Appendix: On the derived subspace-problem

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0.1 (Notation) Let k be a field and A a dg algebra over k. Denote by $\mathcal{C}(A)$ the Abelian k-category of right dg modules, and by $\mathcal{D}(A)$ the corresponding derived category. In the sequel, we consider ordinary algebras as dg algebras concentrated in degree 0.

For $M \in \mathcal{C}(A)$, we consider the one-point extension

$$A[M] := \begin{pmatrix} k & M \\ 0 & A \end{pmatrix},$$

which is a dg algebra with $(A[M])^i = \left(\begin{smallmatrix} k^i & M^i \\ 0 & A^i \end{smallmatrix}\right)$ and differential induced by the components. The objects in $\mathcal{C}(A[M])$ are given by triples (V, X, f) with $V \in \mathcal{C}(k)$, $X \in \mathcal{C}(A)$ and $f \in \mathrm{Hom}_{\mathcal{C}(A)}(V \otimes M, X)$; sometimes we represent f by its adjoint $f_* \in \mathrm{Hom}_{\mathcal{C}(k)}(V, \mathcal{H}om_A(M, X))$. A morphism $h\colon (V, X, f) \longrightarrow (W, Y, g)$ is given by a pair $(h_0, h_1) \in \mathrm{Hom}_{\mathcal{C}(k)}(V, W) \times \mathrm{Hom}_{\mathcal{C}(A)}(X, Y)$ such that $h_1 f = g(h_0 \otimes M)$.

Finally, we denote by $\mathcal{U}(\mathcal{D}(A), M)$ the "derived subspace-category", which has as objects triples (V, X, f) with $V \in \mathcal{D}(k)$, $X \in \mathcal{D}(A)$ and $f \in \mathrm{Hom}_{\mathcal{D}(k)}(V, \mathbb{R}\mathcal{H}om_A(M, X))$. Here, a morphism $h\colon (V, X, f) \longrightarrow (W, Y, g)$ is given by a pair $(h_0, h_1) \in \mathrm{Hom}_{\mathcal{D}(k)}(V, W) \times \mathrm{Hom}_{\mathcal{D}(A)}(X, Y)$ such that $\mathbb{R}\mathcal{H}om_A(M, h_1) \circ f = g h_0$ in $\mathcal{D}(k)$.

For basic notions on representation theory of algebras (in particular, 1-point extensions and subspace-categories) see [4]. For notions on dg-algebras used here, see [2], [3].

0.2 (Main results) Since M is free over k, we can interpret $- \otimes_k M$ also as functor $\mathcal{D}(k) \longrightarrow \mathcal{D}(A)$, which has the right adjoint $\mathbb{R}\mathcal{H}om_A(M, *)\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(k)$. We write in this context also $f_* \in \mathrm{Hom}_{\mathcal{D}(k)}(V, \mathbb{R}\mathcal{H}om\, A(M, X))$ for the morphism which corresponds to $f \in \mathrm{Hom}_{\mathcal{D}(A)}(V \otimes M, X)$.

Proposition 1 *The functor* $\mathbf{P}^M_{\mathbf{u}}\colon \mathcal{D}(A[M]) \longrightarrow \mathcal{U}(\mathcal{D}(A), M)$*, which sends* (V, X, f) *to* (V, X, f_*) *is full, dense and reflects isomorphisms.*

Suppose now, that T is a dg A-B-bimodule such that the functor $- \otimes^{\mathbb{L}}_A T\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is an equivalence. Moreover, let $N := M \otimes^{\mathbb{L}}_A T$ and let $h\colon N \longrightarrow N'$ be an isomorphism in $\mathcal{D}(B)$. The functor $- \otimes^{\mathbb{L}}_A T$ clearly induces an equivalence $\mathbf{T}_{M,h}\colon \mathcal{U}(\mathcal{D}(A), M) \longrightarrow \mathcal{U}(\mathcal{D}(B), N')$ sending (V, X, v) to $(V, X \otimes^{\mathbb{L}}_A T, t_{X,h} f)$, where $t_{X,h}$ is the natural isomorphism $\mathbb{R}\mathcal{H}om_A(M, X) \longrightarrow \mathbb{R}\mathcal{H}om_B(N', X \otimes^{\mathbb{L}}_A T)$ in $\mathcal{D}(k)$.

Proposition 2 *In this situation we have an $A[M]$-$B[N']$-dg bimodule $\widetilde{T}'$ inducing an equivalence $-\otimes^{\mathbb{L}}_{A[M]}\widetilde{T}'\colon \mathcal{D}(A[M]) \longrightarrow \mathcal{D}(B[N'])$, such that the following square commutes,*

$$\begin{array}{ccc} \mathcal{D}(A[M]) & \xrightarrow{?\otimes^{\mathbb{L}}_{A[M]}\widetilde{T}'} & \mathcal{D}(B[N']) \\ \Big\downarrow{\scriptstyle \mathbf{P}^M_{\mathbf{u}}} & & \Big\downarrow{\scriptstyle \mathbf{P}^{N'}_{\mathbf{u}}} \\ \mathcal{U}(\mathcal{D}(A),M) & \xrightarrow{\mathbf{T}_{M,h}} & \mathcal{U}(\mathcal{D}(B),N') \end{array}$$

where $\mathbf{P}^{N'}_{\mathbf{u}}: \mathcal{D}(B[N']) \longrightarrow \mathcal{U}(\mathcal{D}(B),N')$ is the reduction functor.

0.3 (Functors) We have the following natural functors:

$$\mathcal{C}(A) \;\underset{I_\rho}{\overset{I_\lambda}{\underset{\longleftarrow I. -}{\longrightarrow}}}\; \mathcal{C}(A[M]) \;\underset{Q_\lambda}{\overset{Q_\rho}{\underset{- Q.\rightarrow}{\longleftarrow}}}\; \mathcal{C}(k) \quad \text{where}$$

$$\begin{aligned} I_\lambda(X) &= (0,X,0) & Q_\rho(V) &= (V,0,0)\\ I_\rho(X) &= (\mathcal{H}om_A(M,X),X,\mathbb{1}^*) & Q_\lambda(V) &= (V,V\otimes M,\mathbb{1})\\ X &= I.(V,X,f) & V &= Q.(V,X,f) \end{aligned}$$

Note that $(I_\lambda, I.)$, $(I., I_\rho)$ and $(Q_\lambda, Q.)$,$(Q., Q_\rho)$ are pairs of adjoint functors which are exact with the possible exception of I_ρ. Moreover I_λ and Q_ρ are fully faithful, thus the adjunctions

$$\mathbb{1}_{\mathcal{C}(A)} \longrightarrow I.I_\lambda \quad \text{and} \quad Q.Q_\lambda \longrightarrow \mathbb{1}_{\mathcal{C}(k)}$$

are invertible. We conclude that $(I., I_\lambda)$ and $(Q_\lambda, Q.)$ pass to be pairs of adjoint functors between the respective derived categories with the adjunctions still invertible. We use the same names for these functors and conclude, that $I_\lambda\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(A[M])$ and $Q_\rho\colon \mathcal{D}(k) \longrightarrow \mathcal{D}(A[M])$ are fully faithful.

0.4 (Gluing) In $\mathcal{C}(A[M])$, we have a functorial short exact sequence

$$0 \longrightarrow I_\lambda I. \longrightarrow \mathbb{1}_{\mathcal{C}(A[M])} \longrightarrow Q_\rho Q. \longrightarrow 0$$

where the non-trivial maps are the adjunctions. This induces a functorial triangle in $\mathcal{D}(A[M])$

$$I_\lambda I. \longrightarrow \mathbb{1}_{\mathcal{D}(A[M])} \longrightarrow Q_\rho Q. \xrightarrow{\partial} I_\lambda I.[1] \tag{1}$$

where we write ?[1] for the translation-functor. Note, that by adjunction

$$\mathrm{Hom}_{\mathcal{D}(A[M])}(I_\lambda(X), Q_\rho(W)) = \mathrm{Hom}_{\mathcal{D}(k)}(Q.I_\lambda(X), W) = 0$$

since $Q.I_\lambda = 0$, thus (1) is unique up to unique isomorphism, if we demand that the first term factors through I_λ and the third through Q_ρ.

Lemma *We have a natural isomorphism*

$$\sigma\colon \operatorname{Hom}_{\mathcal{D}(k)}(?, \mathbb{R}\mathcal{H}om_A(M,*)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}(A[M])}(Q_\rho(?), I_\lambda(*)[1])$$

of functors $\mathcal{D}(k)^{op} \times \mathcal{D}(A) \longrightarrow$ *mod-k.*

Proof: If we apply (1) to $Q_\lambda(?)$, we obtain the following triangle of functors $\mathcal{D}(k) \longrightarrow \mathcal{D}(A[M])$

$$I_\lambda(? \otimes M) \xrightarrow{\text{incl}} Q_\lambda(?) \xrightarrow{\pi} Q_\rho(?) \xrightarrow{\delta} I_\lambda(? \otimes M)[1] \qquad (2)$$

Note that this triangle may also be obtained directly from the obvious short exact sequence of functors $\mathcal{C}(k) \longrightarrow \mathcal{C}(A[M])$. Now we apply the functor $\operatorname{Hom}_{\mathcal{D}(A[M])}(-, I_\lambda(*)[1])$ to the triangle (2). The resulting long exact sequence yields an isomorphism

$$- \circ \delta_?\colon \operatorname{Hom}_{\mathcal{D}(A[M])}(I_\lambda(? \otimes M)[1], I_\lambda(*)[1]) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}(A[M])}(Q_\rho(?), I_\lambda(*)[1])$$

since by adjunction $\operatorname{Hom}_{\mathcal{D}(A[M])}(Q_\lambda(?), I_\lambda(*)) = 0$. Now, $I_\lambda[1]$ is fully faithful, and our claim follows from the fact that $\mathbb{R}\mathcal{H}om_A(M,*)$ is right adjoint to $(? \otimes M)$. □

0.5 (Proof of Proposition 1) Consider the category $\mathcal{E}(\mathcal{D}(A), M)$ which has as objects the triples (V, X, ∂) with $V \in \mathcal{D}(k)$, $X \in \mathcal{D}(A)$ and $\partial \in \operatorname{Hom}_{\mathcal{D}(A[M])}(Q_\rho(V), I_\lambda(X)[1])$. Like $\mathcal{D}(A[M])$, the category $\mathcal{E}(\mathcal{D}(A), M)$ is the category associated with a bimodule and its morphisms are defined accordingly. By the above Lemma, the bimodule isomorphism σ yields an equivalence

$$\mathbf{S}\colon \mathcal{U}(\mathcal{D}(A), M) \longrightarrow \mathcal{E}(\mathcal{D}(A), M), \quad (V, X, h) \mapsto (V, X, \sigma_{V,X}(h))$$

We have the reduction functor

$$\mathbf{P}^M_{\mathbf{e}}\colon \mathcal{D}(A[M]) \longrightarrow \mathcal{E}(\mathcal{D}(A), M), \quad (V, X, f) \mapsto (V, X, \partial(V, X, f))$$

where $\partial(V, X, f)$ is obtained from the triangle (1). The proof of the above lemma shows that we have $\mathbf{P}^M_{\mathbf{e}} = \mathbf{S}\,\mathbf{P}^M_{\mathbf{u}}$. Thus we may prove the claims for $\mathbf{P}^M_{\mathbf{e}}$.

Here they follow from the axioms of triangulated categories: The diagram describing a morphism in $\mathcal{E}(\mathcal{D}(A), M)$

$$\begin{tikzcd}
(V,0,0) \arrow[r, "\delta"] \arrow[d, "Q_\rho(h_0)"'] & (0,X,0)[1] \arrow[d, "I_\lambda(h_1)[1]"] \\
(V',0,0) \arrow[r, "\delta"] & (0,X',0)[1]
\end{tikzcd}$$

can be completed to a morphism of triangles, which shows us by the uniqueness of (1) that $\mathbf{P}^M_{\mathbf{e}}$ is full and dense. Similarly, we see that $\mathbf{P}^M_{\mathbf{e}}$ reflects isomorphisms. In fact, it is an easy exercise to show that the square of the kernel of $\mathbf{P}^M_{\mathbf{e}}$ vanishes.

0.6 (Homotopically projectives) Let $(V, X, f) \in \mathcal{C}(A[M])$ and consider the projective resolutions $p_M\colon \mathbf{p}\, M \longrightarrow M$ and $p_X\colon \mathbf{p}\, X \longrightarrow X$ in $\mathcal{C}(A)$; we may assume that p_X is surjective in each degree (this can be achieved always by adding an appropriate contractible object to $\mathbf{p}\, M$). Then there exists $\tilde{f}\colon V \otimes \mathbf{p}\, M \longrightarrow \mathbf{p}\, X$ such that $f(V \otimes p_X) = p_M \tilde{f}$.

Lemma *The mapping cone of*

$$(0, V \otimes \mathbf{p}\, M, 0) \xrightarrow{\left(\begin{smallmatrix} (0, & V\otimes p_m) \\ (0, & \tilde{f}\ ) \end{smallmatrix}\right)} (V, V \otimes M, \mathbb{1}) \oplus (0, \mathbf{p}\, X, 0)$$

is a projective resolution of (V, X, f).

Proof: Look first at the co-Cartesian diagram

$$\begin{array}{ccc} (0, V \otimes M) & \xrightarrow{(0,1)} & (V, V \otimes M, \mathbb{1}) \\ {\scriptstyle (0,f)}\downarrow & & \downarrow{\scriptstyle (1,f)} \\ (0, X, 0) & \xrightarrow[(0,1)]{} & (V, X, f) \end{array}$$

which can be reinterpreted as a short exact sequence, thus the natural map from the mapping-cone of

$$(0, V \otimes M, 0) \xrightarrow{\left(\begin{smallmatrix} (0,1) \\ (0,f) \end{smallmatrix}\right)} (V, V \otimes M, \mathbb{1}) \oplus (0, X, 0)$$

to (V, X, f) is a quasi-isomorphism. Next observe, that we have a commutative diagram

$$\begin{array}{ccc} (0, V \otimes \mathbf{p}\, M, 0) & \xrightarrow{\left(\begin{smallmatrix} (0, & V\otimes p_M) \\ (0, & \tilde{f}) \end{smallmatrix}\right)} & (V, V \otimes M, \mathbb{1}) \oplus (0, \mathbf{p}\, X, 0) \\ {\scriptstyle (V\otimes p_M)}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0, & (0,p_X) \end{smallmatrix}\right)} \\ (0, V \otimes M, 0) & \xrightarrow[\left(\begin{smallmatrix} (0,1) \\ (0,f) \end{smallmatrix}\right)]{} & (V, V \otimes M, \mathbb{1}) \oplus (0, X, 0) \end{array}$$

with the top row a morphism between homotopically projective dg modules and the vertical morphisms quasi-isomorphisms. Note, that $Q_\lambda(V) \cong V \otimes (k, M, \mathbb{1}_M)$ is homotopically projective since $(k, M, \mathbb{1}_M)$ is a direct summand of $A[M]_{A[M]}$. We conclude, that the mapping-cone of the first row is homotopically projective, and it admits a quasi-isomorphism to the mapping cone of the second row. $\square$

0.7 (Derived equivalences) We consider first the situation of a standard equivalence $\mathbf{L}$ defined by

$$- \otimes^{\mathbb{L}}_A T\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$$

and $M \in \mathcal{C}(A)$ homotopically projective. With $N := \mathbf{L}(M) = M \otimes_A T$, we obtain on $\widetilde{T} := \left(\begin{smallmatrix} k & N \\ 0 & T \end{smallmatrix}\right)$ a natural $A[M]$-$B[N]$-bimodule structure via $M \otimes_A T \xrightarrow{1} N$ and $\widetilde{T}_{B[N]} = (k, N, \mathbb{1}) \oplus (0, T_B, 0)$. By the observation of Barot and Lenzing [1] we conclude, that

$$- \otimes^{\mathbb{L}}_{A[M]} \widetilde{T} \colon \mathcal{D}(A[M]) \longrightarrow \mathcal{D}(B[N])$$

is a derived equivalence. We calculate $(V, X, f) \otimes^{\mathbb{L}}_{A[M]} \widetilde{T}$ using the projective resolution from 0.6: Take $p_X \colon \mathbf{p}\, X \longrightarrow X$ such that it is surjective in each degree, thus we find a lift $\tilde{f} \colon V \otimes M \longrightarrow \mathbf{p}\, X$ such that $f = p_X \tilde{f}$, and we conclude that the mapping cone of

$$(0, V \otimes M \otimes_A T, 0) \xrightarrow{\left(\begin{smallmatrix} (0,1) \\ (0, \tilde{f} \otimes_A T) \end{smallmatrix}\right)} (0, \mathbf{p}\, X \otimes_A T, 0) \oplus (V, V \otimes N, \mathbb{1})$$

coincides with $(V, X, f) \otimes^{\mathbb{L}}_{A[M]}$. But $\mathbf{p}\, X \otimes_A T = \mathbf{L}(X)$ and $\tilde{f} \otimes_A T = \mathbf{L}(f)$; thus we obtain a co-Cartesian diagram

$$\begin{array}{ccc} (0, V \otimes N, 0) & \xrightarrow{(0,1)} & (V, V \otimes N, \mathbb{1}) \\ \Big\downarrow{\scriptstyle (0,\mathbf{L}(f))} & & \Big\downarrow{\scriptstyle (0,\mathbf{L}(f))} \\ (0, \mathbf{p}\, X \otimes_A, 0) & \xrightarrow[(0,1)]{} & (V, \mathbf{L}(X), \mathbf{L}(f)) \end{array}$$

this means, that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \mathcal{D}(A[M]) & \xrightarrow{? \otimes^{\mathbb{L}}_A \widetilde{T}} & \mathcal{D}(B[N]) \\ \Big\downarrow{\scriptstyle \mathbf{P}^M_{\mathbf{u}}} & & \Big\downarrow{\scriptstyle \mathbf{P}^N_{\mathbf{u}}} \\ \mathcal{U}(\mathcal{D}(A), M) & \xrightarrow{\mathbf{T}_{M,1}} & \mathcal{U}(\mathcal{D}(B), N) \end{array}$$

Now note, that if $h \colon M \longrightarrow M'$ is a quasi-isomorphism, then this induces a quasi-isomorphism of dg algebras $\varphi_h \colon A[M] \longrightarrow A[M']$, and we obtain standard equivalences $\mathbf{L}_\lambda$ and $\mathbf{L}_.$ defined by

$$\begin{aligned} - \otimes^{\mathbb{L}}_{A[M]} A[M'] &\colon \mathcal{D}(A[M]) \longrightarrow \mathcal{D}(A[M']) \quad \text{and} \\ - \otimes^{\mathbb{L}}_{A[M']} A[M'] &\colon \mathcal{D}(A[M']) \longrightarrow \mathcal{D}(A[M]). \end{aligned}$$

Clearly $\mathbf{L}_.(V', X', f') \cong (V', X', f'h)$, and by a similar argument as above $\mathbf{L}_\lambda(V, X, f) \cong (V, \mathbf{i}\, X, i_X f(V \otimes h^{-1}))$ where $i_X \colon X \longrightarrow \mathbf{i}\, X$ is a injective resolution; note that $i_X f(V \otimes h^{-1}) \in \mathrm{Hom}_{\mathcal{D}(A)}(V \otimes M', \mathbf{i}\, X)$ is actually represented by a morphism in $\mathcal{C}(A)$. Proposition 2 follows now from the fact, that the composition of standard equivalences is isomorphic to a standard equivalence (we are working over a field), together with the observation that the required commutativity holds for all the standard equivalences considered here.

0.8 (Coextensions) We have a contravariant functor

$$?^*\colon \mathcal{C}(A) \longrightarrow \mathcal{C}(A^{\mathrm{op}}), X \mapsto \mathcal{H}om_k(X,k).$$

Note, that $H^i(X^*) \xrightarrow{\sim} \mathrm{Hom}_k(H^{-i}(X),k)$. Thus the natural map $i_X\colon X \longrightarrow X^{**}$ is a quasi-isomorphism if $\dim_k H^i(X)$ is finite-dimensional for all $i \in \mathbb{Z}$. Moreover, as a exact functor $?^{**}$ passes directly to a functor $\mathcal{D}(A) \longrightarrow \mathcal{D}(A)$.

For $N \in \mathcal{C}(A)$ the one-point coextension is

$$[N]A := \begin{pmatrix} k & 0 \\ N^* & A \end{pmatrix},$$

which is like the one-point extension a dg algebra. The objects in $\mathcal{C}([N]A)$ are given by triples (V, X, f) with $V \in \mathcal{C}(k)$, $X \in \mathcal{C}(A)$ and $f \in \mathrm{Hom}_{\mathcal{C}(A)}(X \otimes_A N^*, V)$. Dually to the previous situation we have now a full and dense functor

$$\mathbf{P}^N_{\mathbf{q}}\colon \mathcal{D}([N]A) \longrightarrow \mathcal{Q}(\mathcal{D}([N]A), N)$$

which reflects isomorphisms. Here, $\mathcal{Q}(\mathcal{D}([N]A), N)$ is the derived quotient category, which has as objects triples (V, X, g) with $V \in \mathcal{D}(k)$, $X \in \mathcal{D}(A)$ and $g \in \mathrm{Hom}_{\mathcal{D}(k)}(X \otimes^{\mathbb{L}}_A N^*, V)$ and morphisms given by the obvious commutative diagrams.

In fact, we get a similar setup as in 0.3, however with $\mathcal{C}(A)$ and $\mathcal{C}(k)$ interchanged and

$$Q_\lambda\colon \mathcal{C}(A) \longrightarrow \mathcal{C}([N]A), X \mapsto (X \otimes_A N^*, X, \mathbb{1})$$

the only functor which is possibly not exact.

Finally note, that we have in $\mathcal{D}(k)$ the following composition of natural morphisms:

$$X \otimes^{\mathbb{L}}_A N^* \longrightarrow (X \otimes^{\mathbb{L}}_A N^*)^{**} \xrightarrow{\mathrm{adj}^*} (\mathbb{R}\mathcal{H}om_A(X, N^{**}))^* \longrightarrow (\mathbb{R}\mathcal{H}om_A(X, N))^*,$$

which is an isomorphism if N and $X \otimes^{\mathbb{L}}_A N^*$ have finite-dimensional cohomologies. Thus, under these restrictions, we can replace in $\mathcal{Q}(\mathcal{D}(A), N)$ the functor $? \otimes^{\mathbb{L}}_A N^*$ by $(\mathbb{R}\mathcal{H}om_A(X, N))^*$, like in the "classical" situation.

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