REFLECTION GROUP PRESENTATIONS ARISING FROM CLUSTER ALGEBRAS

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ABSTRACT. We give a presentation of a finite crystallographic reflection group in terms of an arbitrary seed in the corresponding cluster algebra of finite type and interpret the presentation in terms of companion bases in the associated root system.

INTRODUCTION

The article [7] proved that cluster algebras of finite type can be classified by the Dynkin diagrams. A matrix (the *Cartan counterpart*) is associated to each seed in such a cluster algebra, and the cluster algebra associated to a Dynkin diagram Δ is characterised by the fact that it contains seeds whose Cartan counterparts coincide with the Cartan matrix of Δ (up to simultaneous permutation of the rows and columns).

It is well known that the Dynkin diagrams also classify the finite crystallographic reflection groups (see e.g. [8]). Our main result is a presentation of the reflection group associated to a Dynkin diagram in terms of an arbitrary seed in the corresponding cluster algebra. The presentation is given in terms of an edge-weighted oriented graph, known as a *diagram*, which is associated by Fomin and Zelevinsky to the seed.

If the Cartan counterpart of the seed is the Cartan matrix, then our presentation reduces to the usual Coxeter presentation of the reflection group. However, in general, the diagram of a seed contains cycles, and the presentation includes extra relations arising from the chordless cycles in the diagram (i.e. induced subgraphs which are cycles).

More precisely, if Γ is the diagram associated to a seed in a cluster algebra of finite type, we define, for vertices i, j of Γ,

$$
m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}
$$

Let $W(\Gamma)$ to be the group with generators s_i, i a vertex of Γ, subject to the following relations (where e denotes the identity element):

(1) $s_i^2 = e$ for all i;
(2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$;
(3) For any chordless cycle C in Γ:

$$
i_0 \xrightarrow{w_1} i_1 \xrightarrow{w_2} \cdots \rightarrow i_{d-1} \xrightarrow{w_0} i_0,
$$

where either all of the weights are 1, or $w_0 = 2$, we have:

$$
(s_{i_0} s_{i_1} \cdots s_{i_{d-2}} s_{i_{d-1}} s_{i_{d-2}} \cdots s_{i_1})^2 = e.
$$

Theorem A *Let Γ be the diagram associated to a seed in a cluster algebra of finite type. Then $W(\Gamma)$ is isomorphic to the corresponding reflection group.*

This work was mainly motivated by the notions of quasi-Cartan companion [1] and companion basis [10, Defn. 4.1] (see also [9]). A companion basis for a skew-symmetric matrix B arising in a seed of the cluster algebra is a set S of roots in the corresponding root system which form a basis for the root lattice and for which the matrix A whose entries are $(\alpha, \beta^\vee)$ for $\alpha, \beta \in S$ is a positive quasi-Cartan companion matrix for B in the sense of [1]. This means that the diagonal entries are equal to 2, that the values of the off-diagonal entries in A coincide with the corresponding entries of B up to sign, and that the symmetrisation of the matrix is positive definite. Quasi-Cartan companions are used in [1] to give criteria for whether a cluster algebra is of finite type and companion bases were used in [9, 10] to describe the dimension vectors of indecomposable modules over cluster-tilted algebras in type A (dimension vectors of indecomposable modules over cluster-tilted algebras in the general concealed case were also independently calculated later in [11]).

The reflections corresponding to the elements of a companion basis associated to a seed in a finite type cluster algebra generate the corresponding reflection group. We show that the generators in a presentation of the reflection group as considered here can be regarded as this set of reflections. This perspective also allows us to show that, in the simply-laced case, the presentations obtained here coincide with root basis presentations of crystallographic reflection groups (amongst others) which were obtained in [5] using signed graphs.

We were also motivated by the article [2], which gives a presentation of a complex semisimple Lie algebra of simply-laced Dynkin type Δ in terms of any positive definite unit form of type Δ (i.e. equivalent to the unit form associated to Δ). This presentation also contains relations arising from chordless cycles, in this case arising in the bigraph describing the unit form.

The article is organised as follows. In Section 1, we recall some of the theory of cluster algebras of finite type from [7]. In Section 2, we collect together some of the properties of the seeds of such cluster algebras. In particular, we consider the diagrams associated to such seeds in [7] and the way in which they mutate, as well as the oriented cycles appearing in them. In Section 3 we associate a group to an arbitrary seed in the cluster algebra, using generators and relations. In Section 4 we find an efficient subset of the relations sufficient to define the group (this is the set of relations given above). In Section 5 we show that the group defined is invariant (up to isomorphism) under mutation of the seed. Since the group associated to a seed whose Cartan counterpart is a Cartan matrix is the reflection group corresponding to the cluster algebra, it follows that the group associated to any seed is isomorphic to this reflection group. In Section 6 we explain the connections with quasi-Cartan companion matrices and companion bases and with the extended Coxeter groups associated to signed graphs in [5]. This also gives an alternative proof of the main result in the simply-laced case.

1. Cluster algebras of finite type

We first recall the definition of a (skew-symmetrisable) *cluster algebra* without coefficients, as defined by Fomin-Zelevinsky [6].

Let $\mathbb{F} = \mathbb{Q}(u_1, u_2, \ldots, u_n)$ be the field of rational functions in n indeterminates over $\mathbb{Q}$. Recall that a square matrix B is said to be *skew-symmetrisable* if there is a diagonal matrix D of the same size with positive diagonal entries such that DB is skew-symmetric. We consider pairs $(\mathbf{x}, B)$, where $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a free generating set of $\mathbb{F}$ and B is an $n \times n$ skew-symmetrisable matrix.

Given $k \in [1, n] = \{1, 2, \ldots, n\}$, a new such pair $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, the *mutation* of $(\mathbf{x}, B)$ at k, is defined as follows. We have:

$$
B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\ B_{ij} + \frac{|B_{ik}|B_{kj} + B_{ik}|B_{kj}|}{2} & \text{otherwise.} \end{cases}
$$

Define an element $x'_k \in \mathbb{F}$ by the *exchange relation*

$$
x'_k x_k = \prod_{B_{ik} > 0} x_i^{B_{ik}} + \prod_{B_{ik} < 0} x_i^{-B_{ik}},
$$

and set $\mathbf{x}' = \{x_1, x_2, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n\}$. We write $\mu_k(B) = B'$.

Now we fix a pair $(\mathbf{x}, B)$, the *initial seed*. We shall call any pair that can be obtained from $(\mathbf{x}, B)$ by a sequence of mutations a *seed*. Two seeds are regarded as the same if there is a permutation of $[1, n]$ which takes the matrix in the first seed to the second (when regarded as a simultaneous permutation of the rows and columns) and the free generating set in the first to the free generating set in the second. Note that mutating twice at k sends a seed to itself.

Each tuple $\mathbf{x} = \{x_1, \ldots, x_n\}$ such that there exists a seed $(\mathbf{x}, B)$ is called a *cluster* and its entries x_i are called *cluster variables*. The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is the

$\mathbb{Q}$-subalgebra of $\mathbb{F}$ generated by the set of all cluster variables. Up to strong iso-morphism (in the sense of [7, 1.2]), $\mathcal{A}(\mathbf{x}, B)$ does not depend on the choice of free generating set $\mathbf{x}$ so we shall write $\mathcal{A}(B)$ for $\mathcal{A}(\mathbf{x}, B)$.

A cluster algebra is said to be of *finite type* if it has finitely many seeds. Given a skew-symmetrisable matrix B, its *Cartan counterpart* $A(B)$ is a matrix of the same size defined by

$$A_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|B_{ij}| & \text{otherwise.} \end{cases}$$

We recall a special case of a result of Fomin-Zelevinsky. Note that the original result deals with more general cluster algebras (with coefficients) and allows more general matrices (sign-skew-symmetric matrices) in the seeds.

Theorem 1.1. [7, 1.4] *There is a canonical bijection between the strong isomorphism classes of skew-symmetrisable cluster algebras of finite type and the Cartan matrices of finite type. A cluster algebra corresponds to a given Cartan matrix A provided it contains a seed for which the Cartan counterpart of the matrix is A.*

A skew-symmetrisable matrix B is said to be 2-*finite* provided $|B_{ij}B_{ji}| \leq 3$ for all $i, j \in [1, n]$. Note that this means the opposite pair of entries (B_{ij}, B_{ji}) in B must be one of $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, 3)$ or $(3, 1)$. Then Fomin-Zelevinsky also show that:

Theorem 1.2. [7, 1.8] *A cluster algebra $\mathcal{A}$ is of finite type if and only if all the matrices appearing in the seeds of $\mathcal{A}$ are 2-finite.*

As observed by Fomin-Zelevinsky [7, §9].

Corollary 1.3. *Let B be a 2-finite matrix. Then any square submatrix of B, obtained by taking a subset of the rows and the corresponding subset of the columns, is again 2-finite. Thus, passing to such a submatrix preserves the fact that the corresponding cluster algebra is of finite type.*

Fomin-Zelevinsky [7, 7.3] define the *diagram* $\Gamma(B)$ of a skew-symmetrisable matrix B to be the weighted oriented graph with vertices $[1, n]$. There is an arrow from i to j if and only if $B_{ij} > 0$, weighted with the positive integer $|B_{ij}B_{ji}|$. If B is 2-finite, then by [7, 7.5], all 3-cycles in the underlying unoriented graph of $\Gamma(B)$ are oriented cyclically. In this case, the mutation rule for diagrams is as follows:

Proposition 1.4. [7, Lemma 8.5] *Let B be a 2-finite skew-symmetrisable matrix. Then $\Gamma(\mu_k(B))$ is uniquely determined by $\Gamma(B)$ as follows:*

  (a) *Reverse the orientations of all edges in $\Gamma(B)$ incident with k (leaving the weights unchanged).*

  (b) *For any path in $\Gamma(B)$ of form $i \xrightarrow{a} k \xrightarrow{b} j$ (i.e. with a, b positive), let c be the weight on the edge $j \to i$, taken to be zero if there is no such arrow. Let c' be determined by $c' \geq 0$ and*

$$c + c' = max(a, b).$$

  Then $\Gamma(B)$ changes as in Figure 1, again taking the case $c' = 0$ to mean no arrow between i and j.

FIGURE 1. Mutation of a diagram.

Note that the formula

$$\sqrt{c} + \sqrt{c'} = \sqrt{ab}$$

appears in [7, Lemma 8.5] but it is easy to check using [7, Prop. 9.7] (see Proposition 2.1 below) that the above formula gives the same answer. We thank the referee for pointing this out to us.

We call $\Gamma(\mu_k(B))$ the *mutation* of Γ at B and use the notation $\mu_k(\Gamma)$.

We say that a skew-symmetrisable matrix B (and the corresponding diagram) is *of finite type* if the corresponding cluster algebra is of finite type. Hence, by Corollary 1.3, we have:

Corollary 1.5. [7, §9] *Let Γ be a diagram of finite type. Then any induced subdiagram of Γ is again of finite type.*

2. DIAGRAMS OF FINITE TYPE

In this section, we collect together some properties of diagrams of finite type. We call an induced subgraph of an unoriented graph which is a cycle a *chordless cycle*. We first recall:

Proposition 2.1. [7, Prop. 9.7] *Let Γ be a diagram of finite type. Then a chordless cycle in the underlying unoriented graph of Γ is always cyclically oriented in Γ. In addition, the unoriented graph underlying the cycle must be one of those in Figure 2.*

We will refer to such cycles as chordless cycles in Γ.
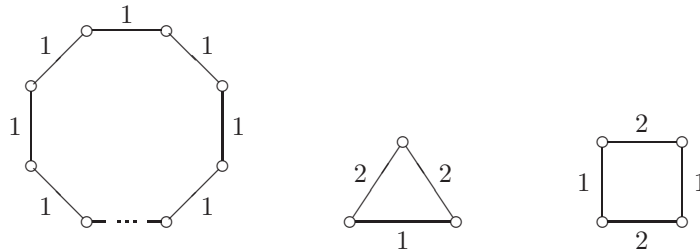

FIGURE 2. The chordless cycles in a diagram of finite type.

Lemma 2.2. *Let Γ be a diagram of finite type and let Ξ be an induced subdiagram with three vertices which is connected. Then the unoriented graph underlying Ξ must be one of those in Figure 3.*

Proof. This follows from Proposition 2.1, together with the fact that a diagram of form $1 \to 2 \to 3$ (with positive weights) cannot have a weight greater than 2 or two weights equal to 2, by [7, Prop. 9.3] and its proof. $\square$
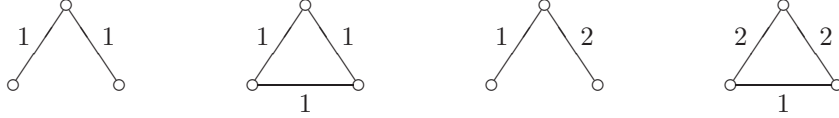
FIGURE 3. 3-vertex connected subdiagrams of diagrams of finite type.

Corollary 2.3. *Let Γ be a graph of finite type and suppose that are 3 distinct vertices i, j, k of Γ with both i and j connected to k. Then the effect of the mutation of Γ at k on the induced subdiagram must be as in Figure 4 (starting on one side or the other), up to switching i and j.*

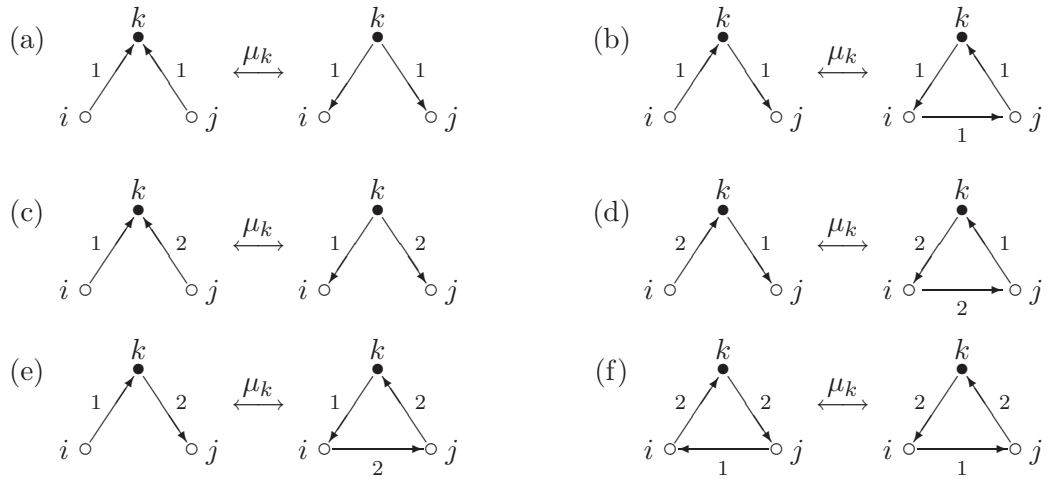*Proof.* This follows from Lemma 2.2 and Proposition 1.4.   □

FIGURE 4. Local picture of mutation of a diagram of finite type.

We say that a vertex in a diagram Γ is *connected* to another if there is an edge between them. The next result describes the way vertices in Γ can be connected to a chordless cycle.

Lemma 2.4. *Suppose that a vertex k is connected to at least one vertex in a chordless cycle in a diagram Γ of finite type. Then k is connected to at most two vertices of the cycle. If it is connected to two vertices, they must be adjacent in the cycle.*

Proof. Consider a chordless cycle C:

$$i_0 \to i_1 \to \cdots \to i_{r-1} \to i_0$$

in Γ, and suppose that k is connected to $i_{a_1}, i_{a_2}, \ldots, i_{a_t}$ for some $t \leq r$, where $0 \leq a_1 < a_2 < \cdots < a_t \leq r - 1$. Suppose first that $t > 2$. Since, by Proposition 2.1, the chordless cycles (in the underlying unoriented graph)

$$i_{a_1} \relbar\joinrel\relbar i_{a_1+1} \relbar\joinrel\relbar \cdots \qquad i_{a_2} \relbar\joinrel\relbar k \relbar\joinrel\relbar i_{a_1}$$

and

$$i_{a_2} \quad\text{———}\quad i_{a_2+1} \quad\text{———}\quad \cdots \qquad i_{a_3} \quad\text{———}\quad k \quad\text{———}\quad i_{a_2}$$

are cyclically oriented in Γ, we must have

$$i_{a_2} \to k \to i_{a_1}$$

from the first and

$$i_{a_3} \to k \to i_{a_2}$$

from the second, giving a contradiction.

If $r = 2$, consider the cycle

$$i_{a_2} \quad\text{———}\quad i_{a_2+1} \quad\text{———}\quad \cdots \qquad i_{a_1} \quad\text{———}\quad k \quad\text{———}\quad i_{a_2}$$

(where the subscripts are written mod r). If it is a chordless cycle, it must be oriented, giving

$$i_{a_1} \to k \to i_{a_2},$$

again giving a contradiction. Hence this last cycle is not a chordless cycle, which implies that i_{a_1} and i_{a_2} are adjacent in C as required. $\square$

Lemma 2.5. *Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. In Figures 5 and 6 we list various types of induced subdiagrams in Γ (on the left) and corresponding cycles C' in Γ' (on the right) arising from the mutation of Γ at k. The diagrams are drawn so that C' is always a clockwise cycle, and in case (i), C' has at least 3 vertices, while in case (j), C' has at least 4 vertices. Then every chordless cycle in Γ' arises in such a way.*

Proof. Let C' be a chordless cycle in Γ'. We consider the different possibilities for the vertex k in relation to C'. If k is a vertex of C', then it follows from Proposition 2.1 that C' arises as in case (a)-(f) or (j). If k does not lie in C' and is not connected to any vertex of C' then C' arises as in case (k). If k does not lie in C' and is connected to exactly one vertex of C', then C' arises as in case (l). Note that the edge linking k to C' must have weight 1 or 2 (since a diagram with two edges with weights 3 and $a = 1, 2$ or 3 is not of finite type; see the proof of [7, Prop. 9.3]).

By Lemma 2.4, the only remaining case is where k does not lie in C' and k is connected to two adjacent vertices of C'. If the 3-cycle C_k containing these 3 vertices in Γ' has all of its weights equal to 1, then C' also has this property, since the vertices in $C_k \cup C'$ form a chordless cycle in Γ containing two consecutive edges of weight 1. Then C' is as in case (i).

If C_k has weights $1, 2, 2$ then it is not possible for the edge of weight 1 to be shared with C', since then in Γ, the vertices of C' form a chordless cycle which is not oriented. If the common edge has weight 2 and C' has more than 3 vertices, then the vertices of $C_k \cup C'$ form a chordless cycle in Γ with at least 5 vertices and at least one edge of weight 2, contradicting Proposition 2.1. The only other possibility is when C' has exactly 3 vertices, and then, by Proposition 2.1, it must arise as in (g) or (h). The result is proved. $\square$
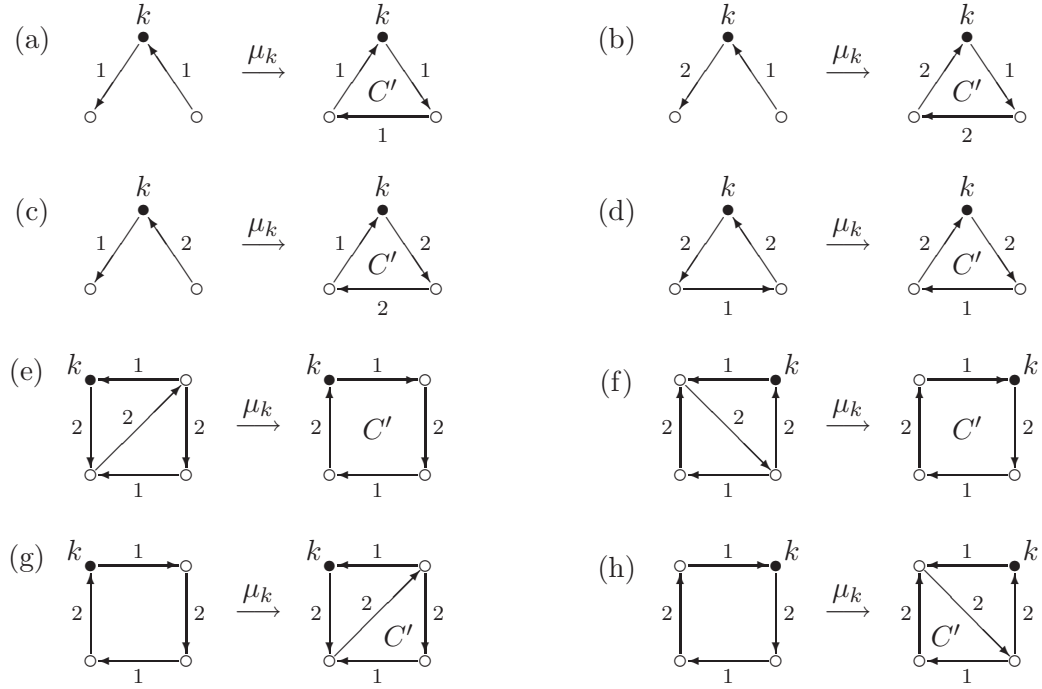
FIGURE 5. Induced subdiagrams of Γ and the corresponding chordless cycles in $\Gamma' = \mu_k(\Gamma)$, part 1.

(k) C is an oriented cycle in Γ not connected to k. Then C' is the corresponding cycle in Γ'.

(l) C is an oriented cycle in Γ with exactly one vertex connected to k (via an edge of unspecified weight). Then C' is the corresponding cycle in Γ'.
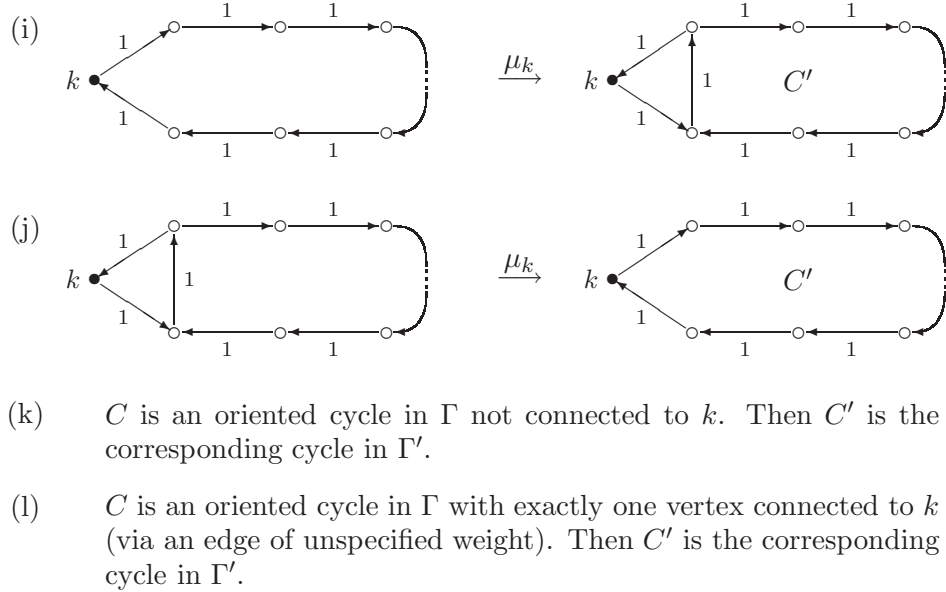
FIGURE 6. Induced subdiagrams of Γ and the corresponding chordless cycles in $\Gamma' = \mu_k(\Gamma)$, part 2.

3. THE GROUP OF A DIAGRAM ARISING IN A CLUSTER ALGEBRA OF FINITE TYPE

In this section we define a group (by generators and relations) for any diagram Γ of finite type. We shall see later that it is isomorphic to the reflection group of the same Dynkin type as the cluster algebra in which Γ arises.

Let Γ be a diagram of finite type. For vertices i, j of Γ define

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}$$

Then we define W_Γ to be the group with generators s_i, $i = 1, 2, \ldots, n$, subject to the following relations:

(R1) $s_i^2 = e$ for all i,
(R2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$,

where e denotes the identity element of W_Γ.

Note that, in the presence of (R1), relation (R2) is symmetric in that $(s_j s_i)^{m_{ji}} = e$ follows from $(s_i s_j)^{m_{ij}} = e$, since m_{ij} is symmetric in i and j. We shall usually use this fact without comment.

For the remaining relations, we suppose that C, given by:

$$i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$$

is a chordless cycle in Γ. For $a = 0, 1, 2, \ldots, d - 1$, define (with subscripts modulo d):

$$r(i_a, i_{a+1}) = s_{i_a} s_{i_{a+1}} \cdots s_{i_{a+d-1}} s_{i_{a+d-2}} \cdots s_{i_{a+1}}.$$

(R3)(a) If all of the weights in the edges of C are equal to 1, then we set $r(i_a, i_{a+1})^2 = e$ for $a = 0, 1, \ldots, d - 1$.
(R3)(b) If C has some edges of weight 2, then, for $a = 0, 1, \ldots, d-1$, set $r(i_a, i_{a+1})^k = e$ where $k = 4 - w_a$ and w_a is the weight of the edge between i_a and i_{a-1}.

Note that if we think of $r(i_a, i_{a+1})$ as a path in Γ, then in (R3)(b), w_a is the weight of the only edge in C not used in the path.

Remark 3.1. We note that if Γ is the diagram associated to a matrix whose Cartan counterpart is the Cartan matrix of a Dynkin diagram Δ, then W_Γ is the reflection group of type Δ. In this case, the relations (R3)(a) and (R3)(b) do not occur, since Γ is a tree.

Remark 3.2. We also note that, as in Coxeter groups, if $i \neq j$ then the relation (R2) for the pair i, j is equivalent (using (R1)) to:

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots,$$

where there are m_{ij} terms on each side of the equation. In particular, if there is no edge in Γ between i and j, then $s_i s_j = s_i s_j$, and if there is an edge in Γ between i and j with weight 1, then $s_i s_j s_i = s_j s_i s_j$. We shall use these relations liberally.

Remark 3.3. We shall see later (using the companion basis perspective) that, in the simply-laced case, the group W_Γ coincides with a group defined in [5, Defn. 6.2]. See the end of Section 6. In this context, the elements $r(i_a, i_{a+1})^2$ are referred to as *cut elements*.

Example 3.4. *We consider the example in Figure 7, which is the diagram of a seed of type D_4.*
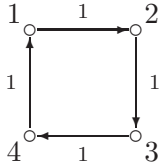

FIGURE 7. A diagram of type D_4.

The relations in this case are:

(R1) $s_i^2 = e$ *for* $i = 1, 2, 3, 4$;
(R2) $(s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^3 = (s_4 s_1)^3 = e$ *and* $(s_1 s_3)^2 = (s_2 s_4)^2 = e$;
(R3) *For each element $r(i, j)$ of the following list we have $r(i, j)^2 = e$.*

$$r(1, 2) = s_1 s_2 s_3 s_4 s_3 s_2;$$
$$r(2, 3) = s_2 s_3 s_4 s_1 s_4 s_3;$$
$$r(3, 4) = s_3 s_4 s_1 s_2 s_1 s_4;$$
$$r(4, 1) = s_4 s_1 s_2 s_3 s_2 s_1.$$

We shall see in the next section that the relations in (R3) can be reduced (in fact, it is enough to choose just one of them).

4. Cyclic Symmetry

It turns out that, in the presence of the relations (R1) and (R2) in the above, many of the relations in (R3)(a) and (b) are redundant. In the sequel, we shall usually use (R1) without comment.

Lemma 4.1. *Let Γ be a diagram of finite type containing a chordless cycle C:*

$$i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$$

with the weights of all of its edges equal to 1. Let W be the group with generators $s_1, s_2, \ldots, s_n$ subject to relations (R1), (R2) and $r(i_a, i_{a+1})^2 = e$ for a single fixed value of a. Then all of the relations in (R3)(a) hold for C.

Proof. Note first that it is enough to consider the case $a = 0$. For simplicity of notation, we renumber the vertices of C so that $i_j = j$ for $j = 0, 1, \ldots, a - 1$. Thus we assume that $r(0, 1)^2 = e$ in W.

We have, using (R2):

$$r(d-1,0) = s_{d-1}s_0s_1 \cdots s_{d-3}s_{d-2}s_{d-3} \cdots s_1s_0$$
$$= s_0s_0s_{d-1}s_0s_1 \cdots s_{d-3}s_{d-2}s_{d-3} \cdots s_1s_{d-1}s_{d-1}s_0$$
$$= s_0s_{d-1}s_0s_{d-1}s_1 \cdots s_{d-3}s_{d-2}s_{d-3} \cdots s_1s_{d-1}s_{d-1}s_0$$
$$= s_0s_{d-1}(s_0s_1 \cdots s_{d-3}s_{d-1}s_{d-2}s_{d-1}s_{d-3} \cdots s_1)s_{d-1}s_0$$
$$= s_0s_{d-1}(s_0s_1 \cdots s_{d-3}s_{d-2}s_{d-1}s_{d-2}s_{d-3} \cdots s_1)s_{d-1}s_0$$
$$= s_0s_{d-1}r(0,1)s_{d-1}s_0.$$

It follows that $r(d-1,0)^2 = e$. Repeating this argument inductively, we see that $r(a, a+1)^2 = e$ for all a, and we are done. $\square$

Lemma 4.2. *Let Γ be a diagram of finite type containing a chordless 3-cycle C as in Figure 8 (where we number the vertices $1, 2, 3$ for convenience). Let W be the group with generators $s_1, s_2, \ldots, s_n$ subject to relations (R1) and (R2) arising from the diagram Γ. Then the following are equivalent:*

(a) $r(1,2)^2 = e$;
(b) $r(2,3)^2 = e$.

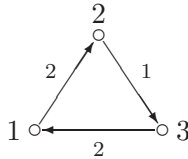*Furthermore, if one of the above holds, then the following holds:*

(c) $r(3,1)^3 = e$.

FIGURE 8. A 3-cycle (see Lemma 4.2).

Proof. Note that $r(1,2) = s_1s_2s_3s_2$, $r(2,3) = s_2s_3s_1s_3$ and $r(3,1) = s_3s_1s_2s_1$. The equivalence of (a) and (b) follows from:

$$s_1s_3r(2,3)^2s_3s_1 = s_1s_3s_2s_3s_1s_3s_3s_2s_3 = s_1s_2s_3s_2s_1s_2s_3s_2 = r(1,2)^2$$

(using (R2)). Suppose that (a) and (b) hold. Then, by (b), $s_3s_1s_3s_2s_3s_1 = s_2s_3$. Hence, using (R2),

$$s_1s_3s_1(s_3s_1s_2s_1)^3s_1s_3s_1 = (s_1s_3s_1s_3s_1s_2s_1s_1s_3s_1)^3 = (s_3s_1s_3s_2s_3s_1)^3 = (s_2s_3)^3 = e,$$

and (c) follows. $\square$

Note that it is not claimed that (c) is equivalent to (a) and (b) (we do not know if this holds, but it seems to be unlikely).

Example 4.3. *Lemma 4.1 implies that in Example 3.4, we can replace the relations in (R3) with any fixed one, e.g.*

$$r(1,2)^2 = (s_1s_2s_3s_4s_3s_2)^2 = e.$$

Lemma 4.4. *Let Γ be a diagram of finite type containing a chordless 4-cycle C as in Figure 9 (where we number the vertices $1, 2, 3, 4$ for convenience). Let W be the group with generators $s_1, s_2, \ldots, s_n$ subject to relations (R1) and (R2) arising from the diagram Γ. Then the following are equivalent:*

   (a) $r(1,2)^2 = e$;
   (b) $r(3,4)^2 = e$;

Furthermore, if one of the above holds, then the following both hold:

   (c) $r(2,3)^3 = e$;
   (d) $r(4,1)^3 = e$.
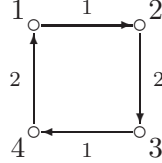

FIGURE 9. A chordless 4-cycle (see Lemma 4.4).

Proof. Note first that $r(1,2) = s_1 s_2 s_3 s_4 s_3 s_2$, $r(2,3) = s_2 s_3 s_4 s_1 s_4 s_3$, $r(3,4) = s_3 s_4 s_1 s_2 s_1 s_4$, and $r(4,1) = s_4 s_1 s_2 s_3 s_2 s_1$.

The equivalence of (a) and (b) follows from:

$$s_3 s_4 s_2 r(1,2) s_2 s_4 s_3 = s_3 s_4 s_2 s_1 s_2 s_3 s_4 s_3 s_4 s_3 = s_3 s_4 s_2 s_1 s_2 s_4 = s_3 s_4 s_1 s_2 s_1 s_4 = r(3,4),$$

using (R2). So suppose that (a) and (b) hold. Applying (R2) to (a), we have:

$$(s_1 s_2 s_4 s_3 s_4 s_2)^2 = e.$$

Thus, noting that s_2 and s_4 commute, we have:

$$(s_1 s_4 s_2 s_3 s_2 s_4)^2 = e.$$

So we get $(s_4 s_1 s_4 s_2 s_3 s_2)^2 = e$ and hence

(1) $$s_4 s_1 s_4 s_2 s_3 s_2 = s_2 s_3 s_2 s_4 s_1 s_4.$$

Therefore, using first $s_3 s_2 s_3 = s_2 s_3 s_2 s_3 s_2$ and then equation (1), we get

$$
\begin{aligned}
s_3 r(2,3) s_3 &= s_3 s_2 s_3 s_4 s_1 s_4 \\
&= s_2 s_3 (s_2 s_3 s_2 s_4 s_1 s_4) \\
&= s_2 s_3 (s_4 s_1 s_4 s_2 s_3 s_2) \\
&= s_2 s_3 s_4 (s_1 s_2) s_4 s_3 s_2,
\end{aligned}
$$

where in the last equation we used that s_2 and s_4 commute. Consequently

$$
\begin{aligned}
(s_3 r(2,3) s_3)^3 &= s_2 s_3 s_4 (s_1 s_2)^3 s_4 s_3 s_2 \\
&= e,
\end{aligned}
$$

since $(s_1 s_2)^3 = e$. This shows that $r(2,3)^3 = e$.

By a similar argument, we see that (b) implies (d), and we are done. $\square$

The following now follows from Proposition 2.1 and Lemmas 4.1, 4.2 and 4.4.

Proposition 4.5. *Let Γ be a diagram of finite type. Then, in the presence of relations (R1) and (R2), the relations (3) considered in the introduction imply the relations (R3) considered above. In particular, W_Γ is isomorphic to the group $W(\Gamma)$ defined in the introduction.*

Finally, in this section, we consider the behaviour of the relations under reversing all of the arrows in Γ.

Proposition 4.6. *Let Γ be a diagram of finite type and let W be a group with generators $s_1, s_2, \ldots, s_n$. Then the s_i satisfy relations (R1), (R2) and (R3) with respect to Γ if and only if they satisfy relations (R1), (R2) and (R3) with respect to Γ^{op}.*

Proof. We assume that the s_i satisfy relations (R1), (R2) and (R3) with respect to Γ, and show that they satisfy relations (R1), (R2) and (R3) with respect to Γ^{op}. The converse follows by replacing Γ with Γ^{op}. Since (R1) and (R2) do not depend on the orientation of Γ, it follows that the s_i satisfy relations (R1) and (R2) with respect to Γ^{op}. In order to show (R3), we note that by Lemmas 4.1, 4.2 and 4.4, we only have to check one of the relations of the form $r(i_a, i_a + 1)^2 = e$ arising from a cycle in order to be sure that all of the relations corresponding to that cycle hold.

Suppose first that Γ contains a chordless cycle C:

$$i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$$

with the weights of all of its edges equal to 1. We renumber the vertices i_j, $j = 0, 1 \ldots, d-1$ so that they become $0, 1, \ldots, d-1$. Then, we have:

$$
\begin{aligned}
r(d-1, d-2) &= s_{d-1} s_{d-2} \cdots s_1 s_0 s_1 \cdots s_{d-3} s_{d-2} \\
&= s_{d-1} s_{d-2} \cdots s_1 (s_0 s_1 \cdots s_{d-3} s_{d-2} s_{d-1} s_{d-2} \cdots s_1) s_1 s_2 \cdots s_{d-1} \\
&= s_{d-1} s_{d-2} \cdots s_1 r(0, 1) s_1 s_2 \cdots s_{d-1}.
\end{aligned}
$$

So $r(d-1, d-2)^2 = e$.

Suppose next that Γ contains a chordless 3-cycle C as in Figure 8 (where we number the vertices $1, 2, 3$ for convenience). Then we have

$$s_2 s_1 r(1, 2) s_1 s_2 = s_2 s_1 s_1 s_2 s_3 s_2 s_1 s_2 = s_3 s_2 s_1 s_2 = r(3, 2),$$

so $r(3, 2)^2 = e$.

Finally, suppose that Γ contains a chordless 4-cycle C as in Figure 9 (where we number the vertices $1, 2, 3, 4$ for convenience). Then, using (R2), we have:

$$
\begin{aligned}
r(2, 1) &= s_2 s_1 s_4 s_3 s_4 s_1 \\
&= s_1 (s_1 s_2 s_1)(s_4 s_3 s_4) s_1 \\
&= s_1 (s_2 s_1 s_2)(s_3 s_4 s_3) s_2 s_2 s_1 \\
&= (s_1 s_2)(s_1 s_2 s_3 s_4 s_3 s_2)(s_2 s_1) \\
&= (s_1 s_2) r(1, 2)(s_2 s_1);
\end{aligned}
$$

hence $r(2, 1)^2 = e$.

Since every chordless cycle in Γ^{op} corresponds to a chordless cycle in Γ, the result follows from Proposition 2.1. $\qquad\square$

Remark 4.7. *Proposition 4.6 shows that W_Γ could be defined using only the underlying unoriented weighted graph of Γ, by taking the relations (R1), (R2) and (R3) corresponding to both Γ and Γ^{op} as the defining relations.*

5. Invariance of the group under mutation

In this section we prove the main result, that the group of a diagram of finite type depends only on the Dynkin type, up to isomorphism. We will prove this by showing that the group of the diagram is invariant (up to isomorphism) under mutation.

We fix a diagram Γ of finite type and a vertex k of Γ. Then we have the mutation $\Gamma' = \mu_k(\Gamma)$.

For a vertex i of Γ, we define an element $t_i \in W_\Gamma$ as follows:

$$t_i = \begin{cases} s_k s_i s_k & \text{if there is an arrow } i \to k \text{ in } \Gamma \text{ (possibly weighted)}; \\ s_i & \text{otherwise.} \end{cases}$$

Our aim is to show that the elements t_i satisfy the relations (R1)-(R3) defining $W_{\Gamma'}$. We denote the values of m_{ij} for Γ' by m'_{ij}. Note that (R1) is clear, since $(s_k s_i s_k)^2 = s_k s_i^2 s_k = e$, so it remains to check (R2) and (R3). We will use the previous section to reduce the number of checks that have to be done.

We first deal with some easy cases:

Lemma 5.1. *Let i, j be distinct vertices of Γ.*

(a) *If $i = k$ or $j = k$ then $(t_i t_j)^{m'_{ij}} = e$.*

(b) *If at most one of i, j is connected to k in Γ (or, equivalently, in Γ'), then $(t_i t_j)^{m'_{ij}} = e$.*

Proof. For (a), suppose first that $i = k$. Note that $m'_{ij} = m_{ij}$. The only non-trivial case is thus when there is an arrow from j to k, so that $t_j = s_k s_j s_k$. But then $t_i t_j = s_k s_k s_j s_k = s_j s_k$ and the result follows. The case when $j = k$ is similar. For (b), suppose first that i is connected to k. The only non-trivial case is when the edge between i and k is oriented towards k, so that $t_i = s_k s_i s_k$. Then $t_i t_j = s_k s_i s_k s_j$ and the result follows from the fact that $s_k s_j = s_j s_k$ (since j is not connected to k). The case when j is connected to k is similar. $\square$

Proposition 5.2. *The elements t_i, for i a vertex of Γ, satisfy the relations (R1)-(R3) defining $W_{\Gamma'}$.*

Proof. After Lemma 5.1, the remaining relations to check are the relations from (R2), that $(t_i t_j)^{m'_{ij}} = e$ when both i and j are connected to k, and the relations (R3) corresponding to the chordless cycles in Γ'. For the former, we need to check that, in going (i) from left to right or (ii) from right to left in Corollary 2.3 (in each case, from Γ to Γ') we have that $(t_i t_j)^{m'_{ij}} = e$. (Note that it is enough to check this for the given labelling of i, j, since the relation for the pair (t_j, t_i) follows from the relation for (t_i, t_j).) At the same time, we will check that the cycle relations in (R3) hold if any cycles are produced; we shall denote by $r'(i, j)$, etc., the elements in the cycle relations for $W_{\Gamma'}$. Note that by Lemmas 4.1, 4.2 and 4.4, we only have

to check one of the relations of the form $r(i_a, i_a + 1)^2 = e$ arising from a cycle in order to be sure that all of the relations corresponding to that cycle hold.

(a)(i)   We have $(t_i t_j)^2 = (s_k s_i s_k s_k s_j s_k)^2 = s_k (s_i s_j)^2 s_k = e$.

(a)(ii)   We have $(t_i t_j)^2 = (s_i s_j)^2 = e$.

(b)(i)   We have: $t_i t_j = s_k s_i s_k s_j = s_i s_k s_i s_j = s_i s_k s_j s_i$ and therefore $s_i (t_i t_j)^3 s_i = (s_k s_j)^3 = e$. And $r'(j, k)^2 = (t_j t_k t_i t_k)^2 = (s_j s_k s_k s_i s_k s_k)^2 = (s_j s_i)^2 = e$.

(b)(ii)   We have $(t_i t_j)^2 = (s_i s_k s_j s_k)^2 = r(i, k)^2 = e$.

(c)(i)   We have $(t_i t_j)^2 = (s_k s_i s_k s_k s_j s_k)^2 = s_k (s_i s_j)^2 s_k = e$.

(c)(ii)   We have $(t_i t_j)^2 = (s_i s_j)^2 = e$.

(d)(i)   We have $s_k (t_i t_j)^4 s_k = s_k (s_k s_i s_k s_j)^4 s_k = (s_i s_k s_j s_k)^4 = (s_i s_j s_k s_j)^4$ and therefore $s_j s_k (t_i t_j)^4 s_k s_j = (s_j s_i s_j s_k)^4 = (s_i s_k)^4 = e$. We also have $r'(j, k) = (t_j t_k t_i t_k)^2 = (s_j s_k s_k s_i s_k s_k)^2 = (s_j s_i)^2 = e$.

(d)(ii)   We have: $(t_i t_j)^2 = (s_i s_k s_j s_k)^2 = r(i, k)^2 = e$.

(e)   Since the s_i satisfy relations (R1), (R2) and (R3) with respect to Γ, they also satisfy these relations with respect to Γ^{op}, by Proposition 4.6. For each vertex i of Γ, let $u_i = s_k t_i s_k$. Then it is easy to check that we have:

$$u_i = \begin{cases} s_k s_i s_k & \text{if } k \to i; \\ s_i & \text{else.} \end{cases}$$

   If the induced subgraph on vertices i, j, k is as in case (e) for Γ, then it is as in case (d) for Γ^{op}. Applying case (d) to the u_i we see that the u_i satisfy (R1),(R2) and (R3) for $\mu_k(\Gamma^{op}) = (\mu_k(\Gamma))^{op}$, and hence so do the t_i, by the definition of the u_i. Thus case (e) follows from case (d).

(f)(i)   We have: $s_j (t_i t_j)^3 s_j = s_j (s_k s_i s_k s_j)^3 s_j = (s_j s_k s_i s_k)^3 = r(j, k)^3 = e$. And $s_k r'(i, j)^2 s_k = s_k (t_i t_j t_k t_j)^2 s_k = s_k (s_k s_i s_k s_j s_k s_j)^2 s_k = (s_i s_k s_j s_k s_j s_k)^2 = (s_i s_j s_k s_j)^2 = r(i, j)^2 = e$.

(f)(ii)   The argument is the same as for (f)(i) by symmetry.

Finally, we need to check that the elements t_i satisfy the cycle relations (R3) for all the chordless cycles in Γ'. Note that Lemma 2.5 describes how the chordless cycles in Γ' arise from induced subdiagrams of Γ. In each case, we need to check that the corresponding cycle relations hold (using Lemmas 4.1, 4.2 and 4.4 to reduce the work). Note that, in the above, we have already checked the cases (a)–(d) in Lemma 2.5. We now check the remaining cases. We also note that, arguing as in case (e) above we see that we only have to check cases (e), (g), (i), (j), (k) and (l) in Lemma 2.5. We choose a labelling of the vertices in each case.
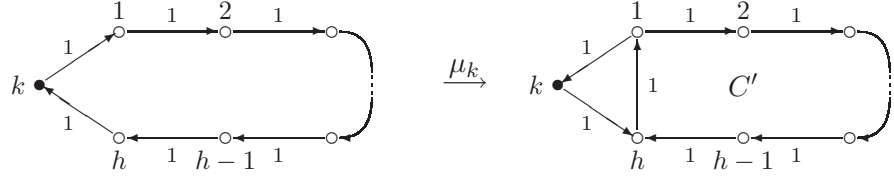
(e)   We label the vertices as follows.
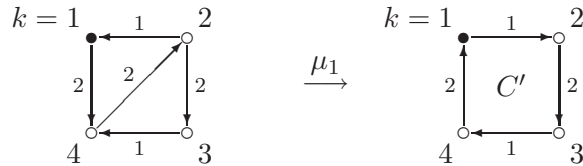
We have $r'(1,2) = t_1t_2t_3t_4t_3t_2 = s_1s_1s_2s_1s_3s_4s_3s_1s_2s_1$ and therefore $s_1s_2s_1r'(1,2)s_1s_2s_1 = (s_1s_2s_1s_2s_1)s_3s_4s_3 = s_2s_3s_4s_3 = r(2,3)$. Hence $r(2,3)^2 = e$ implies $r'(1,2)^2 = e$.

(g)   We label the vertices as follows.
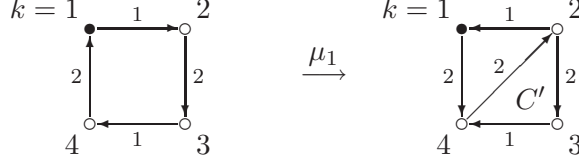
We have $r'(4,3) = t_4t_3t_2t_3 = s_1s_4s_1s_3s_2s_3$. Using that s_1 and s_3 commute we get $s_1r'(4,3)s_1 = s_4s_1s_3s_2s_3s_1 = s_4s_3(s_1s_2s_1)s_3 = s_4s_3s_2s_1s_2s_3 = r(4,3)$ and hence $r(4,3)^2 = e$ implies that $r'(4,3)^2 = e$.
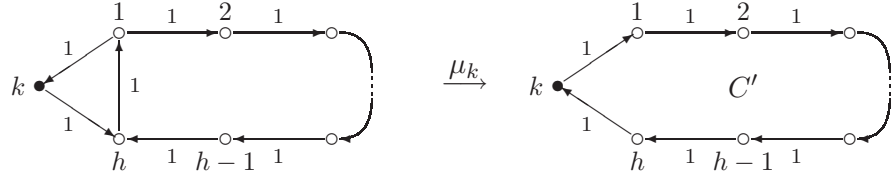
(i)   We use the following labelling of the vertices.

Note that $t_h = s_ks_hs_k$, while $t_i = s_i$ for all $i \neq h$.

$$r'(1,2)^2 = (t_1t_2\cdots t_{h-1}t_ht_{h-1}\cdots t_2)^2$$
$$= (s_1s_2\cdots s_{h-1}s_ks_hs_ks_{h-1}\cdots s_2)^2$$
$$= (s_1s_2\cdots s_{h-1}s_hs_ks_hs_{h-1}\cdots s_2)^2 = r(1,2)^2 = e.$$

(j)   We label the vertices as follows.

Here we have $t_1 = s_ks_1s_k$ and $t_i = s_i$ for all $i \neq 1$.

$$r'(h,h-1)^2 = (t_ht_{h-1}\cdots t_1t_kt_1\cdots t_{h-1})^2$$
$$= (s_hs_{h-1}\cdots s_2s_ks_1(s_ks_k)s_ks_1s_ks_2\cdots s_{h-1})^2$$
$$= (s_hs_{h-1}\cdots s_2(s_ks_1s_ks_1s_k)s_2\cdots s_{h-1})^2$$
$$= (s_hs_{h-1}\cdots s_2s_1s_2\cdots s_{h-1})^2 = r(h,h-1)^2 = e.$$

(k)   This is trivial.

(l)   Suppose that k is connected to exactly one vertex of C via an edge of unspecified weight. We label the vertices as follows.
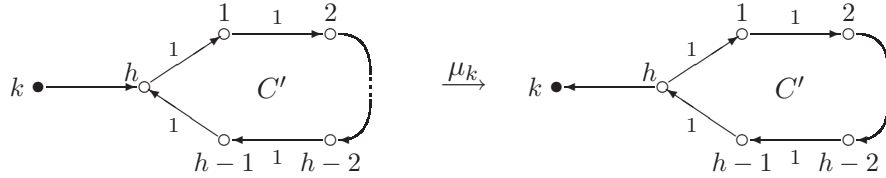
If the edge between k and h points towards h, then $t_h = s_h$ and the relations for C' are the same as the relations for C. If the edge points towards k, then $t_h = s_k s_h s_k$ and we have:

$$
\begin{aligned}
r'(1,2)^2 &= (t_1 t_2 \cdots t_h t_{h-1} \cdots t_2)^2 \\
&= (s_1 s_2 \cdots s_{h-1} s_k s_h s_k s_{h-1} \cdots s_2)^2 \\
&= s_k(s_1 s_2 \cdots s_{h-1} s_h s_{h-1} \cdots s_2)^2 s_k \\
&= s_k r(1,2)^2 s_k = e.
\end{aligned}
$$

This finishes the proof. $\qquad\square$

Note that, in the simply-laced case, some arguments of this kind have been used in the proof of [5, Thm. 6.10]. We explain more about the connection with [5] at the end of Section 6.

Recall that we are in the situation where $\Gamma' = \mu_k(\Gamma)$. We denote by s_i' the defining generators of $W_{\Gamma'}$.

For each vertex i of Γ define the element t_i' in $W_{\Gamma'}$ as follows:

$$
t_i' = \begin{cases} s_k' s_i' s_k' & \text{if there is an arrow } k \to i \text{ in } \Gamma' \text{ (of arbitrary weight)}; \\ s_i' & \text{otherwise.} \end{cases}
$$

Proposition 5.3. *The elements t_i', for i a vertex of Γ', satisfy the relations (R1)-(R3) defining W_Γ.*

Proof. This follows from Proposition 5.2, interchanging Γ and Γ' and using the fact that the definition of the group W_Γ is unchanged under reversing the orientation of all the arrows in Γ (see Proposition 4.6). $\qquad\square$

We can now prove the following:

Theorem 5.4.   (a) *Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at a vertex k. Then $W_\Gamma \cong W_{\Gamma'}$.*
  (b) *Let $\mathcal{A}$ be a cluster algebra of finite type. Then the groups W_Γ associated to the diagrams Γ arising from the seeds of $\mathcal{A}$ are all isomorphic (to the reflection group associated to the Dynkin diagram associated to $\mathcal{A}$).*

Proof. Denote the generators of $W_{\Gamma'}$ (as defined in Section 3) by s_i', for $1 \leq i \leq n$. By Proposition 5.2, there is a group homomorphism $\varphi : W_{\Gamma'} \to W_\Gamma$ such that $\varphi(s_i') = t_i$ for all i. By Proposition 5.3 there is a group homomorphism $\psi : W_\Gamma \to W_{\Gamma'}$ such that $\psi(s_i) = t_i'$. If there is no arrow $i \to k$ in Γ, then there is also no arrow $k \to i$ in Γ' and consequently

$$
\psi(\varphi(s_i')) = \psi(t_i) = \psi(s_i) = t_i' = s_i'.
$$

If there is an arrow $i \to k$ in Γ, then there is an arrow $k \to i$ in Γ' and therefore

$$\psi(\varphi(s'_i)) = \psi(t_i)$$
$$= \psi(s_k s_i s_k)$$
$$= \psi(s_k)\psi(s_i)\psi(s_k)$$
$$= t'_k t'_i t'_k$$
$$= s'_k s'_k s'_i s'_k s'_k = s'_i.$$

So $\psi \circ \varphi = \mathrm{id}_{W_{\Gamma'}}$, and, similarly, $\varphi \circ \psi = \mathrm{id}_{W_\Gamma}$, showing that φ and ψ are isomorphisms and (a) is proved. Part (b) follows from part (a) and Remark 3.1. $\square$

6. COMPANION BASES

We first recall some results from [1]. Recall that a matrix A is said to be *symmetrisable* if there is diagonal matrix D, with positive entries on the diagonal, such that DA is symmetric. A symmetrisable matrix A is said to be *quasi-Cartan* if its diagonal entries are all equal to 2, and it is said to be *positive* if DA is positive definite for some positive diagonal matrix D (this does not depend on the choice of D). If B is a skew-symmetrisable matrix, a *quasi-Cartan companion* of B is a quasi-Cartan matrix A such that $|A_{ij}| = |B_{ij}|$ for all $i \neq j$. Such matrices are used in [1] to give a characterisation of cluster algebras of finite type:

Theorem 6.1. [1] *Let B be a skew-symmetrisable matrix. Then the cluster algebra $\mathcal{A}(B)$ is of finite type if and only if every chordless cycle in $\Gamma(B)$ is cyclically oriented, and B has a positive quasi-Cartan companion.*

We also mention the following result, which we shall need later:

Proposition 6.2. [1, 1.4] *Let A be a positive quasi-Cartan companion of a skew-symmetrisable matrix B. Then, for every chordless cycle*

$$i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$$

in $\Gamma(B)$, the following holds:

$$\prod_{a=0}^{d-1}(-A_{i_a,i_{a+1}}) < 0.$$

Let Φ be a root system of rank n associated to a Dynkin diagram Δ and let B be the exchange matrix in a seed $(\mathbf{x}, B)$ of the cluster algebra of type Δ. Motivated by [1], Parsons [10, Defn. 4.1] defines a *companion basis* of B to be a subset $\beta_1, \beta_2, \ldots, \beta_n$ of Φ which is a $\mathbb{Z}$-basis of $\mathbb{Z}\Phi$ such that the matrix A with i, j entry $A_{ij} = (\beta_i, \beta_j)$ for all i, j is a quasi-Cartan companion of B. In the general case (not necessarily simply-laced), we use $A_{ij} = (\beta_i, \beta_j^\vee)$. Parsons uses companion bases to obtain the dimension vectors of the indecomposable modules over the cluster-tilted algebra associated to $(\mathbf{x}, B)$ by [3] and [4], in the type A case (they are obtained by expanding arbitrary roots in terms of the basis; see [10, Theorem 5.3]). We remark that independent later work of Ringel [11] also obtains such dimension vectors (in a more general setting which includes all the finite type cases).

Suppose that $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is a companion basis for an exchange matrix B in a cluster algebra of finite type as above, and fix $1 \leq k \leq n$. Parsons [10, Thm. 6.1] defines the (inward) *mutation* $\mathcal{B}' = \mu_k(\mathcal{B})$ of $\mathcal{B}$ at k to be the subset $\{\beta_1', \beta_2', \ldots, \beta_n'\}$ of Φ defined by

$$\beta_i' = \begin{cases} s_{\beta_k}(\beta_i) & \text{if there is an arrow from } i \text{ to } k \text{ in } \Gamma(B); \\ \beta_i & \text{otherwise.} \end{cases}$$

Outward mutation of $\mathcal{B}$ is defined similarly.

Proposition 6.3. [10, Thm. 6.1] *Let $\mathcal{B}$ be a companion basis for an exchange matrix B in a cluster algebra of finite simply-laced type and fix $1 \leq k \leq n$. Then $\mu_k(\mathcal{B})$ is a companion basis for $\mu_k(B)$.*

We remark that mutations of quasi-Cartan companion matrices were considered in [1].

Proposition 6.3 can be extended to all finite type cases using a similar argument:

Proposition 6.4. *Let $\mathcal{B}$ be a companion basis for an exchange matrix B in a cluster algebra of finite type and fix $1 \leq k \leq n$. Then $\mu_k(\mathcal{B})$ is a companion basis for $\mu_k(B)$.*

Proof. We must check that if $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is a companion basis for an exchange matrix B underlying a diagram Γ then $\mathcal{B}' = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} = \mu_k(\mathcal{B})$ is a companion basis for the mutation $B' = \mu_k(B)$, which underlies $\mu_k(\Gamma)$. Note that [10, Thm. 6.1] covers the cases (γ_p, γ_q) where $p = k$ or $q = k$ or where at most one of p and q is connected to k by an edge (in this case the assumption of simply-laced type is not used).

Hence, since [10, Thm. 6.1] covers the simply-laced case, we are reduced to cases (d),(e) and (f) in Corollary 2.3. In each case, we are only left with checking that $|(\gamma_i, \gamma_j)| = |B_{ij}'|$.

We denote by A the 3×3 matrix

$$\begin{pmatrix} (\beta_i, \beta_i^\vee) & (\beta_i, \beta_k^\vee) & (\beta_i, \beta_j^\vee) \\ (\beta_k, \beta_i^\vee) & (\beta_k, \beta_k^\vee) & (\beta_k, \beta_j^\vee) \\ (\beta_j, \beta_i^\vee) & (\beta_j, \beta_k^\vee) & (\beta_j, \beta_j^\vee) \end{pmatrix}.$$

We consider first case (d), mutating from left to right. There are two possibilities for A:

$$\begin{pmatrix} 2 & \pm 1 & 0 \\ \pm 2 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & \pm 2 & 0 \\ \pm 1 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}.$$

In the first case, the mutation of B is:

$$\begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

We have

$$(\gamma_i, \gamma_j^\vee) = (s_{\beta_k}(\beta_i), \beta_j^\vee)$$
$$= (\beta_i - (\beta_i, \beta_k^\vee)\beta_k, \beta_j^\vee)$$
$$= (\beta_i, \beta_j^\vee) - (\beta_i, \beta_k^\vee)(\beta_k, \beta_j^\vee) = \pm(\beta_i, \beta_k^\vee) = \pm 1,$$

and

$$(\gamma_j, \gamma_i^\vee) = (\beta_j, (s_{\beta_k}(\beta_i))^\vee)$$
$$= (s_{\beta_k}(\beta_j), \beta_i^\vee)$$
$$= (\beta_j - (\beta_j, \beta_k^\vee)\beta_k, \beta_i^\vee)$$
$$= (\beta_j, \beta_i^\vee) - (\beta_j, \beta_k^\vee)(\beta_k, \beta_i^\vee) = \pm(\beta_k, \beta_i^\vee) = \pm 2,$$

and this case is done.

In the second case, the mutation is:

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Arguing as above, we have $(\gamma_i, \gamma_j^\vee) = \pm 2$ and $(\gamma_j, \gamma_i^\vee) = \pm 1$ as required.

Next, we consider case (d), mutating from right to left. Note that the skew-symmetrisability of B imposes restrictions; see also [7, Lemma 7.6]. The only possibilities for B are thus

$$\begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

In the first case, the mutation is

$$\begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We have $(\gamma_i, \gamma_j^\vee) = (\beta_i, \beta_j^\vee) - (\beta_i, \beta_k^\vee)(\beta_k, \beta_j^\vee)$. Moreover, $|(\beta_i, \beta_j^\vee)| = |(\beta_i, \beta_k^\vee)| = |(\beta_k, \beta_j^\vee)| = 1$. Note that the matrix A must be positive (as the β_i form a basis of $\mathbb{Z}\Phi$). Hence, by Proposition 6.2, an odd number of the signs of $(\beta_i, \beta_j^\vee)$, $(\beta_i, \beta_k^\vee)$ and $(\beta_k, \beta_j^\vee)$ must be positive. We obtain the following table of possible values:

$(\beta_i, \beta_j^\vee)$	$(\beta_i, \beta_k^\vee)$	$(\beta_k, \beta_j^\vee)$	$(\gamma_i, \gamma_j^\vee)$
1	1	1	0
1	−1	−1	0
−1	1	−1	0
−1	−1	1	0

In the second case, the mutation is

$$\begin{pmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

As before, $(\gamma_i, \gamma_j^\vee) = (\beta_i, \beta_j^\vee) - (\beta_i, \beta_k^\vee)(\beta_k, \beta_j^\vee)$. We have $|(\beta_i, \beta_j^\vee)| = |(\beta_i, \beta_k^\vee)| = 2$ and $|(\beta_k, \beta_j^\vee)| = 1$. As in the previous case, we obtain a table of possible values:

$(\beta_i, \beta_j^\vee)$	$(\beta_i, \beta_k^\vee)$	$(\beta_k, \beta_j^\vee)$	$(\gamma_i, \gamma_j^\vee)$
2	2	1	0
2	−2	−1	0
−2	2	−1	0
−2	−2	1	0

and case (d) is complete.

For case (e), we note that replacing B with $-B$ (and keeping the β_i unchanged) gives an instance of case (d). Hence, we have $|\mu_k(-B)_{ij}| = |-B'_{ij}| = |(\gamma'_i, \gamma'_j)|$, where

$$\gamma'_i = \begin{cases} s_{\beta_k}(\beta_i) & i \to k \text{ in } \Gamma(-B); \\ \beta_i & \text{else,} \end{cases}$$

which coincides with $s_{\beta_k}(\gamma_i)$. Therefore

$$|B_{ij}| = |(s_{\beta_k}(\gamma_i), s_{\beta_k}(\gamma_i))| = |(\gamma_i, \gamma_j)|,$$

as required.

We are left with case (f) to consider. By symmetry, we only need to consider the mutation from the left hand side of the figure to the right hand side. Using the skew-symmetrisability of B (see [7, Lemma 7.6]) we conclude that there are only two possible cases for B, namely

$$\begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

In the first case, the mutation is

$$\begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

We have $(\gamma_i, \gamma_j^\vee) = (s_{\beta_k}(\beta_i), \beta_j^\vee) = (\beta_i, \beta_j^\vee) - (\beta_i, \beta_k^\vee)(\beta_k, \beta_j^\vee)$. Again, using Proposition 6.2, we obtain the following table of possible values:

$(\beta_i, \beta_j^\vee)$	$(\beta_i, \beta_k^\vee)$	$(\beta_k, \beta_j^\vee)$	$(\gamma_i, \gamma_j^\vee)$
1	1	2	−1
1	−1	−2	−1
−1	1	−2	1
−1	−1	2	1

A similar argument shows that $(\gamma_j, \gamma_i^\vee) = \pm 1$. In the second case, the mutation is:

$$\begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

As before, $(\gamma_i, \gamma_j^\vee) = (\beta_i, \beta_j^\vee) - (\beta_i, \beta_k^\vee)(\beta_k, \beta_j^\vee)$. and we obtain the following table of possibilities:

$(\beta_i, \beta_j^\vee)$	$(\beta_i, \beta_k^\vee)$	$(\beta_k, \beta_j^\vee)$	$(\gamma_i, \gamma_j^\vee)$
1	2	1	-1
1	-2	-1	-1
-1	2	-1	1
-1	-2	1	1

A similar argument shows that $(\gamma_j, \gamma_i^\vee) = \pm 1$. The proof is complete. $\qquad\square$

Remark 6.5. Every companion basis for $\mu_k(B)$ is of the form $\mu_k(\mathcal{B})$ for some companion basis $\mathcal{B}$ for B, since it can be mutated back to a companion basis for B using outward mutation, and outward and inward mutation are easily seen to be inverses of each other.

Corollary 6.6. [1] *Let B be an exchange matrix in a cluster algebra of finite type. Then B has a companion basis.*

Proof. This follows from results in [1] (see also [10, Cor. 3.10]). Alternatively, it can be proved using Proposition 6.4. $\qquad\square$

Remark 6.7. It is not the case that every subset of Φ forming a basis for $\mathbb{Z}\Phi$ is a companion basis, even if we just restrict to the positive roots. For example, in type A_4, the set of roots $\{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}$ is a such a subset, but is not a companion basis (the inner product of any pair of distinct roots in this subset is ± 1).

We can now prove:

Theorem 6.8. *Let $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_n\} \subseteq \Phi$ be a companion basis for an exchange matrix B in a cluster algebra of finite type Δ (with associated root system Φ). Then there is an isomorphism between W and $W_{\Gamma(B)}$ as defined in Section 3 taking s_{β_i} to s_i. In particular, the defining relations for $W_{\Gamma(B)}$ can be regarded as giving a presentation of W in terms of the generators $s_{\beta_1}, s_{\beta_2}, \dots, s_{\beta_n}$.*

Proof. Note that any simple system in Φ is a companion basis for an exchange matrix B whose Cartan counterpart is a Cartan matrix of type Δ. By Remark 3.1 it follows that the result holds for this case.

Suppose that the result holds for every companion basis associated to an exchange matrix B. Fix $1 \le k \le n$ and let $\mathcal{B}' = \{\beta_1', \beta_2', \dots, \beta_n'\}$ be a companion basis for $\mu_k(B)$. By Remark 6.5, $\mathcal{B}'$ is of the form $\mu_k(\mathcal{B})$ for some companion basis $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_n\}$ for B.

By assumption, the result holds for $\mathcal{B}$, i.e. there is a group isomorphism $\tau_\mathcal{B} : W \to W_{\Gamma(B)}$ taking s_{β_i} to s_i for $1 \le i \le n$. If there is no arrow from i to k in $\Gamma(B)$ then $\beta_i' = \beta_i$. So, $\tau_\mathcal{B}(s_{\beta_i'}) = \tau_\mathcal{B}(s_{\beta_i}) = s_i = t_i$. If there is an arrow from i to k in $\Gamma(B)$, we have:

$$s_{\beta_i'} = s_{s_{\beta_k}(\beta_i)} = s_{\beta_k} s_{\beta_i} s_{\beta_k}^{-1} = s_{\beta_k} s_{\beta_i} s_{\beta_k},$$

so $\tau_\mathcal{B}(s_{\beta_i'}) = s_k s_i s_k = t_i$.

From the proof of Theorem 5.4, there is an isomorphism $\psi : W_{\Gamma(B)} \to W_{\Gamma(B')}$. If the defining generators of $W_{\Gamma'}$ are denoted s_i', then we have $\psi(t_i) = s_i'$. Hence $\tau_{\mathcal{B}'} := \psi \circ \tau_{\mathcal{B}} : W \to W_{\Gamma'}$ is a group isomorphism taking $s_{\beta_i'}$ to s_i'. Therefore, the result holds for $\mathcal{B}'$.

The result follows in general by using induction on the number of mutations needed to obtain B from a fixed exchange matrix whose Cartan counterpart is a Cartan matrix of type Δ. $\qquad\square$

Finally, we explain how the perspective of companion bases allows an alternative proof of Theorem 5.4 in the simply-laced case using results from [5]. Let B be an exchange matrix of a cluster algebra of finite simply-laced type with corresponding diagram Γ. Let $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a companion basis for B and let A be the corresponding matrix with $A_{ij} = (\beta_i, \beta_j)$.

Let G be the unoriented graph on vertices $1, 2, \ldots, n$ with an edge between i and j whenever $A_{ij} \neq 0$. We define a map f from the set of edges of G to $\{1, -1\}$ by setting $f(\{i, j\})$ to be the sign of A_{ij} whenever $A_{ij} \neq 0$. In this way we obtain a signed graph $\mathcal{G}(\mathcal{B}) = (G, f)$ corresponding to $\mathcal{B}$, whose underlying unsigned graph coincides with the underlying unoriented graph of Γ.

Note that A is positive definite, since $\mathcal{B}$ is a companion basis. By [1, Prop. 1.4] or a remark in the proof of [5, Thm. 6.10], this implies that every chordless cycle in (G, f) has an odd number of positive signs.

We recall the definition of local switching from [5]. Since we are in the positive definite case, this takes a simpler form; see the comment after Remark 4.6 in [5].

Definition 6.9. [5, 4.3] *Let (G, f) be a signed graph and suppose that the symmetric matrix $A = A_{ij}$ with $A_{ij} = f(i, j)$ and $A_{ii} = 2$ for all i, j is positive definite. Fix a vertex k of G and let I be a subset of the neighbours of k in G. Let J be the set of neighbours of k in G that do not lie in I. Then the* local switching *of (G, f) at k is given by the following operations:*

(i) *Delete all edges of G between I and J.*
(ii) *For any $i \in I$ and $j \in J$ not originally joined in G, introduce an edge $\{i, j\}$ with sign chosen so that the 3-cycle between i, j, k has an even number (i.e. zero or two) of positive signs.*
(iii) *Change the signs of all edges between k and elements of I.*
(iv) *Leave all other edges and signs unchanged.*

Lemma 6.10. *Fix a vertex k of Γ. Let I be the set of vertices i of Γ for which there is an arrow $i \to k$ in Γ, so that J is the set of vertices j of Γ for which there is an arrow $k \to j$ in Γ. Then $(G', f') = \mathcal{G}(\mu_k(\mathcal{B}))$ coincides with the result of locally switching (G, f) at k with respect to I.*

Proof. By Proposition 6.3, G' is the underlying unoriented graph of $\mu_k(\Gamma)$. Let (G'', f') be the result of locally switching (G, f) at k with respect to I. Then it is easy to see from Definition 6.9 that G'' and G' coincide as unoriented graphs. Let $\mu_k(\mathcal{B}) = (\beta_1', \ldots, \beta_n')$. If $i \in I$ then

$$(\beta_i', \beta_k') = (s_{\beta_k}(\beta_i), \beta_k) = -(\beta_i, \beta_k)$$

and we see that the signs on the edges of G' incident with k and a vertex in I coincide with the signs on the corresponding edges of G''. If $\{i, j\}$ is an edge in G'' arising from step (ii) in Definition 6.9 its sign is chosen to ensure that the 3-cycle on i, j, k has an odd number of positive signs after step (iii) has been applied. Since (G', f') must also have this property (using Proposition 6.2 and the fact that $\mathcal{B}'$ is a companion basis). It is easy to see that the other signs on the edges of G' and G'' coincide, and the result follows. □

Since the signed graph $\mathcal{G}(\mathcal{B})$ has an odd number of positive signs in every cycle (and noting Proposition 4.6) we have that the group $\mathrm{Cox}(G, f)$ defined in [5, Defn. 6.2] coincides with the group W_Γ defined in Section 3, in the simply-laced case. Then Theorem 5.4 (in the simply-laced case) follows by combining [5, Thm. 6.10] and [10, Thm. 6.1] (see Proposition 6.3) with Lemma 6.10.

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