf{e}, \mathbf{f} \rangle = 0 = \langle \mathbf{f}, \mathbf{e} \rangle$. Now, if $\langle \mathbf{e}, \mathbf{f} \rangle > 0$ and $\langle \mathbf{f}, \mathbf{e} \rangle = 0$ (or $\langle \mathbf{e}, \mathbf{f} \rangle = 0$ and $\langle \mathbf{f}, \mathbf{e} \rangle > 0$) then E and F lie on the same ray or coray and (i) or (ii) is satisfied. If $\langle \mathbf{e}, \mathbf{f} \rangle = 0 = \langle \mathbf{f}, \mathbf{e} \rangle$ then let j be minimal with $\langle \tau^j \mathbf{e}, \mathbf{f} \rangle \neq 0$. If $\langle \tau^j \mathbf{e}, \mathbf{f} \rangle > 0$ then again (i) or (ii) is satisfied whereas if $\langle \tau^j \mathbf{e}, \mathbf{f} \rangle < 0$ then (iii) holds if and only if $\mathrm{ql}(\mathbf{e}) + \mathrm{ql}(\mathbf{f}) < |\tau^{\mathbb{Z}} \mathbf{f}|$.

Altogether we have proved the following statement.

Proposition 2.10. *Let E and F be two indecomposable rigid sheaves with classes $\mathbf{e} = [E]$ and $\mathbf{f} = [F]$. Then E and F are Ext-orthogonal if and only if one of the following conditions is satisfied.*

(a) $\operatorname{slope}(\mathbf{e}) < \operatorname{slope}(\mathbf{f})$ *and* $\langle \mathbf{f}, \mathbf{e}\rangle = 0$,
(b) $\operatorname{slope}(\mathbf{e}) > \operatorname{slope}(\mathbf{f})$ *and* $\langle \mathbf{e}, \mathbf{f}\rangle = 0$,
(c) $\operatorname{slope}(\mathbf{e}) = \operatorname{slope}(\mathbf{f})$ *and* $\langle \tau^j \mathbf{e}, \mathbf{f}\rangle = 0$ *for* $j = 0, \dots, \operatorname{ql}(\mathbf{e}) - 1$,
(d) $\operatorname{slope}(\mathbf{e}) = \operatorname{slope}(\mathbf{f})$ *and* $\langle \mathbf{e}, \mathbf{f}\rangle \geq 0$, $\langle \mathbf{f}, \mathbf{e}\rangle \geq 0$ *but not both zero,*
(e) $\operatorname{slope}(\mathbf{e}) = \operatorname{slope}(\mathbf{f})$ *and* $\langle \mathbf{e}, \mathbf{f}\rangle = 0 = \langle \mathbf{f}, \mathbf{e}\rangle$ *and there exists a j such that* $\langle \tau^j \mathbf{e}, \mathbf{f}\rangle \neq 0$. *If this j is minimal then either*
  (e1) $\langle \tau^j \mathbf{e}, \mathbf{f}\rangle > 0$, *or*
  (e2) $\langle \tau^j \mathbf{e}, \mathbf{f}\rangle < 0$ *and* $\operatorname{ql}(\mathbf{e}) + \operatorname{ql}(\mathbf{f}) < |\tau^{\mathbb{Z}}\mathbf{e}|$.

2.7. **Rigid objects and cluster-tilting objects.** A rigid sheaf $E \in \operatorname{coh}\mathbb{X}$ is called a *tilting sheaf* (or *maximal rigid*) if it is maximal among the rigid sheaves in the following sense: any sheaf F such that $E \oplus F$ is again rigid belongs to the additive closure $\operatorname{Add} E$. Note that the number of (pairwise non-isomorphic) indecomposable direct summands of any tilting sheaf equals the rank $n = 2 + \sum_{i=1}^{t}(p_i - 1)$ of the Grothendieck group. A rigid sheaf with $n - 1$ (pairwise non-isomorphic) indecomposable direct summands is called an *almost complete tilting sheaf*. An indecomposable sheaf E' such that $E \oplus E'$ is a tilting sheaf is called a *complement* of the almost complete tilting sheaf E. Two indecomposable rigid objects E and E' are called *compatible* if $E \oplus E'$ is rigid.

Following Keller [22], the orbit category $\mathcal{C}_{\mathbb{X}} := \mathcal{D}^b(\operatorname{coh}\mathbb{X})/\langle \tau^{-1}[1]\rangle$ associated to a weighted projective line is a triangulated 2-Calabi-Yau category.

Remark 2.11. As explained in [5], the composition of the canonical functors

$$\operatorname{coh}\mathbb{X} \xrightarrow{\text{incl.}} \mathcal{D}^b(\operatorname{coh}\mathbb{X}) \xrightarrow{\text{proj.}} \mathcal{C}_{\mathbb{X}}$$

allows to think of $\operatorname{coh}\mathbb{X}$ as a non-full subcategory of $\mathcal{C}_{\mathbb{X}}$ which has the same isoclasses of indecomposable resp. rigid objects. It follows that the tilting objects in $\operatorname{coh}\mathbb{X}$ correspond bijectively with the *cluster-tilting objects* in $\mathcal{C}_{\mathbb{X}}$, that is, the maximal rigid objects in $\mathcal{C}_{\mathbb{X}}$.

By [21, Prop. 5.14] each almost complete tilting sheaf in $\operatorname{coh}\mathbb{X}$ has precisely two complements. It follows that the exchange graph for tilting objects in $\operatorname{coh}\mathbb{X}$ and the exchange graph for cluster tilting objects can be identified. By [5], the cluster category $\mathcal{C}_{\mathbb{X}}$ has a cluster structure in the sense of [6].

By Proposition 2.3 the positive real Schur roots parametrize the indecomposable rigid objects in $\mathcal{C}_{\mathbb{X}}$.

3. Tubular cluster categories

In case the weighted projective line $\mathbb{X}$ is tubular we call $\mathcal{C}_{\mathbb{X}}$ the corresponding *tubular cluster category*. Moreover, we note, that by [5, Thm. 8.8] in this case the exchange graph of cluster-tilting objects is connected.

3.1. **GLS-cluster categories and -character.** Let us review some of the results of [1] and [17]: Let Q be a quiver without oriented cycles and n vertices and denote by $\mathbb{C}Q$ the path algebra of Q. Moreover, let $M_1, \dots, M_r$ be a family of indecomposable, pairwise non-isomorphic preinjective representations of Q, closed under successors and such that $M = \oplus_{i=1}^{r} M_i = \bar{M} \oplus I$ for an injective cogenerator I of $\mathbb{C}Q$- mod. Note, that $E_{\bar{M}} = \mathrm{End}_Q(\bar{M})$ is a basic algebra of global dimension 2. The Gabriel quiver $\widetilde{Q}_{\bar{M}}$ of $E_{\bar{M}}$ is given by the full subquiver of the preinjective component of $\mathbb{C}Q$ which is supported in the summands of $\bar{M}$.

Let Λ be the preprojective algebra associated to the path algebra $\mathbb{C}Q$. Since $\mathbb{C}Q$ is a subalgebra of Λ we have the restriction functor $?|_Q \colon \Lambda\text{- mod} \to \mathbb{C}Q\text{- mod}$. Recall from [20] that a finite dimensional algebra A is called *piecewise hereditary* if $\mathcal{D}^b(A\text{- mod})$ is triangle-equivalent to $\mathcal{D}^b(\mathcal{H})$ for $\mathcal{H}$ a connected abelian hereditary k-category and a tilting object. Note that by [19], the only relevant cases are where $\mathcal{H} = H$- mod for some hereditary algebra H or $\mathcal{H} = \mathrm{coh}\,\mathbb{X}$ for some weighted projective line $\mathbb{X}$.

Theorem 3.1. *For the subcategory* $\mathcal{C}_M := \{X \in \Lambda\text{- mod} \mid X|_Q \in \mathrm{Add}(M)\}$ *the following holds:*

(a) $\mathcal{C}_M$ *is Frobenius category. The stable category* $\underline{\mathcal{C}}_M$ *is a triangulated 2-Calabi-Yau category with a basic cluster-tilting object* $T_M = \oplus_{i=1}^{r-n} T_i$ *such that the quiver* $\widehat{Q}_{\bar{M}}$ *of* $\mathrm{End}_{\underline{\mathcal{C}}_M}(T_M)$ *is obtained from* $\widetilde{Q}_{\bar{M}}$ *by inserting extra arrows* $M_i \to M_j$ *whenever* $M_j \cong \tau_Q M_i$.
(b) *We have a cluster character* $\varphi_? \colon \mathrm{obj}(\underline{\mathcal{C}}_M) \to A_M$ *in the sense of* [25, Def. 2] *with the following additional properties:*
 (i) A_M *is a finitely generated cluster algebra with trivial coefficients and initial seed* $((\varphi_{T_i})_{i=1,\dots,r-n}, \widehat{Q}_{\bar{M}})$.
 (ii) *The family* (φ_X) *where* X *runs over the isoclasses of rigid objects in* $\underline{\mathcal{C}}_M$, *is linearly independent in* A_M.
(c) *If* $E_{\bar{M}}$ *is piecewise hereditary, then* $\underline{\mathcal{C}}_M$ *is triangle equivalent to the cluster category* $\mathcal{D}^b(E_{\bar{M}}\text{- mod})/\langle \tau_{\mathcal{D}}^{-1}[1]\rangle$.

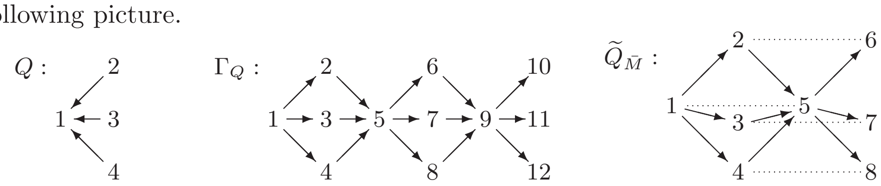
Remark 3.2. Part (a) of the theorem is a consequence of Theorems 2.1, 2.2 and 2.3 in [17]. Part (b) follows easily from the discussion in the Sections 3.2, 3.4. and 3.6 of [17]. Part (c) follows from [1, Theorem 5.15]. Note, that Amiot's category $\mathcal{C}_A$ is by construction the triangulated hull of the orbit category $\mathcal{D}^b(E_{\bar{M}}\text{- mod})/\langle \tau_{\mathcal{D}}^{-1}[1]\rangle$, however if A is piecewise hereditary already the orbit category is triangulated by [22].

Remark 3.3. The above theorem implies the following: Suppose $E_{\bar{M}}$ is derived equivalent to $\mathrm{coh}\,\mathbb{X}$ for a tubular weighted projective line $\mathbb{X}$, then by (c) the stable category $\underline{\mathcal{C}}_M$ is triangle equivalent to the cluster category $\mathcal{C}_{\mathbb{X}}$. Since in this case the exchange graph of the cluster tilting object is connected we obtain by (b) a bijection between the positive real Schur roots and cluster variables. We will see in the next subsection that this situation is given for the tubular E-types $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$.

Moreover, two cluster variables belong to a common cluster if and only if the corresponding positive real Schur roots are Ext-orthogonal.

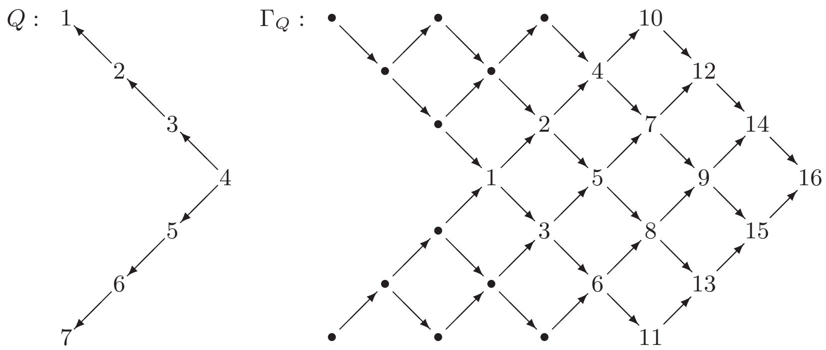
3.2. **The tubular E-cases.** We are now ready to give an explicit proof in each of the three E-cases that the tubular cluster algebra is categorified by $\mathrm{coh}\,\mathbb{X}$ for $\mathbb{X}$ a weighted projective line of weight type $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$ respectively.

3.2.1. Let Q be the quiver as shown in the following figure on the left. The Auslander-Reiten quiver Γ_Q of Q has then the shape as shown in the middle of following picture.

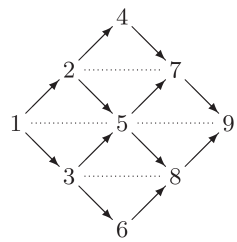


We choose $M = \oplus_{i=1}^{12} M_i$ the direct sum of all indecomposable modules. Thus, $E_{\bar{M}}$ is given by the quiver $\widetilde{Q}_{\bar{M}}$ with relations as shown in the previous picture on the right. Use [2, Thm. A] to check that this algebra is derived equivalent to a tubular algebra of type $(3,3,3)$. Note that condition (ii) can be verified computationally, see [3]. The algebra $E_{\bar{M}}$ is in fact tubular as can be seen using the techniques of tubular extensions (which later also were called branch-enlargements) of tame concealed algebras described in [26, Sect. 4.7], although we don't need that. It follows that the bounded derived category of $E_{\bar{M}}$ is triangle-equivalent to coh $\mathbb{X}$ for $\mathbb{X}$ of weight type $(3,3,3)$ by [15, Thm. 3.2, Prop. 4.1]. In particular $E_{\bar{M}}$ is piecewise hereditary. Recall that $\widehat{Q}_{\bar{M}}$ denotes the quiver of $\mathrm{End}_{\mathcal{C}_{\mathbb{X}}}(\bar{M})$. A straightforward check shows that $\mu_2\mu_3\mu_4(\widehat{Q}_{\bar{M}}) = \Delta$, where Δ is the quiver in Figure 1 associated to $\mathsf{E}_6^{(1,1)}$.

3.2.2. Let Q be the quiver as shown in the following figure on the left. The Auslander-Reiten quiver Γ_Q of Q has then the shape as shown in the following picture on the right.

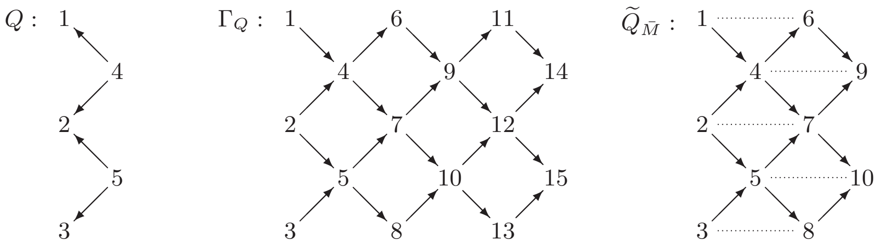


We choose $M = \oplus_{i=1}^{16} M_i$ with M_i corresponding to the vertex marked by i in the Auslander-Reiten quiver. Then $E_{\bar{M}}$ is given by the following quiver with relations:



Again use [2] and [15] to check that $E_{\bar{M}}$- mod is derived equivalent to coh $\mathbb{X}$ for $\mathbb{X}$ of weight type $(4,4,2)$. In particular, $E_{\bar{M}}$ is piecewise hereditary. A straightforward check shows that $\mu_5\mu_2\mu_8\mu_1\mu_9\mu_5\mu_7\mu_3(\widehat{Q}_{\bar{M}}) = \Delta$, where Δ is the quiver in Figure 1 associated to $\mathsf{E}_7^{(1,1)}$.

3.2.3. Let Q be the quiver as shown in the following figure on the left. The Auslander-Reiten quiver Γ_Q of Q has then the shape as shown in the middle of the following picture.



We choose $M = \oplus_{i=1}^{15} M_i$ the direct sum of all indecomposable modules. Thus, $E_{\bar{M}}$ is given by the following quiver $\widetilde{Q}_{\bar{M}}$ with relations as shown in the previous picture on the right. Again, [2] and [15] can be used to check that $E_{\bar{M}}$- mod is derived equivalent to coh $\mathbb{X}$ where $\mathbb{X}$ is a weighted projective line of weight type $(6,3,2)$ (in fact $E_{\bar{M}}$ is tubular as can be seen using [26, Sect. 4.7]), in particular it is piecewise hereditary. A straightforward check shows that $\mu_2\mu_5\mu_4\mu_{10}\mu_9\mu_8\mu_3\mu_5\mu_7\mu_5\mu_9\mu_8\mu_3\mu_6\mu_1(\widehat{Q}_{\bar{M}}) = \Delta$, where Δ is the quiver in Figure 1 associated to $\mathsf{E}_8^{(1,1)}$.

3.3. **The tubular D-case.** Let us point out first that in this case the method from 3.1 can't work: The derived tubular algebras of type $(2,2,2,2)$ are well-known. The 9 types are listed for example in [2, p. 652] as quivers with relations. A quick check shows that in each case there is either a vertex in the quiver where 2 relations start resp. end, or there is a relation of length 3. Thus, none of these algebras can be of the form $E_{\bar{M}}$ for any quiver Q and a terminal $\mathbb{C}Q$-module M. Moreover, we were unable to find a Hom-finite Frobenius category $\mathcal{F}$ such that the stable category $\underline{\mathcal{F}}$ is equivalent to the cluster category $\mathcal{C}_{\mathbb{X}}$ where $\mathbb{X}$ is a weighted projective line of type $(2,2,2,2)$.

In any case, we can use Palu's cluster character [25]

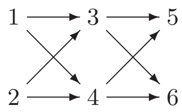
$$X_?\colon \mathcal{C}_{\mathbb{X}} \to \mathbb{C}[x_1^{\pm 1}, \cdots, x_6^{\pm 1}].$$

Since the exchange graph for cluster tilting objects in $\mathcal{C}_{\mathbb{X}}$ is connected [5, Thm. 8.8], $X_?$ induces a *surjection* from the indecomposable rigid objects in $\mathcal{C}_{\mathbb{X}}$ (which are in bijection with the positive real Schur roots) and the corresponding tubular cluster algebra.

4. Combinatorics of real Schur roots in the case $(2,2,2,2)$

If not otherwise mentioned, we have in this section $\boldsymbol{\lambda} = ([1:0],[0:1],[1:1],[\rho:1])$ and $\mathbb{X} = (\mathbb{P}^1, \boldsymbol{\lambda}, (2,2,2,2))$.

4.1. **A coordinate system.** It is well-known that in this situation there exists a tilting object $T \in \operatorname{coh}\mathbb{X}$ such that the Gabriel quiver of $A = \operatorname{End}_{\mathbb{X}}(T)$ has the following shape



(4.1) ,

see for example [2]. Then the triangle equivalence

$$\mathbf{R}\operatorname{Hom}_{\mathbb{X}}(T,-)\colon \mathcal{D}^b(\operatorname{coh}\mathbb{X}) \to \mathcal{D}^b(A\text{-}\operatorname{mod})$$

induces an isometry between $(K_0(\mathbb{X}), \langle -,-\rangle)$ and $(K_0(A\text{-}\operatorname{mod}), \langle -,-\rangle_A)$ where

$$\langle \underline{\dim} X, \underline{\dim} Y\rangle_A = \dim \operatorname{Hom}_A(X,Y) - \dim \operatorname{Ext}^1_A(X,Y) + \dim \operatorname{Ext}^2_A(X,Y)$$

is the homological form for A. As usual we will use in $K_0(A\text{-}\operatorname{mod})$ the basis given by the simple A-modules. We will describe the positive real Schur roots and their combinatorics in this coordinate system. We denote a vector $\mathbf{v} \in \mathbb{Z}^6$ usually by $\mathbf{v} = \begin{bmatrix} \mathbf{v}(1) & \mathbf{v}(3) & \mathbf{v}(5) \\ \mathbf{v}(2) & \mathbf{v}(4) & \mathbf{v}(6) \end{bmatrix}$.

4.2. **Description of the basic roots.** In the the following we shall use the quaternion group

$$H = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. Let $H^+ = 1, i, j, k$.

Let

$$\mathbf{h}_0 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{h}_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_\infty = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Observe that $\langle \mathbf{h}_0, \mathbf{h}_\infty\rangle = -\langle \mathbf{h}_\infty, \mathbf{h}_0\rangle = 2$.

Furthermore we define vectors $\mathbf{v}_q^x$ for $q = 0, 1, \infty$ and $x \in H^+$ as follows:

$$\mathbf{v}_0^1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_1^1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{v}_\infty^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{v}_0^i = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_1^i = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad \mathbf{v}_\infty^i = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{v}_0^j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{v}_1^j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_\infty^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{v}_0^k = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{v}_1^k = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{v}_\infty^k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $x \in H^+$ and $q = 0, 1, \infty$ define

$$\mathbf{v}_q^{-x} = \mathbf{h}_q - \mathbf{v}_q^x.$$

The homological form can be calculated explicitly by

$$\langle x, y\rangle = x^\top \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} y.$$

The following result is crucial for the forthcoming.

Lemma 4.1. *The following formula hold for all $x, h \in H$.*

$$\langle \mathbf{v}_0^x, \mathbf{v}_1^{hx} \rangle = \langle \mathbf{v}_1^x, \mathbf{v}_\infty^{hx} \rangle = \langle \mathbf{v}_0^{-hx}, \mathbf{v}_\infty^x \rangle = \begin{cases} 1, & \text{if } h \in H^+, \\ 0, & \text{if } h \notin H^+; \end{cases} \tag{4.2}$$

and also

$$\langle \mathbf{v}_0^x, \mathbf{h}_\infty \rangle = \langle \mathbf{v}_1^x, \mathbf{h}_\infty \rangle = \langle \mathbf{h}_0, \mathbf{v}_1^x \rangle = \langle \mathbf{h}_0, \mathbf{v}_\infty^x \rangle = 1. \tag{4.3}$$

Proof. This is an immediate (although lengthy) calculation. □

4.3. **Description of all real Schur roots.** Let $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ and for $q \in \mathbb{Q}_\infty$ the numbers $a(q) \in \mathbb{Z}$ and $b(q) \in \mathbb{Z}_{\geq 0}$ are defined by the properties $\gcd(a(q), b(q)) = 1$ and $q = \frac{a(q)}{b(q)}$. Furthermore, for $q \in \mathbb{Q}_\infty$ we define its *type* by

$$\mathrm{t}(q) = \begin{cases} 0, & \text{if } a(q) \equiv 0,\, b(q) \equiv 1 \mod 2, \\ 1, & \text{if } a(q) \equiv 1,\, b(q) \equiv 1 \mod 2, \\ \infty, & \text{if } a(q) \equiv 1,\, b(q) \equiv 0 \mod 2. \end{cases}$$

For $n \in \mathbb{Z}$ let

$$n^{(+)} = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

For $q \in \mathbb{Q}_\infty$ and $x \in H$ define

$$\mathbf{v}_q^x = \mathbf{v}_{\mathrm{t}(q)}^x + b(q)^{(+)}\mathbf{h}_0 + a(q)^{(+)}\mathbf{h}_\infty \quad \text{and} \quad \mathbf{h}_q = b(q)\mathbf{h}_0 + a(q)\mathbf{h}_\infty. \tag{4.4}$$

Lemma 4.2. *We have* $\mathrm{slope}(\mathbf{h}_q) = \mathrm{slope}(\mathbf{v}_q^x) = q$ *for each* $q \in \mathbb{Q}_\infty$ *and* $x \in H$. *In particular* $\langle \mathbf{v}_q^x, \mathbf{h}_q \rangle = \langle \mathbf{h}_q, \mathbf{v}_q^x \rangle = 0$. *Furthermore, we have* $\mathbf{v}_q^x + \mathbf{v}_q^{-x} = \mathbf{h}_q$ *and* $\mathbf{v}_p^x = \mathbf{v}_q^y$ *implies* $p = q$ *and* $x = y$.

Proof. $\mathrm{slope}(\mathbf{h}_q) = q$ follows directly from the definition. Moreover, we have $\langle \mathbf{h}_0, \mathbf{v}_q^x \rangle = \langle \mathbf{h}_0, \mathbf{v}_{\mathrm{t}(q)}^x \rangle + a(q)^{(+)} \langle \mathbf{h}_0, \mathbf{h}_\infty \rangle = \langle \mathbf{h}_0, \mathbf{v}_{\mathrm{t}(q)}^x \rangle + 2a(q)^{(+)} = a(q)$, where the last equation can easily be verified in each of the three cases $q = 0, 1, \infty$. Similarly $\langle \mathbf{v}_q^x, \mathbf{h}_\infty \rangle = b(q)$ and therefore $\mathrm{slope}(\mathbf{v}_q^x) = q$.

If $\mathrm{t}(q) = 0$ then $\mathbf{v}_q^x + \mathbf{v}_q^{-x} = \mathbf{v}_0^x + \mathbf{v}_0^{-x} + (b(q) - 1)\mathbf{h}_0 + a(q)\mathbf{h}_\infty = \mathbf{h}_q$, the latter since $\mathbf{v}_0^x + \mathbf{v}_0^{-x} = \mathbf{h}_0$. Similarly $\mathbf{v}_q^x + \mathbf{v}_q^{-x} = \mathbf{h}_q$ is verified in the two remaining cases where $\mathrm{t}(q) = 1, \infty$. □

We denote by $\mathcal{E}$ the set of indecomposable rigid objects in $\operatorname{coh} \mathbb{X}$.

Proposition 4.3. *The set of positive real Schur roots is* $\{\underline{\dim} E \mid E \in \mathcal{E}\} = \{\mathbf{v}_q^x \mid q \in \mathbb{Q}_\infty, x \in H\}$.

Proof. By Proposition 2.9 $\{\underline{\dim} E \mid E \in \mathcal{E}\}$ is precisely the set of the real positive Schur roots. Since $\mathbb{X}$ is of type $(2,2,2,2)$ we have by Proposition 2.9 and Theorem 2.5 that $0 \prec \mathbf{e} \in K_0(\mathbb{X})$ is a real positive Schur root if and only if $\langle \mathbf{e}, \mathbf{e} \rangle = 1$ and $\mathrm{ql}(\mathbf{e}) = 1$. Moreover, for each $q \in \mathbb{Q}_\infty$ there are precisely 8 positive real Schur roots $\mathbf{e}$ with $\mathrm{slope}(\mathbf{e}) = q$. By Lemma 4.2 the $\mathbf{v}_q^x$ with $x \in H$ are precisely 8 elements with the required properties. □

Remark 4.4. Note, that for our description of the real Schur roots the adequate choice of $24 = 4 \times (2^2 - 1) \times 2$ basic roots was essential. This is the number of roots for D_4.

In [16, Sec. 15] a similar description of the real Schur roots was discussed for the tubular case $(6, 3, 2)$. In that case $240 = (6^2 - 1) \times 6 + (3^2 - 1) \times 3 + (2^2 - 1) \times 2$ have to be chosen properly. This is the number of roots for E_8.

For the tubular cases $(4, 4, 2)$ and $(3, 3, 3)$ one would need to find $126 = 2 \times (4^2 - 1) \times +(2^2 - 1) \times 2$ respectively $3 \times (3^2 - 1) \times 3 = 72$ basic roots, that is, precisely the number of roots in a root system of type E_7 respectively E_6.

4.4. **Orthogonality.** In the forthcoming we shall need the following definition of "distance" between different slopes.

Definition 4.5. For $p, q \in \mathbb{Q}_\infty$ we define

$$\Delta(p, q) = |a(q)b(p) - a(p)b(q)|.$$

Lemma 4.6. *Let $p, q \in \{0, 1, \infty\}$ and $x, y \in H$. Then*

$$\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = 0 \Leftrightarrow \begin{cases} x = y, & \text{if } \mathrm{t}(p) = \mathrm{t}(q), \\ y \in -H^+x, & \text{if } (p, q) = (0, 1), (1, \infty), (\infty, 0), \\ x \in H^+y, & \text{if } (p, q) = (1, 0), (\infty, 1), (0, \infty). \end{cases}$$

Proof. This follows directly from Lemma 4.1 and the fact that $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = \langle \Phi \mathbf{v}_q^y, \mathbf{v}_p^x \rangle = \langle \mathbf{v}_q^{-y}, \mathbf{v}_p^x \rangle$. □

Proposition 4.7. *Let $p, q \in \mathbb{Q}_\infty$ and $x, y \in H$. If $p > q$ then $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = 0$ if and only if the condition of the corresponding cell in the following table is satisfied.*

(4.5)

	$\mathrm{t}(q) = 0$	$\mathrm{t}(q) = 1$	$\mathrm{t}(q) = \infty$
$\mathrm{t}(p) = 0$	$x = y$ $\Delta(p, q) = 2$	$y \in H^+x$ $\Delta(p, q) = 1$	$x \in -H^+y$ $\Delta(p, q) = 1$
$\mathrm{t}(p) = 1$	$x \in H^+y$ $\Delta(p, q) = 1$	$x = y$ $\Delta(p, q) = 2$	$y \in H^+x$ $\Delta(p, q) = 1$
$\mathrm{t}(p) = \infty$	$y \in -H^+x$ $\Delta(p, q) = 1$	$x \in H^+y$ $\Delta(p, q) = 1$	$x = y$ $\Delta(p, q) = 2$

Proof. First suppose $\mathrm{t}(p) = \mathrm{t}(q) = 0$. To calculate $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle$ use Definition 4.4, bilinearity and then Lemma 4.2 and equations (4.3) of Lemma 4.1 in order to obtain

$$\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = \langle \mathbf{v}_0^x, \mathbf{v}_0^y \rangle + \left(2b(q)^{(+)} + 1\right) a(p)^{(+)} - \left(2b(p)^{(+)} + 1\right) a(q)^{(+)}.$$

Observe that $\left(2b(p)^{(+)} + 1\right) = b(p)$, $\left(2b(q)^{(+)} + 1\right) = b(q)$ and $a(p)^{(+)} = \frac{a(p)}{2}$, $a(q)^{(+)} = \frac{a(q)}{2}$. Hence

$$\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = \langle \mathbf{v}_0^x, \mathbf{v}_0^y \rangle - \tfrac{a(p)b(q) - a(q)b(p)}{2} = \langle \mathbf{v}_0^x, \mathbf{v}_0^y \rangle - \tfrac{\Delta(p,q)}{2},$$

the latter since $p > q$, which implies that $a(p)b(q) - a(q)b(p) > 0$. Therefore $\langle \mathbf{v}_0^x, \mathbf{v}_0^y \rangle > 0$. This in turn implies $\langle \mathbf{v}_0^x, \mathbf{v}_0^y \rangle = 1$ thus $x = y$ and $\Delta(p, q) = 2$. This shows the result in that case. The other two cases where $\mathrm{t}(p) = \mathrm{t}(q)$ are quite similar.

Similarly all other cases are calculated. For instance if $\mathrm{t}(p) = 0$ and $\mathrm{t}(q) = 1$ then we calculate using similar arguments as above that $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = \langle \mathbf{v}_0^x, \mathbf{v}_1^y \rangle + \frac{-\Delta(p,q)-1}{2}$. Since the first summand is at most 1 and the second at most -1 we get that $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = 0$ if and only if $\langle \mathbf{v}_0^x, \mathbf{v}_1^y \rangle = 1$ and $\Delta(p,q) = -1$, which happens if and only if $y \in H^+x$ and $\Delta(p,q) = 1$. □

Corollary 4.8. *Let $p, q \in \mathbb{Q}_\infty$ and $x, y \in H$. If $p < q$ then $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = 0$ if and only if the condition of the corresponding cell in the following table is satisfied.*

	$\mathrm{t}(q) = 0$	$\mathrm{t}(q) = 1$	$\mathrm{t}(q) = \infty$
$\mathrm{t}(p) = 0$	$x = -y$ $\Delta(p,q) = 2$	$y \in -H^+x$ $\Delta(p,q) = 1$	$x \in H^+y$ $\Delta(p,q) = 1$
$\mathrm{t}(p) = 1$	$x \in -H^+y$ $\Delta(p,q) = 1$	$x = -y$ $\Delta(p,q) = 2$	$y \in -H^+x$ $\Delta(p,q) = 1$
$\mathrm{t}(p) = \infty$	$y \in H^+x$ $\Delta(p,q) = 1$	$x \in -H^+y$ $\Delta(p,q) = 1$	$x = -y$ $\Delta(p,q) = 2$

Proof. Observe that $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = -\langle \Phi \mathbf{v}_q^y, \mathbf{v}_p^x \rangle = -\langle \mathbf{v}_q^{-y}, \mathbf{v}_p^x \rangle$ and therefore the result follows from the previous proposition. □

Lemma 4.9. *We have $\langle \mathbf{v}_p^x, \mathbf{v}_p^y \rangle = 0$ if and only if $x \neq -y$.*

Proof. This follows by direct calculations for $p = 0, 1, \infty$ and in the general case using (4.4) from $\langle \mathbf{v}_p^x, \mathbf{v}_p^y \rangle = \langle \mathbf{v}_{\mathrm{t}(p)}^x, \mathbf{v}_{\mathrm{t}(p)}^y \rangle$ since the other summands cancel each other. □

Proposition 4.10. *Let $p, q \in \mathbb{Q}_\infty$ and $x, y \in H$. If $\langle \mathbf{v}_p^x, \mathbf{v}_q^y \rangle = 0 = \langle \mathbf{v}_q^y, \mathbf{v}_p^x \rangle$ then $p = q$ and $x \neq \pm y$.*

Proof. It follows from Proposition 4.7 and Corollary 4.8 that $p = q$. Then the result follows from the previous lemma. □

5. Triangulations of the sphere with 4 punctures

5.1. **Definition of triangulated surfaces.** In [11], Fomin, Shapiro and Thurston established a connection between cluster algebras and triangulated surfaces. We will use this approach for tubular cluster algebras of type $(2, 2, 2, 2)$.

Let $\mathbf{S}$ be an oriented 2-dimensional Riemann surface (possibly with boundary) and $\mathbf{M}$ a finite set of points in the closure of $\mathbf{S}$. An *arc* in $(\mathbf{S}, \mathbf{M})$ is (the homotopy class in $\mathbf{S} \setminus \mathbf{M}$ of) a curve without self-intersections connecting two marked points which is not contractible in $\mathbf{S} \setminus \mathbf{M}$ nor deformable into the boundary of $\mathbf{S}$. Also a curve and its inverse will be identified. Two arcs are *compatible* if they contain curves which do not intersect in $\mathbf{S} \setminus \mathbf{M}$. Each arc is compatible with itself. A maximal collection of pairwise compatible curves is called an *ideal triangulation* of $(\mathbf{S}, \mathbf{M})$.

Some of these arcs (namely those which are not loops enclosing a single marked point) are then enhanced to *tagged arcs*, that is, to each of its two endpoints one of the two labels "plain" or "notched", is attached, see [11] for details. Tagged arcs should be seen as a generalization of "plain" arcs. If β is a tagged arc we denote by β° the "underlying" untagged arc obtained from β by removing the labels on the endpoints. Recall from [11, Definition 7.4] that if $\beta^\circ \neq \gamma^\circ$ then β and γ are compatible if and only if the tags on common endpoints are the same and β° and γ° are compatible; and if $\beta^\circ = \gamma^\circ$ then then β and γ are compatible if and only if they share the same tag on at least one endpoint. A *tagged triangulation* is, by definition, a maximal collection of pairwise compatible tagged arcs. The advantage of the tags is that for each tagged triangulation T and each arc γ there exists a unique tagged arc $\gamma' \neq \gamma$ such that $\mu_\gamma(T) = T \setminus \{\gamma\} \cup \{\gamma'\}$ is again a tagged triangulation, called the *flip of T along γ*.

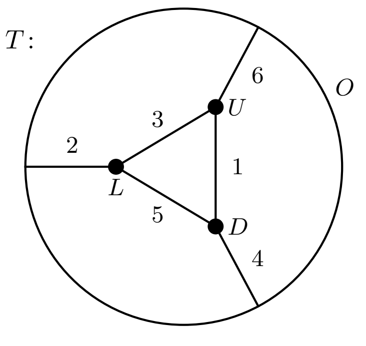
To each tagged triangulation T of a marked surface $(\mathbf{S}, \mathbf{M})$ an integer square matrix $B(T)$, indexed by the arcs of T, is assigned in such a way that the matrix mutation μ_k, as defined by Fomin-Zelevinsky in [12], corresponds to the *flip* of tagged triangulation, that is, $B(\mu_\gamma(T)) = \mu_\gamma(B(T))$. Consequently, the set of matrices $\mathcal{B}(\mathbf{S}, \mathbf{M})$ of matrices $B(T)$ as T varies through the tagged triangulation of $(\mathbf{S}, \mathbf{M})$, is a single mutation class of matrices.

Denote by $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ the *tagged arc complex*, that is, the clique complex on the set of tagged arcs given by the compatibility relation. The *tagged exchange graph* $\mathbf{E}^{\bowtie}(\mathbf{S}, \mathbf{M})$ is by definition the dual graph of $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$. We quote part of the main result, Theorem 7.11, of [11].

Theorem 5.1. *If $\mathcal{A}$ is a cluster algebra whose set of exchange matrices is $\mathcal{B}(\mathbf{S}, \mathbf{M})$ for some marked surface $(\mathbf{S}, \mathbf{M})$ then the cluster complex of $\mathcal{A}$ is isomorphic to $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ and the exchange graph of $\mathcal{A}$ is isomorphic to $\mathbf{E}^{\bowtie}(\mathbf{S}, \mathbf{M})$.*

As a consequence, in the situation of the previous result, the cluster variables of $\mathcal{A}$ are in bijection with the tagged arcs of $(\mathbf{S}, \mathbf{M})$.

Example 5.2. Let $(\mathbf{S}, \mathbf{M})$ be the 2-sphere with 4 punctures called L, U, D and O. We draw the situation as a disk with 3 punctures L, U, D, and represent the fourth puncture O by its boundary. Let T be the tagged triangulation (all tags are plain) showed in the picture below. The matrix $B(T)$ is shown on the right hand side. Notice that the associated quiver is exactly the one given in (4.1).



$$B(T) = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

5.2. **Special untagged arcs.** We now want to associate for each rational number p two untagged arcs α_q^+ and α_q^-, namely an *inner* arc α_q^+, connecting two of the three vertices L, U, D, and an *outer* arc α_q^-, connecting one of the vertices L, U, D with O. We start by doing so for $q = -1, 0$ and ∞.

Definition 5.3. We define α_q^+ and α_q^- for $q = -1, 0, \infty$ as shown in Figure 3.

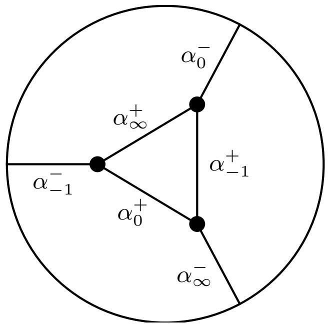


FIGURE 3. Untagged arcs of low complexity

To simplify notations we redefine here the *type* of $q \in \mathbb{Q}_\infty$ as follows:

$$\mathrm{t}'(q) = \begin{cases} 0, & \text{if } a(q) \equiv 0,\ b(q) \equiv 1 \mod 2, \\ -1, & \text{if } a(q) \equiv 1,\ b(q) \equiv 1 \mod 2, \\ \infty, & \text{if } a(q) \equiv 1,\ b(q) \equiv 0 \mod 2. \end{cases}$$

Notice that $\mathrm{t}'(q) = -\mathrm{t}(q)$.

We start by describing first the arc α_p^+ in the case where $p = \frac{r}{s}$ with $r, s > 0$ are coprime. In the first step we draw $r^{(+)}$ different semicircles with center U and $s^{(+)}$ different semicircles with center D, all of them open to the left and mutually non-intersecting. We also draw $(r+s)^{(+)}$ different semicircles with center L open to the right.

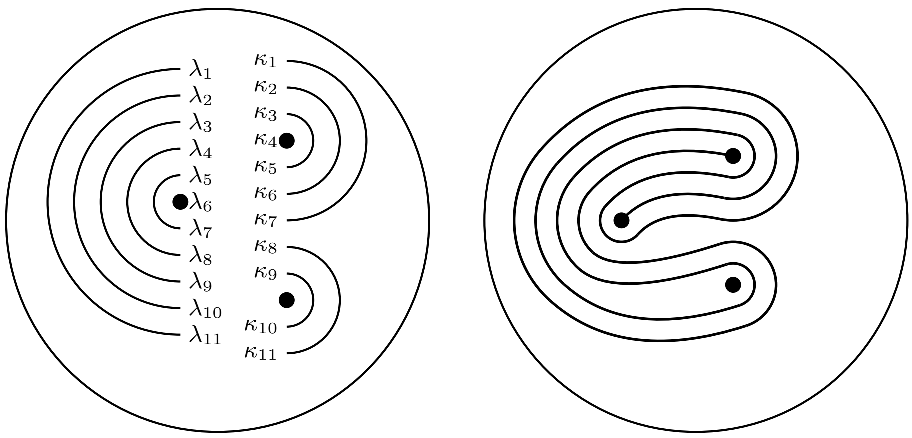


FIGURE 4. Construction of α_p^+ for $p = \frac{7}{4}$

The endpoints of the semicircles around U and D together with U (resp. D) in case r (resp. s) is odd, define $r+s$ points $\kappa_1, \ldots, \kappa_{r+s}$, enumerated consecutively from the top to the bottom. Similarly, the endpoints of the semicircles around L together with L in case $r+s$ is odd, define $r+s$ endpoints $\lambda_1, \ldots, \lambda_{r+s}$, enumerated consecutively from the top to the bottom. In the second step, for each i, the point κ_i is joined with λ_i by a segment. We illustrate this construction in Figure 4.

We will show in section 5.3 below that each such arc is connected.

To define the outer arcs, it will be convenient to use the abbreviation $n^{(-)} = (n-1)^{(+)}$ for positive n, that is,

$$n^{(-)} = \begin{cases} 0, & \text{if } n = 0, \\ \frac{n-2}{2}, & \text{if } n > 0 \text{ is even}, \\ \frac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

Notice that for each arc connecting two different marked points there exists a unique non-tagged arc connecting the other two vertices. In this way α_p^+ defines α_p^- in case $p = \frac{r}{s}$ with $r, s > 0$. However, we shall give a more explicit construction, again in two steps. In the first step we draw $r^{(-)}$ (resp. $s^{(-)}$) semicircles around U (resp. D). Their endpoints define (possibly together with U, resp D) $r-1$ (resp. $s-1$) points denoted by $\kappa_1, \ldots, \kappa_{r-1}$ (resp. $\kappa_{r+1}, \ldots, \kappa_{r+s-1}$). An additional point κ_r is introduced vertically between κ_{r-1} and κ_{r+1}. Also $(r+s)^{(-)}$ semicircles are drawn around L with endpoints $\lambda_1, \ldots, \lambda_{r+s-1}$. Then λ_i is joined with κ_i for all i.

Additionally κ_r is joined with O by a horizontal segment with left endpoint κ_r. The resulting picture is the arc α_p^-. We illustrate this construction in Figure 5.

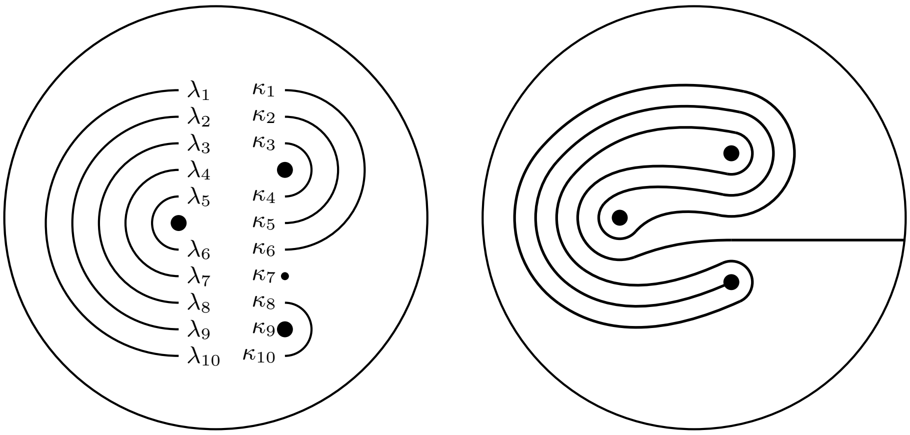


FIGURE 5. Construction of α_p^- for $p = \frac{7}{4}$

To define α_p^+ and α_p^- for negative p we generalize the above construction. In general, to construct α_p^+ let $u = |r|^{(+)}$, $d = |s|^{(+)}$ and $l = |r+s|^{(+)}$. Take the two smallest values among u, d, l and draw as many semicircles around the two corresponding marked points. These are then joined around the third marked point. Similarly the construction of α_p^- is generalized, see Figure 6.

Figure 7 shows how all inner arcs can be displayed nicely in the plane.

5.3. **Unfolding of arcs.** We now describe three procedures μ_D, μ_U and μ_L to modify an arc. They will be called *unfolding maps*. For this, let D' (resp. U', L') be the reflection of D (resp. U, L) on the line LU (resp. LD, UD).

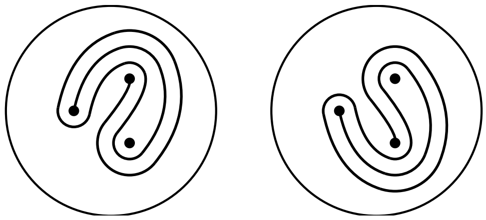


FIGURE 6. α_p^+ for $p = \frac{-7}{4}$ (left) and $p = \frac{-4}{7}$ (right)

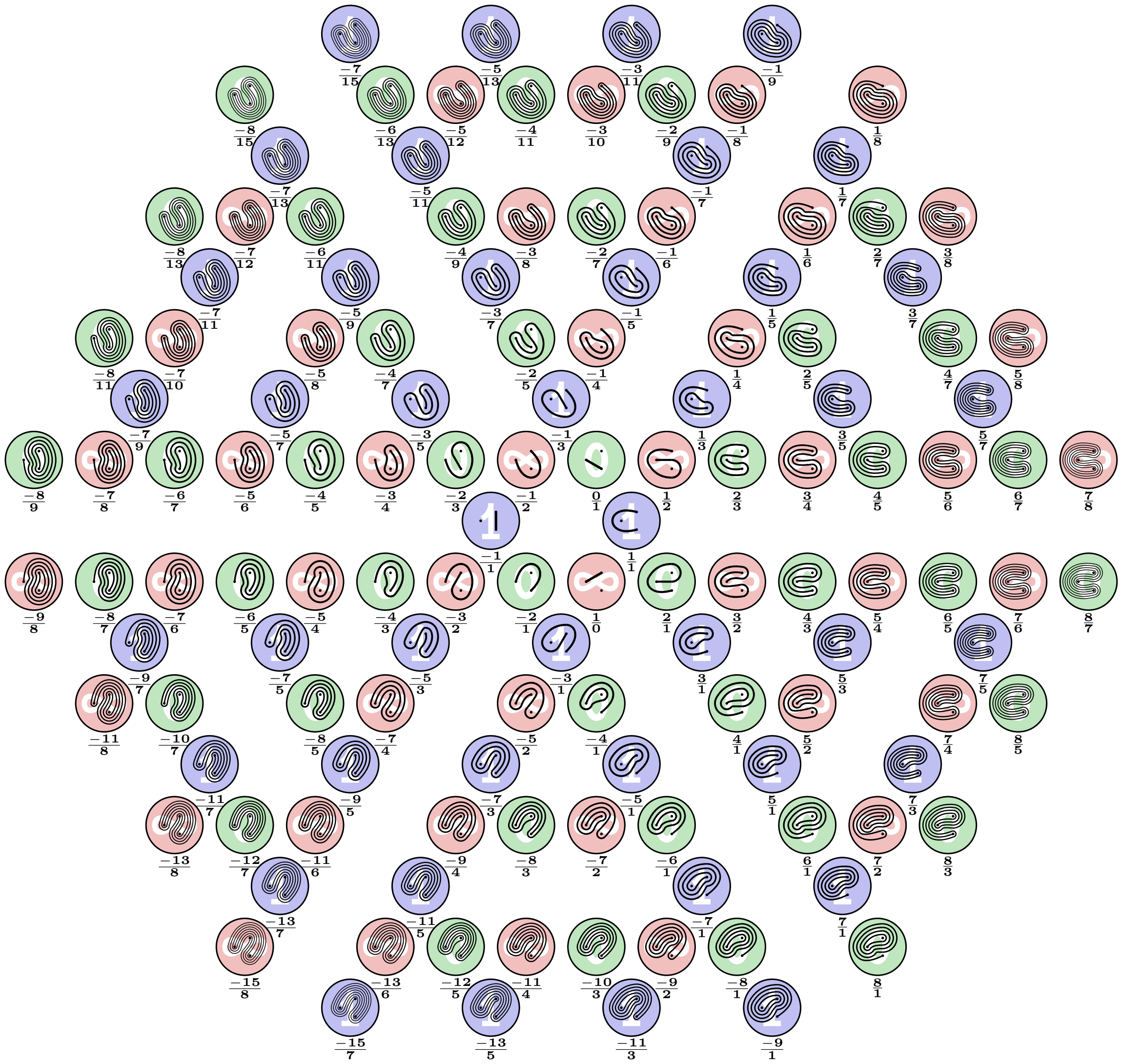


FIGURE 7. Display of inner arcs α_p^+ for $p \in \mathbb{Q}_\infty$

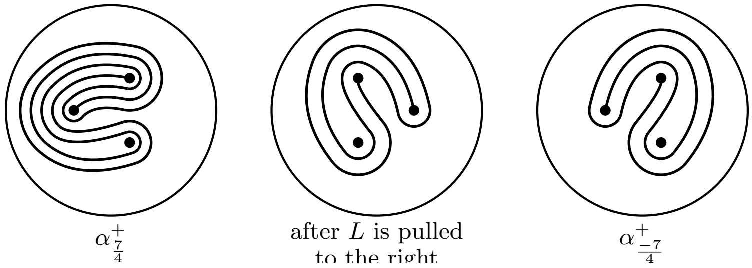


FIGURE 8. Effect of μ_L on the arc $\alpha^+_{\frac{7}{4}}$.

The procedure μ_L consists in pulling the point L together with the semicircles surrounding it between the other two points and place it at the position L' and then reflect the whole situation on the line UD such that L' is reflected back to the position L.

Similarly the other two procedures μ_U and μ_D are defined.

Definition 5.4. For each $\frac{r}{s} \in \mathbb{Q}_\infty$ we define its *complexity* by

$$\gamma(\tfrac{r}{s}) = |r| + |s| + |r+s|.$$

We shall also say that the arc α^ε_p *has complexity* $\gamma(p)$. The lowest complexity is 2 and occurs if and only if $\frac{r}{s} \in \{-1, 0, \infty\}$.

Lemma 5.5. *Let α^ε_p be an arc.*

(a) *We have $\mu_L(\alpha^\varepsilon_p) = \alpha^\varepsilon_{p'}$ where $p' = -p$. Suppose that $0 \le p$. Then, $\gamma(p) \le 2$ implies $\gamma(p') \le 2$ and if $\gamma(p) > 2$ then $\gamma(p') < \gamma(p)$.*
(b) *We have $\mu_D(\alpha^\varepsilon_p) = \alpha^\varepsilon_{p'}$ where $p' = \frac{-p}{2p+1}$. Suppose that $-1 \le p \le 0$. Then, $\gamma(p) \le 2$ implies $\gamma(p') \le 2$ and if $\gamma(p) > 2$ then $\gamma(p') < \gamma(p)$.*
(c) *We have $\mu_U(\alpha^\varepsilon_p) = \alpha^\varepsilon_{p'}$ where $p' = 2 - p$. Suppose that $p \le -2$. Then, $\gamma(p) \le 2$ implies $\gamma(p') \le 2$ and if $\gamma(p) > 2$ then $\gamma(p') < \gamma(p)$.*

Hence, for each arc α^ε_p with $\gamma(p) > 2$ there exists an unfolding map μ such that $\mu(\alpha^\varepsilon_p) = \alpha^\varepsilon_{p'}$ with $\gamma(p') < \gamma(p)$.

Proof. We shall give the arguments only for the case (a) since it is completely similar for (b) and (c). We start by considering first an inner arc α^+_p for some $p = \frac{r}{s} \ge 0$. Notice that there are $m = \min(r, s)$ semicircles around L which connect the points $\kappa_1, \ldots, \kappa_m$ above U with the points $\kappa_{r+s-m+1}, \ldots, \kappa_{r+s}$ below D, see Figure 9.

After pulling L between U and D to the position L' only $\max(r, s) - \min(r, s)$ semicircles remain around L', whereas the number of semicircles around U and D remains unchanged. Hence the resulting situation is obtained by our construction $\alpha^+_{p'}$ where $p' = \frac{-r}{s}$. It is also clear that the construction is involutive, hence $\mu_L(\alpha^+_p) = \alpha^+_{p'}$ even if $p < 0$. The argument is completely similar for outer arcs, that is $\mu_L(\alpha^-_p) = \alpha^-_{-p}$.

Observe that $\gamma(p) = 2$ implies for $p \geq 0$ that $p = 0$ or $p = \infty$. Hence $\mu_L(\alpha_p^\varphi) = \alpha_p^\varphi$ for such values. Now assume that $p = \frac{r}{s} > 0$ and $\mu_L(\alpha_p^\varepsilon) = \alpha_{p'}^\varepsilon$. Then we have by construction $\gamma(p') = |r| + |s| + |s - r| < |r| + |s| + |s + r| = \gamma(p)$. □

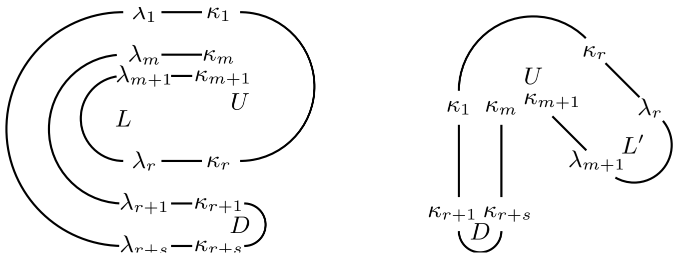


FIGURE 9. Reducing complexity (assume $r > s$)

Proposition 5.6. *For each $p \in \mathbb{Q}_\infty$ the arc α_p^ε is a connected curve which connects the same endpoints than $\alpha_{\mathfrak{t}'(p)}^\varepsilon$ as defined in Definition (5.3).*

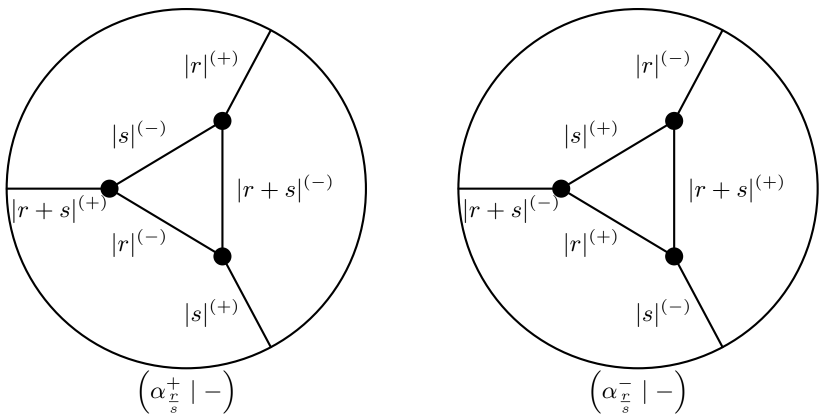
Proof. The proof is done by induction on the complexity of p. Clearly, if $\gamma(p) = 2$ then $p \in \{-1, 0, \infty\}$ and the arc α_p^ε is connected. Otherwise we apply the unfolding map μ_L, μ_D or μ_U depending whether $0 < p$, $-1 < p < 0$ or $p < -1$ respectively. By the preceding Lemma the resulting arc has lower complexity and is therefore connected by induction. But it is clear from the definition of the unfolding maps that they do not change the connectivity of the arcs nor the endpoints. Hence the result. □

5.4. **Compatibility of untagged arcs.** We now study when two arcs α_p^ε and α_q^φ are compatible. For this we start with the simple case when $q = -1, 0, \infty$.

Lemma 5.7. *The untagged arc α_p^ε intersects the untagged arc α_q^φ (for $q = -1, 0, \infty$) precisely*

$$(\alpha_p^\varepsilon \mid \alpha_q^\varphi) = \Delta(p, q)^{(-\varphi\varepsilon)} \tag{5.6}$$

times. We indicated the function $(\alpha_p^\varepsilon \mid -)$ by writing the values on α_q^φ close to these arcs in the following pictures. For this let $p = \frac{r}{s}$ with r, s coprime and $s > 0$.



Proof. Let $\beta = \alpha_p^+$ and assume that $p > 0$. Then, clearly $(\beta \mid \alpha_0^-)$ equals the number of semicircles drawn around the point U, that is $(\beta \mid \alpha_0^-) = r^{(+)} = \Delta(p,0)^{(+)}$. Similarly $(\beta \mid \alpha_\infty^-) = s^{(+)} = \Delta(p,\infty)^{(+)}$ and since there are $(r+s)^{(+)}$ semicircles around L we have also $(\beta \mid \alpha_{-1}^-) = (r+s)^{(+)} = \Delta(p,-1)^{(+)}$. For the following, we use the notations as introduced in Figure 4. The segments $\kappa_i\lambda_i$ intersect LU if and only if λ_i lies above L and κ_i lies below U, that is, if and only if $i \leq (r+s)^{(+)}$ and $i \geq r - r^{(+)}$. Therefore, we have $(\beta \mid \alpha_\infty^+) = (r+s)^{(+)} - r + r^{(+)} \geq 0$ if such an index exists and $(\beta \mid \alpha_\infty^-) = 0$ otherwise, when $(r+s)^{(+)} - r + r^{(+)} < 0$. Thus, in case $\mathrm{t}'(p) = 0$, we have

$$(\beta \mid \alpha_\infty^+) = \max(0, \tfrac{r+s-1}{2} - r + \tfrac{r-1}{2}) = \max(0, \tfrac{s}{2} - 1) = s^{(-)} = \Delta(p,\infty)^{(-)}$$

Similarly, in case $\mathrm{t}'(p) = -1$ or $\mathrm{t}'(p) = \infty$, we have

$$(\beta \mid \alpha_\infty^+) = \max(0, \tfrac{r+s}{2} - r + \tfrac{r-1}{2}) = \max(0, \tfrac{s-1}{2} - 1) = s^{(-)} = \Delta(p,\infty)^{(-)}.$$

Similarly one verifies that $(\beta \mid \alpha_0^+) = \Delta(p,0)^{(-)}$. This shows that $(\alpha_p^+ \mid \alpha_q^\varphi) = \Delta(p,q)^{(-\varphi)}$ whenever $p > 0$. The cases where $\beta = \alpha_p^+$ with $p < 0$ are dealt with similarly. Also, the case where $\beta = \alpha_p^-$ can be solved in much the same way. This shows the claim. □

Proposition 5.8. *Two untagged arcs α_p^ε and α_q^φ are compatible if and only if*

$$\Delta(p,q) \leq \begin{cases} 2, & \text{if } \varepsilon = \varphi, \\ 1, & \text{if } \varepsilon \neq \varphi. \end{cases}$$

Proof. Let $p = \frac{r}{s}$. The proof is done by induction on the complexity $\gamma(q)$. Assume first that $\gamma(q) = 2$, that is, $q = -1, 0$ or ∞ and let $x := \Delta(p,q) = |r+s|$, $|r|$ or $|s|$ respectively.

By definition, α_p^ε is compatible with α_q^φ if and only if $(\alpha_p^\varepsilon \mid \alpha_q^\varphi) = 0$, that is by Lemma 5.7, if and only if $x^{(-\varphi\varepsilon)} = 0$.

In case $\varphi = \varepsilon$, this is equivalent to $x^{(-)} = 0$, which happens if and only if $x = 0$ or $x > 0$ is even and $x \leq 2$ or x is odd and $x \leq 1$, that is, if and only if $x \leq 2$. Similarly, if $\varphi \neq \varepsilon$, then the two arcs are compatible if and only if $x^{(+)} = 0$, that is, if and only if $x \leq 1$.

If $\gamma(q) > 2$ then we apply an appropriate unfolding map to both arcs to get $\alpha_{p'}^\varepsilon$ and $\alpha_{q'}^\varphi$ with $\gamma(q') < \gamma(q)$ and the result follows since the unfolding does not change the number of intersections. □

5.5. **Tagged arcs.** We are now able to define a tagged arc $\alpha_{p,x}$ for each $p \in \mathbb{Q}_\infty$ and each $x \in H$. First we define such arcs for slopes of low complexity, that is for $p = -1, 0, \infty$ as shown in Figure 10.

Furthermore, for $p = 0,\ -1,\ \infty$, we define $\alpha_{p,-x}$ to be the arc obtained from $\alpha_{p,x}$ by switching the tags on both ends from plain to notched and vice versa. In this way we have defined 24 arcs $\alpha_{p,x}$ for $p = 0,\ -1,\ \infty$ and $x \in H$. We shall identify the untagged arcs for these slopes in the obvious way, that is, $\alpha_{-1}^+ = \alpha_{-1,-j}$ and $\alpha_\infty^- = \alpha_{\infty,i}$, for instance.

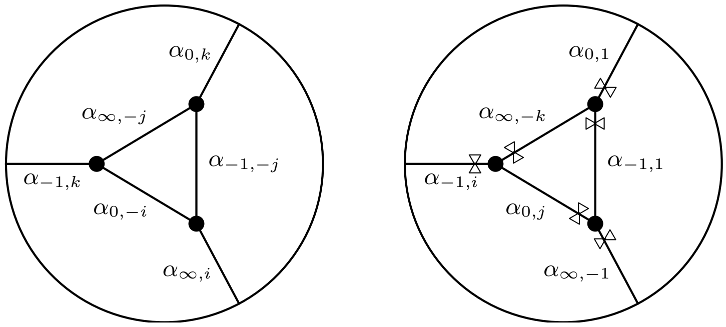


FIGURE 10. Tagged arcs of low complexity

Definition 5.9. For each $p \in \mathbb{Q}_\infty$ and $x \in H$ define the arc $\alpha_{p,x}$ to be the arc connecting the same vertices as $\alpha_{\mathrm{t}'(p),x}$ having the same tags at the endpoints as $\alpha_{\mathrm{t}'(p),x}$, and the untagged version of $\alpha_{p,x}$ is either α_p^+ or α_p^-. More precisely, the untagged version $\alpha_{p,x}^\circ$ of $\alpha_{p,x}$ is as follows:

$$\alpha_{p,x}^\circ = \begin{cases} \alpha_p^+, & \text{if } (\mathrm{t}'(p),x) \in \{(0,\pm i),(0,\pm j),(-1,\pm 1),(-1,\pm j),(\infty,\pm j),(\infty,\pm k)\}, \\ \alpha_p^-, & \text{else.} \end{cases}$$

Theorem 5.10. *Let $p,q \in \mathbb{Q}_\infty$ and $x,y \in H$. Then the two arcs $\alpha_{p,x}$ and $\alpha_{q,y}$ are compatible if and only if the vectors $\mathbf{v}_p^x$ and $\mathbf{v}_q^y$ are compatible.*

Proof. Let β and γ be two tagged arcs with underlying non-tagged arcs β° and γ°.

First we study the case where $p = q$. Then the underlying untagged arcs are always compatible and hence we only have to care for the condition on the tags, which can already be done by looking at the case where $p,q \in \{-1,0,\infty\}$. Hence $\alpha_{p,x}$ is compatible with $\alpha_{p,y}$ if and only if $x \neq -y$, that is, if and only if $\mathbf{v}_p^x$ is compatible with $\mathbf{v}_p^y$ by Lemma 4.9.

Now, we assume that $p \neq q$ and deal first with the case where $p,q \in \{-1,0,\infty\}$. By direct inspection of Figure 10 we get that $\alpha_{p,x}$ is compatible with $\alpha_{q,y}$ if and only if the condition of the corresponding cell in the following table is satisfied.

(5.7)

	$q=-1$	$q=0$	$q=\infty$
$p=-1$	$x \neq -y$	$x \in H^+y$	$y \in H^+x$
$p=0$	$y \in H^+x$	$x \neq -y$	$x \in -H^+y$
$p=\infty$	$x \in H^+y$	$y \in -H^+x$	$x \neq -y$

Since $\Delta(p,q) \leq 2$ in any such case we get by Proposition 4.7 that this happens if and only if $\mathbf{v}_p^x$ is compatible with $\mathbf{v}_q^y$.

Now, let $p,q \in \mathbb{Q}_\infty$ be arbitrary with $p \neq q$. Notice that therefore $\alpha_{p,x}^\circ \neq \alpha_{q,y}^\circ$. Observe also that $\mathrm{t}'(p) = \mathrm{t}'(q)$ holds if and only if $\Delta(p,q)$ is even. Therefore the two arcs $\alpha_{p,x}$ and $\alpha_{q,y}$ are compatible by Proposition 5.8 if and only if either $\Delta(p,q) = 2$ and the tags on both ends coincide, that is $x = y$, or $\Delta(p,q) = 1$ and the arcs $\alpha_{\mathrm{t}'(p),x}$ and $\alpha_{\mathrm{t}'(q),y}$ are compatible, which happens if and only if the condition in (5.7) is satisfied. In both cases, this is equivalent to the condition that $\mathbf{v}_p^x$ and $\mathbf{v}_q^y$ are compatible by Proposition 4.5 and Corollary 4.8. $\square$

5.6. **Connection to real Schur roots.** Recall from Proposition 4.3 that there exists a bijection

$$\Phi : \{\mathbf{v}_p^x \mid p \in \mathbb{Q}_\infty, x \in H\} \longrightarrow \{\text{isoclass of } E \in \operatorname{coh}\mathbb{X} \mid E \text{ is indecomposable rigid}\}.$$

Theorem 5.11. *The map $\Psi : \alpha_{p,x} \mapsto \Phi(\mathbf{v}_p^x)$ is a bijection between the set of all tagged arcs of the 2-sphere with 4 punctures $(\mathbf{S}, \mathbf{M})$ to the set of isoclasses of indecomposable rigid objects in $\operatorname{coh}\mathbb{X}$, where $\mathbb{X}$ has weight type $(2, 2, 2, 2)$. Moreover, two tagged arcs are compatible if and only if their images under Ψ are compatible. Therefore Ψ induces an isomorphism between the exchange graph $\mathbf{E}^{\bowtie}(\mathbf{S}, \mathbf{M})$ and the mutation-exchange graph of tilting objects in $\operatorname{coh}\mathbb{X}$.*

Proof. By Theorem 5.10, we know that there is a bijection Ψ' from the set $\{\alpha_{p,x} \mid p \in \mathbb{Q}_\infty, x \in H\}$ to the set of indecomposable rigid objects in $\operatorname{coh}\mathbb{X}$, with $\Psi'(\alpha_{p,x}) = \Phi(\mathbf{v}_p^x)$. Furthermore, two tagged arcs are compatible if and only if their images are Ext-orthogonal in $\operatorname{coh}\mathbb{X}$. It remains to see that each tagged arc of $(\mathbf{S}, \mathbf{M})$ is of the form $\alpha_{p,x}$ for some $p \in \mathbb{Q}_\infty$ and some $x \in H$. By [11], the exchange graph $\mathbf{E}^{\bowtie}(\mathbf{S}, \mathbf{M})$ is connected and each tagged triangulation consists of precisely 6 tagged arcs. Consequently if β is any tagged arc, we can complete β to a tagged T triangulation of $(\mathbf{S}, \mathbf{M})$. Let $\mu_{i_1}, \ldots, \mu_{i_L}$ be a sequence of mutations such that $\mu_{i_L} \cdots \mu_{i_1}(T) = \{\alpha_q^\varepsilon \mid q = -1, 0, \infty; \quad \varepsilon = \pm\} =: T'$. Then $\mu_{i_1} \cdots \mu_{i_L}(\Psi'(T'))$ is a tilting object in $\operatorname{coh}\mathbb{X}$ and therefore of the form $\Psi'(T'')$ with $T'' = \mu_{i_1} \cdots \mu_{i_L}(T') = T$ and consequently $\beta = \alpha_{p,x}$ for some $p \in \mathbb{Q}_i$ and $x \in H$. □

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