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Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifold

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In the winter of 1980-81 it was found that the totality of solutions of the Kadomtsev - Petviashvili equation as well as of its multi-component generalization forms an infinite dimensional Grassmann manifold [1]. In this picture the time evolution of a solution is interpreted as the dynamical motion of a point on this manifold. A generic solution corresponds to a generic point whose orbit (in the infinitely many time variables) is dense in the manifold, whereas degenerate solutions corresponding to points bound on those closed submanifolds which are stable under the time evolution describe the solutions to various specialized equations such as KdV, Boussinesq, nonlinear Schrödinger, sine-Gordon, etc.

We foresee that a similar structural theory should hold also for multi-dimensional 'integrable' systems.

§1. The universal Grassmann manifold

For a vector space $V=V(N)$ (say, over $\mathbb{C}$) of dimension N $(=m+n)$ the Grassmann manifold $GM(m,V)$ $(=GM(m,n))$ is by definition the parameter space for the totality of m-dimensional subspaces in V. We can write

$$GM(m,V) = \{m\text{-frames in } V\} \,/\, GL(m)$$

where an m-frame means an m-tuple of linearly independent vectors. $GM(m,V)$ is a homogeneous space of the general linear group $GL(V)$.

Further, it is viewed as an algebraic submanifold (of dimension mn) of the $\binom{N}{m}-1$ dimensional projective space $\mathbb{P}(\wedge^m V)$ by letting an m-frame $(\xi^{(0)},\cdots,\xi^{(m-1)})$ correspond to the exterior product $\xi^{(0)}\wedge\cdots\wedge\xi^{(m-1)}\in\wedge^m V$ (the canonical projective embedding). If $\xi^{(i)}=\xi_{0i}e_0+\cdots+\xi_{N-1,i}e_{N-1}$ where $e_0,\cdots,e_{N-1}$ denote a basis of V, then $\xi^{(0)}\wedge\cdots\wedge\xi^{(m-1)}=\sum_{0\leq\ell_0<\cdots<\ell_{m-1}<N}\xi_{\ell_0\cdots\ell_{m-1}}e_{\ell_0}\wedge\cdots\wedge e_{\ell_{m-1}}$ with $\xi_{\ell_0\cdots\ell_{m-1}}=\det(\xi_{\ell_i j})_{i,j=0,\cdots,m-1}$. These $\xi_{\ell_0\cdots\ell_{m-1}}$, $0\leq\ell_i<N$, (which are antisymmetric in suffixes) satisfy the Plücker's relations:

$$\sum_{i=0}^{m}(-)^i\xi_{k_0\cdots k_{m-2}\ell_i}\xi_{\ell_0\cdots\hat{\ell}_i\cdots\ell_m}=0$$

and vice versa; i.e. a point in the ambient $\mathbb{P}(\wedge^m V)$ lies in the embedded $GM(m,V)$ if and only if its projective coordinates $\xi_{\ell_0\cdots\ell_{m-1}}$, $0\leq\ell_i<N$, satisfy the Plücker's relations (i.e. are Plücker coordinates).

To each set of suffixes $(\ell_0,\cdots,\ell_{m-1})$, $0\leq\ell_0<\cdots<\ell_{m-1}<N$, we associate a Young diagram Y consisting of rows of length $\ell_{m-1}-(m-1),\cdots,\ell_1-1,\ell_0$, respectively (cf. H. Weyl, The Classical Groups, Princeton, 1939) and often identify them; e.g. Plücker coordinates are also written ξ_Y, the diagrams Y being those contained in the $m\times n$ rectangular diagram Δ_{mn}.

After Weyl's celebrated work Young diagrams (of vertical size $\leq N$) classify irreducible tensor representations of $GL(V)$. Denoting by R_{ij} the contragredient of the irreducible representation space labeled by the $i\times j$ rectangular diagram Δ_{ij}, our $GM(m,V)$ is the projective algebraic manifold corresponding to the graded algebra $\bigoplus_{j=0}^{\infty}R_{mj}$. (Here multiplication is unambiguously defined because $R_{mi}\otimes R_{mj}$ containes $R_{m,i+j}$ exactly once.) We can also write:

$$GM(m,V)=(\widetilde{GM}(m,V)-\{0\})\,/\,GL(1),$$

where $\widetilde{GM}(m,V)=\{(\xi_Y)_{Y\subset\Delta_{mn}}\,|\,\xi_Y$ satisfy the Plücker's relations$\}\subset\wedge^m V$.

Let $m\leq m'$ and $n\leq n'$. Then: (i) if $(\xi'_Y)_{Y\subset\Delta_{m'n'}}$ satisfies the Plücker's relations, so does its restriction to Y's within Δ_{mn} (whence $\widetilde{GM}(m',n')\to\widetilde{GM}(m,n)$). On the other hand, (ii) $(\xi_Y)_{Y\subset\Delta_{mn}}$ satisfies the Plücker's relations if and only if $(\xi'_Y)_{Y\subset\Delta_{m'n'}}$ does, ξ'_Y being defined by $\xi'_Y = \xi_Y$ or $= 0$ according as $Y\subset\Delta_{mn}$ or not (whence $\widetilde{GM}(m,n)\hookrightarrow\widetilde{GM}(m',n')$. (i) and (ii) combined give the commutative diagram

$$\begin{array}{ccc} \widetilde{GM}(m',n') & \longrightarrow & \widetilde{GM}(m,n) \quad \text{(restriction)} \\ \text{id} \uparrow\wr & & \text{id} \uparrow\wr \\ \widetilde{GM}(m',n') & \hookleftarrow & \widetilde{GM}(m,n) \quad \text{(embedding)}. \end{array}$$

Hence, defining the universal Grassmann manifold $GM = (\widetilde{GM}-\{0\}) / GL(1)$ and its dense submanifold $GM^{fin} = (\widetilde{GM}^{fin} - \{0\}) / GL(1)$ by

$$\widetilde{GM} = \{(\xi_Y)_{Y:\text{all diagrams}} \mid \xi_Y \text{ satisfy all the Plücker's relations}\},$$

$$\widetilde{GM}^{fin} = \{(\xi_Y)_Y \in\widetilde{GM} \mid \xi_Y = 0 \text{ for almost all } Y\}$$

respectively, we have

$$\widetilde{GM} = \{(\xi_Y)_{Y:\text{all diagrams}} \mid (\xi_Y)_{Y\subset\Delta_{mn}} \in\widetilde{GM}(m,n) \text{ for any } m \text{ and } n\},$$

$$\widetilde{GM}^{fin} = \bigcup_{m,n} \widetilde{GM}(m,n), \text{ and}$$

$$\begin{array}{ccc} \widetilde{GM} & \xrightarrow{\text{surjective}} & \widetilde{GM}(m,n) \\ \text{dense} \uparrow\cup & & \text{id} \uparrow\wr \\ \widetilde{GM}^{fin} & \hookleftarrow & \widetilde{GM}(m,n). \end{array}$$

To each $\xi\in GM(m,n)$ (resp. $\in GM$) uniquely corresponds a diagram $Y\subset\Delta_{mn}$ (resp. an unrestricted Y) in such a way that, for the Plücker coordinates of ξ, $\xi_Y \neq 0$ while $\xi_{Y'} = 0$ unless $Y'\supset Y$; and, denoting by $GM^Y(m,n)$ those points to which the given Y corresponds, we have a cellular decomposition $GM(m,n) = \coprod_{Y\subset\Delta_{mn}} GM^Y(m,n)$, with $GM^Y(m,n) \simeq \mathbb{C}^{mn-|Y|}$, $|Y|$ = size of $Y = \ell_0+\cdots+\ell_{m-1}-\frac{1}{2}m(m-1)$ (resp. $GM = \coprod_Y GM^Y$).

Consider the infinite dimensional vector space $\mathbf{V}$ (resp. $\dot{\mathbf{V}}$) consisting of elements $\xi = (\xi_\nu)_{\nu\in\mathbb{Z}}$, with $\xi_\nu \in \mathbb{C}$, $\xi_\nu = 0$ for $\nu \ll 0$ (resp. for $\nu \gg 0$). (Setting $e_\mu = (\delta_{\mu\nu})_{\nu\in\mathbb{Z}} \in \mathbb{V}$ one also writes $\xi = \sum_{-\infty\ll\nu<\infty} \xi_\nu e_\nu$ (resp. $\xi = \sum_{-\infty<\nu\ll\infty} \xi_\nu e_\nu$).) Further, by introducing the dual (or contragredient) basis $(e^*_\mu)_{\mu\in\mathbb{Z}}$ to $(e_\mu)_{\mu\in\mathbb{Z}}$ and the dual space $\mathbb{V}^* = \{\xi^* = \sum_{-\infty<\nu\ll\infty} \xi^*_\nu e^*_\nu \mid \xi^*_\nu \in \mathbb{C}\}$ (resp. $\dot{\mathbb{V}}^* = \{\xi^* = \sum_{-\infty\ll\nu<\infty} \xi^*_\nu e^*_\nu \mid \xi^*_\nu \in \mathbb{C}\}$) to $\mathbb{V}$ (resp. to $\dot{\mathbb{V}}$) so that their pairing is given by the effectively finite sum: $\langle\xi^*,\xi\rangle = \sum \xi^*_\nu \xi_\nu$, our vector space naturally acquires the weak topology (or rather, S. Lefschetz's linear topology, in which our space is locally linearly compact). (Any locally convex topology on a vector space induces via its dual a linear topology there, and its subsapce is closed by the latter if and only if it is so by the former.)

Define subspaces $\mathbb{V}^{(m)}$ of $\mathbb{V}$ (resp. subspaces $\dot{\mathbb{V}}^{(m)}$ of $\dot{\mathbb{V}}$), $m \in \mathbb{Z}$, by $\mathbb{V}^{(m)}$ (resp. $\dot{\mathbb{V}}^{(m)}$) $= \{(\xi_\nu)_{\nu\in\mathbb{Z}} \in \mathbb{V}$ (resp. $\dot{\mathbb{V}}$) $\mid \xi_\nu = 0$ for $\nu<m\}$. Then we have

$$\begin{aligned} GM(\text{resp. } GM^{fin}) &= \{\text{closed subspaces } V \text{ of } \mathbb{V} \text{ (resp. } \dot{\mathbb{V}}) \mid \text{The dimensions of Ker and Coker of the natural map } V \to \mathbb{V}/\mathbb{V}^{(0)} \text{ (resp. } \to \dot{\mathbb{V}}/\dot{\mathbb{V}}^{(0)}) \text{ are both finite and coincide.}\} \\ &= \{\text{closed subspaces } V \text{ of } \mathbb{V} \text{ (resp. } \dot{\mathbb{V}}) \mid \dim V\cap\mathbb{V}^{(\nu)} \text{ (resp. } \dim V\cap\dot{\mathbb{V}}^{(\nu)}) = |\nu| \text{ for } \nu \ll 0\}, \end{aligned}$$

where the closedness of V is a consequence of the other conditions and the qualifier is dispensable for $V\subset\mathbb{V}$, while it is not for $V\subset\dot{\mathbb{V}}$. Also we have, for any diagram Y parametrized by $(\ell_0,\cdots,\ell_{m-1})$,

$$GM^Y = \{V\subset\mathbb{V} \mid \dim V\cap\mathbb{V}^{(\nu)} \leq k \text{ if and only if } \nu > -m+\ell_{m-1-k} \ (\nu\in\mathbb{Z},\ k\in\mathbb{N})\},$$

understanding that $\ell_\nu = \nu$ for $\nu < 0$.

Between these extremes, GM and GM^{fin}, come various intermediates. For example we define, for $r=1,2,\cdots$,

$$GM^{ana(r)} = \{(\xi_Y)_Y \in GM \mid \sqrt[|Y|]{|\xi_Y|/(|Y|/r)!} \text{ are bounded as } |Y| \to \infty\}$$

and for $0 \leq a < \infty$,

$$GM^{\exp(a)} = \{(\xi_Y)_Y \in GM \mid \overline{\lim_{|Y|\to\infty}} \sqrt[|Y|]{|\xi_Y|} \leq a\},$$

so that we have

$$GM \supset GM^{ana(1)} \supset GM^{ana(2)} \supset \cdots \supset GM^{\exp(a)} \supset GM^{\exp(0)} \supset GM^{fin}.$$

Then

$GM^{ana(r)}$ (resp. $GM^{\exp(a)}$) = {closed subspaces V of $\mathbb{V}^{ana(r)}$ (resp. of $\mathbb{V}^{\exp(a)}$) | The dimensions of Ker and Coker of the natural map $V \to \mathbb{V}^{ana(r)}/(\mathbb{V}^{ana(r)})^{(0)}$ (resp. $\to \mathbb{V}^{\exp(a)}/(\mathbb{V}^{\exp(a)})^{(0)}$) are both finite and coincide.}

where

$\mathbb{V}^{ana(r)}$ (resp. $\mathbb{V}^{\exp(a)}$) = $\{(\xi_\nu)_{\nu\in Z} \mid \sqrt[\nu]{|\xi_\nu|/(\nu/r)!}$ are bounded and $\sqrt[\nu]{|\xi_{-\nu-1}|(\nu/r)!}$ tend to 0 as $\nu\to\infty$ (resp. $\overline{\lim_{\nu\to\infty}}\sqrt[\nu]{|\xi_\nu|} \leq a$, $\overline{\lim_{\nu\to\infty}}\sqrt[\nu]{|\xi_{-\nu-1}|} < a^{-1}$)},

$\mathbb{V}^{ana(r)*}$ (resp. $\mathbb{V}^{\exp(a)*}$) = $\{(\xi_\nu)_{\nu\in Z} \mid (\xi_{-\nu-1})_{\nu\in Z} \in \mathbb{V}^{ana(r)}$ (resp. $\mathbb{V}^{\exp(a)}$)}

and

$(\mathbb{V}^{ana(r)})^{(m)}$ (resp. $(\mathbb{V}^{\exp(a)})^{(m)}$) = $\{(\xi_\nu)_{\nu\in Z} \in \mathbb{V}^{ana(r)}$ (resp. $\mathbb{V}^{\exp(a)}$) | $\xi_\nu = 0$ for $\nu < m\}$.

§2. Time evolution on GM

Denoting by Λ the shift operator:

$$\Lambda e_\nu = e_{\nu-1}, \quad \Lambda\sum\xi_\nu e_\nu = \sum\xi_\nu e_{\nu-1},$$

we define for $\xi \in \widetilde{GM}$ its evolution in time variables $t = (t_1, t_2, \cdots)$ by $\xi(t) = e^{t_1\Lambda + t_2\Lambda^2+\cdots}\xi$.

In the case of $\xi \in \widetilde{GM}^{fin}$, $\xi(t)$ is again in $\widetilde{GM}^{fin}$ for any $t_\nu \in \mathbb{C}$, $\nu=1,2,\cdots$. For general $\xi \in \widetilde{GM}$, however, $\xi(t)$ should be understood as a generalized element whose components are formal power series in $(t_1, t_2, \cdots)$ rather than complex numbers. (In the case of $\xi \in \widetilde{GM}^{ana(r)}$, one has $\xi(t) \in \widetilde{GM}^{ana(r)}$ if $|t_\nu|$ is sufficiently small for $\nu = r$ and are 0 for $\nu > r$. For $\xi \in \widetilde{GM}^{\exp(a)}$, one has $\xi(t) \in \widetilde{GM}^{\exp(a)}$ for $t_\nu \in \mathbb{C}$ subject to the condition $\overline{\lim}\sqrt[\nu]{|t_\nu|} < a^{-1}$.)

In any case we have, for the Plücker coordinates $\xi_Y(t)$ of $\xi(t)$,

$$\xi_Y(t) = \chi_Y(\partial_t)\xi_\phi(t) \quad \text{and} \quad \xi_\phi(t) = \sum_Y \xi_Y \cdot \chi_Y(t),$$

where ϕ denotes the empty Young diagram, $\chi_Y(t)$ denotes the character polynomial for the general linear group, and $\chi_Y(\partial_t)$ denotes the differential operator obtained from $\chi_Y(t)$ by replacing t_ν by $\frac{1}{\nu}\frac{\partial}{\partial t_\nu}$. (After H. Weyl, $\chi_Y(t)$ admits various expressions, one of which is

$$\chi_Y(t) = \sum_{\nu_1+2\nu_2+\cdots=|Y|} \pi_Y(1^{\nu_1}2^{\nu_2}\cdots)\frac{t_1^{\nu_1}t_2^{\nu_2}\cdots}{\nu_1!\nu_2!\cdots},$$

where $\pi_Y(1^{\nu_1}2^{\nu_2}\cdots)$ is the irreducible character of the symmetric permutation group of $|Y|$ letters, labeled by the Young diagram Y and evaluated at the conjugacy class consisting of ν_1 cycles of size 1, ν_2 cycles of size 2, etc.)

We call $\xi_\phi(t)$ the τ function of ξ (Notation: $\tau(t;\xi)$ or $\tau(t)$). The above formulae show that $\tau(t)$ plays the role of generating function for Plücker coordinates:

$$\xi_Y(t) = \chi_Y(\partial_t)\tau(t;\xi), \quad \xi_Y = \chi_Y(\partial_t)\tau(t;\xi)\,|_{t\mapsto 0},$$

$$\tau(t'+t;\xi) = \tau(t';\xi(t)) = \sum_Y \xi_Y(t)\chi_Y(t'),$$

and that the Plücker's relations for $(\xi_Y(t))_Y$ assume the form of quadratic differential equations, or, what amounts to the same, the form of 'bilinear' equations of R. Hirota.

Summing up, we have

Theorem 1. Although any $f(t) \in \mathbb{C}[[t_1,t_2,\cdots]]$ admits the formal expansion of the form: $f(t) = \sum_Y c_Y\chi_Y(t)$, where the coefficients are uniquely given by $c_Y = \chi_Y(\partial_t)f(t)\,|_{t\mapsto 0}$, it represents the τ function of some $\xi \in \widetilde{GM}$ if and only if its coefficients c_Y satisfy the Plücker's relations.

Theorem 2. An $f(t) \in \mathbb{C}[[t_1,t_2,\cdots]]$ is the τ function of some $\xi \in GM$ if and only if it satisfies the Hirota bilinear equations of the form

$$\sum_{i=0}^{m} (-)^i \chi_{k_0\cdots k_{m-2}\ell_i}\left(\frac{D_t}{2}\right)\chi_{\ell_0\cdots\hat{\ell}_i\cdots\ell_m}\left(-\frac{D_t}{2}\right)\tau\cdot\tau = 0.$$

Moreover these exhaust all the Hirota equations to be satisfied by τ.

These quadratic differential equations are also equivalent to the quadratic difference equations. Namely,

Theorem 3. (Addition formulae) For any $\alpha \in \mathbb{C}$ we set $[\alpha] = (\alpha,\frac{1}{2}\alpha^2,\frac{1}{3}\alpha^3,\cdots)$ so that $t+[\alpha] = (t_1+\alpha,\ t_2+\frac{1}{2}\alpha^2,\cdots)$. Let $\alpha_i \in \mathbb{C}$ for $i = 0,\cdots,N-1$ and define

$$\zeta_{\ell_0\cdots\ell_{m-1}}(t) = \Delta(\alpha_{\ell_{m-1}},\cdots,\alpha_{\ell_0})\tau(t+[\alpha_{\ell_0}]+\cdots+[\alpha_{\ell_{m-1}}]),\ 0 \leq \ell_i < N$$

with $\Delta(\alpha_{m-1},\cdots,\alpha_0) = \prod_{m>i>j\geq 0}(\alpha_i-\alpha_j)$. Then $\zeta_{\ell_0\cdots\ell_{m-1}}(t)$ satisfy the Plücker's relations for $GM(m,V(N))$. This property again characterizes the function τ.

E.g. we have

$$(\alpha_1-\alpha_0)(\alpha_3-\alpha_2)\tau(t+[\alpha_0]+[\alpha_1])\tau(t+[\alpha_2]+[\alpha_3])$$
$$-(\alpha_2-\alpha_0)(\alpha_3-\alpha_1)\tau(t+[\alpha_0]+[\alpha_2])\tau(t+[\alpha_1]+[\alpha_3])$$
$$+(\alpha_3-\alpha_0)(\alpha_2-\alpha_1)\tau(t+[\alpha_0]+[\alpha_3])\tau(t+[\alpha_1]+[\alpha_2]) = 0.$$

Denote by $E_{\nu\mu}$ the linear operator on $\mathbb{V}$ sending e_μ to e_ν and all the other e_κ, $\kappa \neq \mu$, to 0 (i.e. $E_{\nu\mu}\sum\xi_\kappa e_\kappa = \xi_\mu e_\nu$), and by $L_{\mu\nu}$ the vector field on $\widetilde{GM}$ induced by $E_{\nu\mu}$ (i.e. $(1+\varepsilon L_{\mu\nu})F(\xi) \equiv F((1+\varepsilon E_{\nu\mu})\xi) \mod \varepsilon^2$ for any function F on $\widetilde{GM}$). Since any $F(\xi)$ is a function of the Plücker coordinates ξ_Y's of ξ, $L_{\mu\nu}$ is also characterized by: $L_{\mu\nu}\xi_{\ell_0\cdots\ell_{m-1}} = \sum_{0\leq i<m}\delta_{\nu+m,\ell_i}\times \xi_{\ell_0\cdots\mu+m\cdots\ell_{m-1}}$ assuming $\nu+m$ and $\mu+m \geq 0$. (This poses no restriction on the diagram Y labelled by $(\ell_0,\cdots,\ell_{m-1})$ since $(0,1,\cdots,k-1,\ell_0+k,\cdots,\ell_{m-1}+k)$ also labels the same Y for any $k \in \mathbb{N}$.)

For the shift operator Λ we have: $\Lambda^n = \sum_{\nu\in\mathbb{Z}} E_{\nu,\nu+n}$, $n \in \mathbb{Z}$. Further, define the operator K s.t. $\Lambda K - K\Lambda = 1$ by $K\sum\xi_\nu e_\nu = \sum\nu\xi_{\nu-1}e_\nu$ to have $f(K\Lambda)\Lambda^n = \sum_\nu f(\nu)E_{\nu,\nu+n}$ for any polynomial $f(\nu)$ of ν, and in particular, $\frac{K^k}{k!}\Lambda^{k+n} = \sum_\nu \binom{\nu}{k}E_{\nu,\nu+n}$.

For $n\neq 0$, the infinitesimal operator $1+\varepsilon f(K\Lambda)\Lambda^n$, mod ε^2, induces the well-defined infinitesimal transformation on $\widetilde{GM}$, and one can write

$$(1+\varepsilon U_n^{(k)})F(\xi) \equiv F((1+\varepsilon\frac{K^k}{k!}\Lambda^{k+n})\xi) \quad \text{mod } \varepsilon^2$$

with $U_n^{(k)} = \sum_\nu\binom{\nu}{k}L_{\nu+n,\nu}$, while for $n=0$ we introduce another vector field M defined by $M\xi_Y = \xi_Y$ and set: $U_0^{(k)} = \sum_{\nu<0}\binom{\nu}{k}(L_{\nu\nu}-M) + \sum_{\nu\geq 0}\binom{\nu}{k}L_{\nu\nu}$ to have an well-defined vector field on $\widetilde{GM}$. (Indeed, $U_0^{(k)}\xi_Y = f_k(Y)\xi_Y$ where $f_k(Y) = \sum_{0\leq i<m}\left(\binom{\ell_i-m}{k} - \binom{i-m}{k}\right)$. In particular $U_0^{(0)} = 0$.) M commutes $L_{\mu\nu}$ and $U_\nu^{(k)}$, and $U_\nu^{(k)}$'s satisfy the commutation relation

$$[U_\nu^{(k)},U_\mu^{(\ell)}] = \sum_{j\geq 0}\left(\binom{\ell+\mu}{j}\binom{k+\ell-j}{\ell}-\binom{k+\nu}{j}\binom{k+\ell-j}{k}\right)U_{\nu+\mu}^{(k+\ell-j)}+\delta_{\nu,-\mu}\cdot(-)^\ell\binom{\ell+\mu}{\ell+k+1}M.$$

<u>Theorem 4.</u> $\tau(t;\xi)$, as the function of t and $\xi\in\widetilde{GM}$, satisfies, and is characterized up to an arbitrary constant factor by, the following holonomic system of linear differential equations:

$$((L_{\mu'\nu'}-\delta_{\mu'\nu'})L_{\mu\nu} + (L_{\mu\nu'}-\delta_{\mu\nu'})L_{\mu'\nu})\tau = 0 \quad \text{for } \mu,\mu',\nu,\nu'\in\mathbf{Z},$$

$$(U_n^{(0)} - \frac{\partial}{\partial t_n})\tau = 0,\ (U_{-n}^{(0)} - nt_n)\tau = 0 \quad \text{for } n=1,2,\cdots.$$

Indeed, the first equations (which are of the second order) restrict the solution to a linear form $\sum_Y c_Y(t)\xi_Y$ of the Plücker coordinates ξ_Y while the remaining (first order) equations fix the coefficients $c_Y(t)$ to $c\cdot\chi_Y(t)$.

Here we see that the holonomic system of these linear equations on $\{t\}\times\widetilde{GM}$ produces no linear equation but the system of non-linear (quadratic or Hirota) equations of Theorem 2, upon elimination of the variables $\xi\in\widetilde{GM}$ (i.e. upon taking the direct image by the projection $\{t\}\times\widetilde{GM}\to\{t\}$), in a sharp contrast to the finite dimensional case [2].

Also remarkable is the close resemblance between this holonomic system and the system characterizing theta functions [3]. (Theorem 3 also suggests analogy between τ and θ.)

The holonomic system generated by these equations in Theorem 4 contains also the equations of the form: $(U_\nu^{(k)}-T_\nu^{(k)})\tau = 0$, $k\in\mathbb{N}$, $\nu\in\mathbb{Z}$, where $T_\nu^{(k)}$ is a differential operator in t of the form: $T_\nu^{(k)} = \frac{1}{k!}\sum_{\nu_0,\ldots,\nu_{k-1}\in\mathbb{Z},\nu_0+\cdots+\nu_{k-1}=\nu} s_{\nu_0}s_{\nu_1}\cdots s_{\nu_{k-1}}$ + terms of lower degree, with $s_\nu = \frac{\partial}{\partial t_\nu}$, 0, or $|\nu|t_{|\nu|}$ according as $\nu>0$, $=0$, or <0. ($T_0^{(0)}=0$, $T_\nu^{(0)}=s_\nu$.)

§3. Soliton equations and their solutions

Consider the totality $\mathcal{E}_{\mathcal{R}}$ of the microdifferential operators in the formal category $P = \sum_{-\infty<\nu\ll\infty} a_\nu(x)(\frac{d}{dx})^\nu$, where the coefficients $a_\nu(x)$ are taken from a given differential ring $\mathcal{R}$ (i.e. an associative algebra endowed with the derivation $\frac{d}{dx}:\mathcal{R}\to\mathcal{R}$). If $a_\nu(x)=0$ for $\nu>m$ we write $P\in\mathcal{E}_{\mathcal{R}}^{(m)}$. Together with P its adjoint $P^* = \sum(-\frac{d}{dx})^\nu a_\nu(x)$ is again in $\mathcal{E}_{\mathcal{R}}$, and for $P,Q\in\mathcal{E}_{\mathcal{R}}$ their product $PQ\in\mathcal{E}_{\mathcal{R}}$ is well-defined by employing the Leibniz rule $(\frac{d}{dx})^\nu a(x) = \sum_{k\geq 0}\binom{\nu}{k}a^{(k)}(x)(\frac{d}{dx})^{\nu-k}$ for $\nu\in\mathbb{Z}$. Setting $a_{-1}(x) = \mathrm{Res}\,P\,dx$, we have $\mathrm{Res}\,Pdx = -\mathrm{Res}\,P^*dx$. Thus $\mathcal{E}_{\mathcal{R}}$ constitutes a (non-commutative) ring including $\mathcal{D}_{\mathcal{R}}$ = {differential oprators} as a subring. We have: $P = P_+ + P_-$ with $P_+ = \sum_{0\leq\nu\ll\infty} a_\nu(x)(\frac{d}{dx})^\nu\in\mathcal{D}_{\mathcal{R}}$, $P_- = \sum_{\nu<0}a_\nu(x)(\frac{d}{dx})^\nu\in\mathcal{E}_{\mathcal{R}}^{(-1)}$, yielding the decomposition

$$\mathcal{E}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}}\oplus\mathcal{E}_{\mathcal{R}}^{(-1)}.$$

In the following we choose $\mathcal{R} = \mathbb{C}[[x]]$, the ring of formal power series in x, and simply write $\mathcal{E}_{\mathbb{C}[[x]]} = \mathcal{E}$; similarly with $\mathcal{E}^{(m)}$ and $\mathcal{D}$. Then $\mathbb{V}$ of §1 is canonically isomorphic to the quotient module of $\mathcal{E}$ by its maximal left ideal $\mathcal{E}x$ as left $\mathcal{E}$ modules, by letting $\xi = \sum_{-\infty\ll\nu<\infty}\xi_\nu e_\nu\in\mathbb{V}$ correspond to the residue class of $\sum\xi_\nu(\frac{d}{dx})^{-\nu-1}$ mod $\mathcal{E}x$ and the action of $P(x,\frac{d}{dx})\in\mathcal{E}$ on ξ be defined by $\xi \mapsto P(K,\Lambda)\xi$. Hereafter we identify them: $\mathbb{V} = \mathcal{E}/\mathcal{E}x$. Further we write $\mathbb{V} = \mathbb{V}^*$ by identifying $\sum \xi_\nu e_\nu \in \mathbb{V}$ with $\sum (-)^\nu \xi_{-\nu-1} e^*_\nu \in \mathbb{V}^*$, so that we have: $\langle \xi', \xi \rangle = -\langle \xi, \xi' \rangle$, $\langle \xi', P\xi \rangle = \langle P^*\xi', \xi \rangle$.

We also set

$$\mathcal{E}^{ana} = \{\sum_\nu a_\nu(x)(\tfrac{d}{dx})^\nu \mid \exists \delta > 0 \text{ s.t. } a_\nu(x) \text{ are holomorphic in } |x| < \delta,$$
$$\sqrt[\nu]{|a_\nu(x)|/\nu!} \to 0 \text{ and } \sqrt[\nu]{\nu!\,|a_{-\nu-1}(x)|} \text{ are bounded as } \nu \to \infty,$$
$$\text{both uniformly in } |x| < \delta\},$$

$$\dot{\mathcal{E}} = \{\sum_{-\infty \ll \nu < \infty} a_\nu(x)(\tfrac{d}{dx})^\nu \mid \exists k \in \mathbb{N} \text{ s.t. } a_\nu(x) \text{ are polynomials of } x \text{ of degree} \leq k\},$$

and get: $\mathbb{V}^{ana(1)} = \mathcal{E}^{ana}/\mathcal{E}^{ana}x$, $\dot{\mathbb{V}} = \dot{\mathcal{E}}/\dot{\mathcal{E}}x$.

Consider the operator $W = \sum_{\nu \geq 0} w_\nu \cdot (\frac{d}{dx})^{-\nu} \in \mathcal{E}^{(0)}_{\mathcal{R}}$ which is monic (i.e. $w_0 = 1$) so that W^{-1} is again an operator of the same kind which we shall write $W^{-1} = \sum_{\nu \geq 0} (\frac{d}{dx})^{-\nu} \cdot w^*_\nu$ with $w^*_0 = 1$. Let $\mathcal{W}$ denote the totality of such monic operators $W \in \mathcal{E}^{(0)}_{\mathcal{R}}$ with $\mathcal{R} = \mathbb{C}((x))$ (= the field of formal Laurent series in x, which is the field of quotients of $\mathbb{C}[[x]]$), satisfying the additional condition that there exists $m, n \in \mathbb{N}$ s.t. $x^m W$ and $W^{-1}x^n$ both $\in \mathcal{E}^{(0)}_{\mathbb{C}[[x]]}$ (i.e. $x^m w_\nu$ and $x^n w^*_\nu \in \mathbb{C}[[x]]$ for $\nu = 1, 2, \ldots$). Set $V^\phi = \bigoplus_{\nu < 0} \mathbb{C} e_\nu \subset \mathbb{V}$; V^ϕ is also characterized by the property that its Plücker coordinates $\xi_Y = 1$ or 0 according as $Y = \phi$ or not. For $W \in \mathcal{W}$ we set $\gamma(W) = (W^{-1}x^n)V^\phi$, where n is so chosen that $W^{-1}x^n \in \mathcal{E}^{(0)}_{\mathbb{C}[[x]]}$. This definition of $\gamma(W)$ does not depend on the choice of such n. (This is because $xV^\phi = V^\phi$.)

<u>Theorem 5</u>. For $W \in \mathcal{W}$, $\gamma(W) \in GM$ and this map is bijective, namely

$$\gamma : \mathcal{W} \xrightarrow{\sim} GM.$$

In this correspondence, the inverse images of GM^{fin} and $GM^{ana(1)}$ are given by $\mathcal{W}^{fin}$ and $\mathcal{W}^{ana(1)}$, respectively, where $\mathcal{W}^{ana(1)} = \mathcal{W} \cap \mathcal{E}^{ana(1)}$ and

$$\mathcal{W}^{fin} = \mathcal{W} \cap \dot{\mathcal{E}} = \{W \in \mathcal{W} \mid \exists m, n \in \mathbb{N}, \text{ s.t. } W(\tfrac{d}{dx})^m \text{ and } (\tfrac{d}{dx})^n W^{-1} \in \mathcal{D}_{\mathbb{C}((x))}\}.$$

Theorem 6. Let $\xi(t) = e^{t_1\Lambda + t_2\Lambda^2 + \cdots}\xi$ as in §2, and let $W = W(t) = \gamma^{-1}(\xi(t))$ $\in \mathcal{W}$ be the corresponding microdifferential operator. Then the evolution of $W(t)$ in t is given by

$$[W] \qquad \frac{\partial W}{\partial t_n} = B_n W - W(\frac{d}{dx})^n, \text{ with } B_n = (W(\frac{d}{dx})^n W^{-1})_+ .$$

(B_n is a differential operator of the n-th order.)

Theorem 5 tells that conversely any solution of [W] is given in the above form in a unique way. More explicitly we have

$$w_\nu = \left.\frac{p_\nu(-\partial_t)\tau(t;\xi)}{\tau(t;\xi)}\right|_{t_1 \mapsto x+t_1}, \quad w^*_\nu = \left.\frac{p_\nu(\partial_t)\tau(t;\xi)}{\tau(t;\xi)}\right|_{t_1 \mapsto x+t_1},$$

where p_ν represents the character polynomial χ_Y for $Y = \Delta_{1,\nu}$.

Put $L = W\frac{d}{dx}W^{-1}$. Then $L = \sum_{\nu\geq 0} u_\nu(\frac{d}{dx})^{1-\nu}$ with $u_0=1$, $u_1=0$, and u_ν are differential polynomials of $w_1,\cdots,w_{\nu-1}$, and the above system of evolution equations for W immediately implies that for L as follows:

$$[L] \qquad \frac{\partial L}{\partial t_n} = B_n L - LB_n, \text{ with } B_n = (L^n)_+$$

which is also equivalent to the following system:

$$[B] \qquad \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + B_m B_n - B_n B_m = 0,$$

which constitutes the integrability condition for

$$[\psi] \qquad \frac{\partial \psi}{\partial t_n} = B_n \psi.$$

(Incidentally, the explicit solution for ψ is given by $\psi = \left.\frac{\tau(t;\xi')}{\tau(t;\xi)}\right|_{t_1 \mapsto t_1+x}$, where ξ' is any element of GM containing $\Lambda\xi$ (as subspaces in $\mathbb{V}$).)

[L] (or [B]) gives infinite number of non-linear equations for $u_2, u_3, \cdots$, known as the equations of Kadomtsev - Petviashvili hierarchy (e.g. $3u_{2,t_2t_2} + (-4u_{2,t_3} + u_{2,t_1t_1t_1} + 12u_2u_{2,t_1})_{t_1} = 0$).

Explicit forms of the equations are easier to obtain for $v_2, v_3, \cdots$ than for

$u_2, u_3, \cdots$ where v_n's are defined as coefficients of $\frac{d}{dx} = L + v_2 L^{-1} + v_3 L^{-2} + \cdots$. ($v_n$ is a differential polynomial of $u_2, \cdots, u_n$ and conversely u_n is that of $v_2, \cdots, v_n$; e.g. $v_2 = -u_2$, $v_3 = -u_3$, etc. and $u_2 = -v_2$, $u_3 = -v_3$, etc.) Namely we have

$$[\psi'] \qquad p_n(-\partial_t)\psi = v_n\psi, \text{ for } n \geq 2,$$

and its integrability condition

$$[v] \qquad p_n(-\partial_t)v_{m+1} + (J_{n,m}[v])_x = 0, \text{ for } m,n \geq 1,$$

with

$$J_{n,m}[v] = v_{m+n} + \frac{1}{2}\sum_{\substack{i,i',j,j' \geq 1 \\ i+i'=m,\, j+j'=n}} v_{i+j}v_{i'+j'} + \frac{1}{3}\sum_{\substack{i,i',i'',j,j',j'' \geq 1 \\ i+i'+i''=m,\, j+j'+j''=n}} v_{i+j}v_{i'+j'}v_{i''+j''} + \cdots$$

as the equivalents of $[\psi]$ and $[L]$, respectively.

Again, v_n is explicitly given by $v_n = \frac{\partial}{\partial t_1}(p_{n-1}(-\partial_t)\log\tau)\Big|_{t_1 \mapsto t_1 + x}$, for $n \geq 2$.

So far, accounts are given for the 1-component case. To generalize it to the r-component case we shall modify the notations as follows. For $\nu \in \mathbf{Z}$ and $0 \leq i < r$ the basis element $e_{r\nu+i} \in \mathbf{V}$ is rewritten as $e_\nu^{(i)}$ and operators Λ^r and $\sum_{\nu\in\mathbf{Z}} E_{r\nu+i,r\nu+i}$ as Λ and E_{ii}, respectively, so that we now have

$$\Lambda e_\nu^{(i)} = e_{\nu+1}^{(i)}, \quad E_{ii}e_\nu^{(j)} = \delta_{ij}e_\nu^{(j)}.$$

For $\xi \in \widetilde{GM}$ we define its evolution in the new set of time variables $t = (t_\nu^{(i)})_{0\leq i<r,\,\nu=1,2,\cdots}$ by $\xi(t) = e^{\sum t_\nu^{(i)} E_{ii}\Lambda^\nu}\xi$.

Let the Young diagram Y be labelled by $(\ell_0, \cdots, \ell_{mr-1})$, and for each $i=0, \cdots, r-1$, suppose that there are m_i of ℓ_ν's s.t. $\ell_\nu \equiv i \pmod r$, whom we rewrite as $(\ell_\nu^{(i)} r + i)_{\nu=0,\cdots,m_i-1}$. Set $m_i' = m_i - m$ to have $\sum m_i' = 0$, and call Y_i the Young diagram labelled by $(\ell_0^{(i)}, \cdots, \ell_{m_i-1}^{(i)})$. Then we see that the single diagram Y and the composite object $((Y_0, m_0'), \cdots, (Y_{r-1}, m_{r-1}'))$ correspond to each other in 1 to 1 manner. Accordingly, we rewrite ξ_Y as $\pm\xi_{(Y_0,m_0'),\cdots,(Y_{r-1},m_{r-1}')}$, with the possible change of sign caused by rearrangement of the suffixes ℓ_ν. If $m_i' = 0$ for $i=0,\cdots,r-1$, it is simply written as $\pm\xi_{Y_0,\cdots,Y_{r-1}}$.

All the results for the 1-component case are, mutatis mutandis, generalized to the r-component case. For example,

$$\tau(t;\xi) = \xi_{\phi,\cdots,\phi}(t) = \sum_{Y_0,\cdots,Y_{r-1}} \xi_{Y_0\cdots Y_{r-1}} \chi_{Y_0}(t^{(0)})\cdots\chi_{Y_{r-1}}(t^{(r-1)})$$

with $t^{(i)} = (t_1^{(i)}, t_2^{(i)},\cdots)$, and, as for $W = \gamma^{-1}\xi(t)$,

$$\frac{\partial W}{\partial t_n^{(i)}} = B_n^{(i)} W - W E_{ii}\left(\frac{d}{dx}\right)^n \quad \text{with} \quad B_n^{(i)} = \left(W E_{ii}\left(\frac{d}{dx}\right)^n W^{-1}\right)_+.$$

References

[1] M. Sato, Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, RIMS Kokyuroku 439 (1981), 30-46.

[2] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations, Lecture Notes in Math. 287, Springer (1973), 265-529.

[3] M. Sato, Pseudo-differential equations and theta functions, Astérisque 2 et 3 (1973), 286-291.