MAPS BETWEEN REPRESENTATIONS OF CLANS

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Abstract. We give a description of the homomorphisms between representations of a clan [CB2], more or less as it was done in [Kr] for string-algebras. As a consequence of our findings we can establish the irreducibility of all maps which are expected to be so. This allows us, after defining a natural exact structure on the category of representations, to calculate in many situations Auslander-Reiten sequences like in the string-algebra case.

Introduction

In [CB2] Crawley-Boevey introduced clans, a class of problems which includes several well-known problems from representation theory, like the Gelfand-problem [Gf], self-reproducing systems [NRo] or semi-dihedral algebras [CB3]. In [CB2] Crawley-Boevey gives also a complete classification of the indecomposable representations of such a clan, similar to the well known classification of indecomposable modules for string-algebras (see for example [BR]). The original description of the indecomposable representations can be improved by the work of Bondarenko [Bd] and Deng [De].

For string algebras, there is moreover a well established description of the homomorphisms between representations, see [CB4] and [Kr], as well as a handy description of Auslander-Reiten sequences. In the present paper we supply analogous results for representations of clans and clannish algebras.

In section 1 we recall the definition of a clan $\mathcal{C}$ and of the category of its representations $\mathsf{R}(\mathcal{C})$, and introduce by the way some conventions about (bi-) quivers, words and letters.

Section 2 is dedicated to the description of the indecomposable representations of the clan $\mathcal{C}$. We follow essentially [CB2], though the proof that this particularly nice description works is due to [Bd] or [De]. We define a set $\mathfrak{X}$ of words, and functors $S_{\mathfrak{w}}: A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$ which parameterize the indecomposable representations. An important feature of this description is, that the functors $S_{\mathfrak{w}}$ are symbolized by certain almost linear quivers $H^{\mathfrak{w}}$ for $\mathfrak{w} \in \mathfrak{X}$ plus some extra data.

In section 3 we present the system of linear equations to solve for the description of $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$, codify this data in a quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) and state several auxiliary combinatorial facts. In section 4 we

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present and prove our findings, in fact it turns out, that the morphisms in $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ can be read off conveniently from the quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) plus some extra data. Section 5 is dedicated to Auslander-Reiten theory: Despite the fact, that in case of asymmetric bands $\mathfrak{v}, \mathfrak{w}$ we can not write down an explicit base for $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$, we have enough information to prove that that all "candidates" for irreducible morphisms are in fact irreducible. After introducing a natural exact structure on $\mathsf{R}(\mathcal{C})$ we present canonical exact sequences involving the irreducible maps, and obtain thus the existence of Auslander-Reiten sequences under reasonable hypothesis on $\mathcal{C}$.

The remaining two sections contain the proofs of the more technical results from 3, those involving linear algebra in 6 and combinatorics in 7.

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1. Basic constructions

1.1. **Quivers and letters.** A *biquiver* Q is a tuple (Q_0, Q_1, Q_2, s, t) with Q_0 the set of vertices, Q_1 the set of arrows, Q_2 the set of special arrows, and $s, t\colon (Q_1 \cup Q_2) \to Q_0$ functions defining the start- and terminal point of the arrows, thus $s(a) \xrightarrow{a} t(a)$ for each arrow $a \in Q_1$ and $s(b) \overset{b}{\dashrightarrow} t(b)$ for each $b \in Q_2$.

For technical reasons we need frequently a *polarization* of Q, i.e. a pair of functions $\sigma, \tau\colon (Q_1 \cup Q_2) \to \{-1, 1\} \subset \mathbb{Z}$.

The set of *letters* Q_* of the quiver Q consists of the following three types of symbols l

type of l	symbol l		$s(l)$	$t(l)$	$\sigma(l)$	$\tau(l)$
direct	$\underline{\alpha}$	$\alpha \in Q_1$	$s(\alpha)$	$t(\alpha)$	$\sigma(\alpha)$	$\tau(\alpha)$
inverse	$\underline{\beta^{-1}}$	$\beta \in Q_1$	$t(\beta)$	$s(\beta)$	$\tau(\beta)$	$\sigma(\beta)$
special	$\underline{\varepsilon^*}$	$\varepsilon \in Q_2$	$s(\varepsilon)$	$t(\varepsilon)$	$\sigma(\varepsilon)$	$\tau(\varepsilon)$

We have also indicated the definition of the functions s, t, σ and τ for the letters.

1.2. **Definition.** A *clan* $\mathcal{C}$ is a tuple $(\mathrm{k}, Q, (\sigma, \tau)\ \geq)$ where k is a field, Q a biquiver with polarization (σ, τ), such that $s(\varepsilon) = t(\varepsilon)$ and $\sigma(\varepsilon) = \tau(\varepsilon)$ for all $\varepsilon \in Q_2$, and $\geq$ is a linear order on each set of letters

$$Q_*(p, \rho) := \{l \in Q_* \mid t(l) = p, \tau(l) = \rho\}.$$

A *representation* M of $\mathcal{C}$ consists of the following data:

(R1) a k-vector-space M_p for each $p \in Q_0$,

(R2) a filtration of vector-spaces

$$M_p = l_1^+(M) \geq l_1^-(M) = l_2^+(M) \geq \cdots$$
$$\cdots \geq l_{n-1}^-(M) = l_n^+(M) \geq l_n^-(M) = 0$$

for each of the linearly ordered sets $Q_*(p, \rho) = \{l_1 > l_2 > \cdots > l_n\}$, (R3) an isomorphism

$$M_\alpha\colon (\underline{\alpha}^{-1})^+(M)/(\underline{\alpha}^{-1})^-(M) \xrightarrow{\sim} (\underline{\alpha})^+(M)/(\underline{\alpha})^-(M)$$

for every arrow $\alpha \in Q_1$

(R4) An *idempotent* endomorphism M_ε of $(\underline{\varepsilon}^*)^+(M)/(\underline{\varepsilon}^*)^-(M)$ for each special loop $\varepsilon \in Q_2$.

A *homomorphism* $\mathbf{f}$ between representations M and N of $\mathcal{C}$ consists of a collection of linear maps $\mathbf{f}_p\colon M_p \to N_p$ for each vertex $p \in Q_0$ such that

(H1) $\mathbf{f}_{t(l)}(l^+(M)) \leq l^+(N)$ for each letter $l \in Q_*$, (i.e. $\mathbf{f}$ respects the filtration (R2)), and thus inducing linear maps

$$\mathbf{f}_l\colon l^+(M)/l^-(M) \to l^+(N)/l^-(N),$$

(H2) $N_\alpha \mathbf{f}_{\underline{\alpha}^{-1}} = \mathbf{f}_{\underline{\alpha}} M_\alpha$ for each arrow $\alpha \in Q_1$,

(H3) $N_\varepsilon \mathbf{f}_{\underline{\varepsilon}^*} = \mathbf{f}_{\underline{\varepsilon}^*} M_\varepsilon$ for each special loop $\varepsilon \in Q_2$.

We denote by $\mathsf{R}(\mathcal{C})$ the category of representations of $\mathcal{C}$ with the above described homomorphisms. This is a Krull-Schmidt k-category, [CB2, 1.4].

Remark. We usually describe a clan, by displaying the corresponding biquiver Q and the linearly ordered sets of letters $Q_*(q,\rho)$ which contain more than one element.

1.3. **Words.** Let Q be a biquiver. A *word* $\mathfrak{w}$ in Q is a formal sequence $w_1 w_2 \cdots w_n$ of letters in Q_* with $s(w_i) = t(w_{i+1})$ and $\sigma(w_i) = -\tau(w_{i+1})$ for $i = 1, 2, \ldots, n-1$. We define $s(\mathfrak{w}) := s(w_n), t(\mathfrak{w}) := t(w_1), \sigma(\mathfrak{w}) := \sigma(w_1), \tau(\mathfrak{w}) := \tau(w_1)$. We write $l(\mathfrak{w}) := n$. We also allow *two* trivial words $1_{(q,\rho)}$ with $\rho \in \{-1, 1\}$ for each vertex $q \in Q_0$ so that $l(1_{(q,\rho)}) = 0$, and $s(1_{(q,\rho)}) = q = t(1_{(q,\rho)})$, $\sigma(1_{(q,\rho)}) = \rho = -\tau(1_{(q,\rho)})$.

We define the *product* $\mathfrak{w}\mathfrak{v}$ of two words $\mathfrak{w} = w_1 \cdots w_n$ and $\mathfrak{v} = v_1 \cdots v_m$ by $\mathfrak{w}\mathfrak{v} := w_1 \cdots w_n v_1 \cdots v_m$ if this is again a word i.e. if $s(\mathfrak{w}) = t(\mathfrak{v})$ and $\sigma(\mathfrak{w}) = -\tau(\mathfrak{v})$. If $\mathfrak{w}\mathfrak{w}$ exists, one can define arbitrary powers $\mathfrak{w}^n$ of $\mathfrak{w}$.

We say that a word $\mathfrak{w}$ is *left* (resp. *right*) *inextensible* if there is no letter $x \in Q_*$ with $x\mathfrak{w}$ (resp. $\mathfrak{w}x$) a word.

For $0 < i \leq n$ we write $\mathfrak{w}_{[i]} := w_1 \cdots w_i$; for $i > n$ we set $\mathfrak{w}_{[i]} := \mathfrak{w}$, while for $i \leq 0$, we set $\mathfrak{w}_{[i]} := 1_{(t(\mathfrak{w}), -\tau(\mathfrak{w}))}$. We define $\mathfrak{w}^{[i]}$ to be the unique word with $\mathfrak{w} = \mathfrak{w}_{[i]}\mathfrak{w}^{[i]}$, and for $j \geq i$ we define $\mathfrak{w}^{[i]}_{[j]}$ to be the unique word with $\mathfrak{w}_{[i]}\mathfrak{w}^{[i]}_{[j]}\mathfrak{w}^{[j]} = \mathfrak{w}$. By a *rotation* of $\mathfrak{w}$ we mean a word of the form $\mathfrak{w}^{[i]}\mathfrak{w}_{[i]}$

Finally, if all arrows in $\varepsilon \in Q_2$ are loops (i.e. $s(\varepsilon) = t(\varepsilon)$) and fulfill $\sigma(\varepsilon) = \tau(\varepsilon)$, we define a formal *inverse* l^{-1} for each letter l as follows:

$$l^{-1} := \begin{cases} \underline{\alpha}^{-1} & \text{if } l = \underline{\alpha} \text{ is a direct letter} \\ \underline{\alpha} & \text{if } l = \underline{\alpha}^{-1} \text{ is an inverse letter} \\ \underline{\varepsilon}^* & \text{if } l = \underline{\varepsilon}^* \text{ is a special letter} \end{cases}$$

In this situation we define the *inverse word* of $\mathfrak{w} = w_1 \cdots w_n$ as $\mathfrak{w}^{-1} := w_n^{-1} \cdots w_1^{-1}$ and $1_{p,\rho}^{-1} := 1_{p,-\rho}$.

If there is a (linear) order $>$ on each of the sets $Q_*(p,\rho) = \{w \in Q_* \mid t(w) = p, \tau(w) = \rho\}$ there is a partial ordering $\geq$ on the set of words defined by $\mathfrak{w} > \mathfrak{w}'$ if and only if $\mathfrak{w} = \mathfrak{x} y \mathfrak{u}$ and $\mathfrak{w}' = \mathfrak{x} z \mathfrak{v}$ where $\mathfrak{x}, \mathfrak{u}$ and $\mathfrak{v}$ are words and y and z are letters with $y > z$; in this situation we define

$$\Delta(\mathfrak{w}, \mathfrak{w}') := l(\mathfrak{x})$$

Observe, that the sets

$$\mathfrak{W}(p,\rho) := \{\mathfrak{w} \text{ word in } Q_* \mid t(\mathfrak{w}) = p,\ \tau(\mathfrak{w}) = \rho,\ \mathfrak{w} \text{ right inextensible}\}$$

are linearly ordered for all vertices $p \in Q_0$ and $\rho \in \{-1, 1\}$.

2. Description of the indecomposable representations

2.1. **Strings and bands.** Let $\mathfrak{S}$ be a set of representatives of words $\mathfrak{w}$ which are both right and left inextensible, under the equivalence relation which identifies $\mathfrak{w}$ with $\mathfrak{w}^{-1}$. We call the elements of $\mathfrak{S}$ *strings* and say that a string $\mathfrak{w}$ is *symmetric* if $\mathfrak{w} = \mathfrak{w}^{-1}$, and otherwise *asymmetric*.

Let $\mathfrak{B}$ be a set of representatives of nontrivial words $\mathfrak{w}$ such that $\mathfrak{w}\mathfrak{w}$ is defined and such that $\mathfrak{w}$ is not a power under the equivalence relation which identifies all rotations of $\mathfrak{w}$ and $\mathfrak{w}^{-1}$. We call the elements of $\mathfrak{B}$ *bands* and say that a band $\mathfrak{w}$ is *symmetric* if $\mathfrak{w}$ is equal to some rotation of $\mathfrak{w}^{-1}$, and otherwise *asymmetric*.

We define $\mathfrak{X} := \mathfrak{S} \cup \mathfrak{B}$. For each $\mathfrak{w} \in \mathfrak{X}$, we shall define an algebra $A_\mathfrak{w}$ and a functor $S_\mathfrak{w} \colon A_\mathfrak{w}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$.

2.2. **From words to quivers.** We assign to each $\mathfrak{w} \in \mathfrak{X}$ a biquiver $H^\mathfrak{w}$, a quiver morphism $G_\mathfrak{w} \colon H^\mathfrak{w} \to Q$, a k-algebra $A_\mathfrak{w}$ and a map $U_\mathfrak{w} \colon H_1^\mathfrak{w} \cup H_2^\mathfrak{w} \to A_\mathfrak{w}$. These data symbolize a functor $S_\mathfrak{w} \colon A_\mathfrak{w}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$, as we will explain in 2.4.

2.2.1. If $\mathfrak{w} = w_1 \cdots w_n$ is an *asymmetric string*, let $H^\mathfrak{w}$ be the quiver

$$0 \overset{\nu_1}{\text{———}} 1 \overset{\nu_2}{\text{———}} 2 \overset{\nu_3}{\text{———}} \cdots \overset{\nu_{n-1}}{\text{———}} n{-}1 \overset{\nu_n}{\text{———}} n$$

in which, if w_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_\mathfrak{w}(\nu_i) = \alpha \in Q_1$; if w_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_\mathfrak{w}(\nu_i) = \alpha \in Q_1$; while if w_i is a special letter $\underline{\varepsilon}^*$ we set $G_\mathfrak{w}(\nu_i) = \varepsilon \in Q_2$ and ν_i points to the left if and only if $\mathfrak{w}_{[i-1]}^{-1} > \mathfrak{w}^{[i]}$. We define $A_\mathfrak{w} := \mathrm{k}$ and $U_\mathfrak{w}(\nu_i) = 1$.

2.2.2. If $\mathfrak{w}$ is a *symmetric string*, then $\mathfrak{w} = \mathfrak{z}\, \underline{\delta}^*\, \mathfrak{z}$ for a unique word $\mathfrak{z} = z_1 \cdots z_n$. Let $H^\mathfrak{w}$ be the quiver

$$0 \overset{\nu_1}{\text{———}} 1 \overset{\nu_2}{\text{———}} 2 \overset{\nu_3}{\text{———}} \cdots \overset{\nu_{n-1}}{\text{———}} n{-}1 \overset{\nu_n}{\text{———}} n \circlearrowright \eta$$

in which, if z_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_\mathfrak{w}(\nu_i) = \alpha \in Q_1$; if z_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_\mathfrak{w}(\nu_i) = \alpha \in Q_1$; while if z_i is a special letter $\underline{\varepsilon}^*$ we set $G_\mathfrak{w}\nu_i = \varepsilon \in Q_2$ and ν_i points to the left if and only if $\mathfrak{z}_{[i-1]}^{-1} > \mathfrak{z}^{[i]} \underline{f}^* \mathfrak{z}$. Finally, $G_\mathfrak{w}(\eta) := \delta$

We define $A_{\mathfrak{w}} := \mathrm{k}[z]/(z^2 - z)$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i = 1, \dots n$, and $U_{\mathfrak{w}}(\eta) = z$.

2.2.3. If $\mathfrak{w} = w_1 \cdots w_n$ is an *asymmetric band*, let $H^{\mathfrak{w}}$ be the quiver

$$\begin{tikzcd}
0 \arrow[r, dash, "\nu_1"] \arrow[rrrrrr, dash, bend right, "\nu_n"'] & 1 \arrow[r, dash, "\nu_2"] & 2 \arrow[r, dash, "\nu_3"] & \cdots \arrow[r, dash, "\nu_{n-2}"] & n-2 \arrow[r, dash, "\nu_{n-1}"] & n-1
\end{tikzcd}$$

in which, if w_i is a direct letter $\underline{\alpha}$, then ν_i points anti-clockwise, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if w_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points clockwise, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$. If w_i is a special letter $\underline{\varepsilon}^*$ then $G_{\mathfrak{w}}(\nu_i) = \varepsilon \in Q_2$ and ν_i points anti-clockwise if and only if $\mathfrak{w}_{[i-1]}^{-1}(\mathfrak{w}^{[i]})^{-1} > \mathfrak{w}^{[i]}\mathfrak{w}_{[i-1]}$.

We define $A_{\mathfrak{w}} := \mathrm{k}[x, x^{-1}]$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i \neq 1$ and $U_{\mathfrak{w}}(\nu_1) = x$ if ν_1 is anti-clockwise, $U_{\mathfrak{w}}(\nu_1) = x^{-1}$ otherwise.

2.2.4. If $\mathfrak{w}$ is a *symmetric band*, it can be written uniquely in the form $\mathfrak{t}\,\underline{\delta_2}^*\,\mathfrak{t}^{-1}\mathfrak{r}^{-1}\,\underline{\delta_1}^*\,\mathfrak{r}$ where δ_1, δ_2 are special loops, and $\mathfrak{r}, \mathfrak{t}$ are words. Set $\mathfrak{z} := \mathfrak{r}\mathfrak{t} = z_1 \cdots z_n$ and let $H^{\mathfrak{w}}$ be the quiver

$$\begin{tikzcd}
0 \arrow[loop left, dotted, "\eta_1"] \arrow[r, dash, "\nu_1"] & 1 \arrow[r, dash, "\nu_2"] & 2 \arrow[r, dash, "\nu_3"] & \cdots \arrow[r, dash, "\nu_{n-1}"] & n-1 \arrow[r, dash, "\nu_n"] & n \arrow[loop right, dotted, "\eta_2"]
\end{tikzcd}$$

in which, if z_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if z_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; while if z_i is a special letter $\underline{\varepsilon}^*$, we set $G_{\mathfrak{w}}(\nu_i) = \beta \in Q_2$ and ν_i points to the left if and only if

$$\mathfrak{z}_{[i-1]}^{-1}\,\underline{\delta_1}^*\,\mathfrak{z}\,\underline{\delta_2}^*\,(\mathfrak{z}^{[i]})^{-1} > \mathfrak{z}_{[i]}\underline{\delta_2}^*\,\mathfrak{z}^{-1}\underline{\delta_1}^*\,\mathfrak{z}_{i-1}.$$

Finally, $G_{\mathfrak{w}}(\eta_1) = \delta_1,\ G_{\mathfrak{w}}(\eta_2) = \delta_2$

We define $A_{\mathfrak{w}} := \mathrm{k}\langle x, y\rangle/(x^2 - x, y^2 - y)$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i = 1, \dots n$, and $U_{\mathfrak{w}}(\eta_1) = y,\ U_{\mathfrak{w}}(\eta_2) = x$.

2.3. **Note.** For $\mathfrak{w} \in \mathfrak{X}$ we consider the quiver $H^{\mathfrak{w}}$ as a biquiver, with $H_1^{\mathfrak{w}}$ consisting of the solid arrows and $H_2^{\mathfrak{w}}$ consisting of the dotted loops. Moreover $H^{\mathfrak{w}}$ has the following induced polarization:

$$\sigma(\nu) := \sigma(G_{\mathfrak{w}}(\nu)) \text{ and } \tau(\nu) := \tau(G_{\mathfrak{w}}(\nu))$$

for all arrows in $\nu \in H_1^{\mathfrak{w}} \cup H_2^{\mathfrak{w}}$. Thus, our conventions about letters and words apply for $H^{\mathfrak{w}}$. In particular, the quiver homomorphism $G_{\mathfrak{w}} \colon H^{\mathfrak{w}} \to Q$ induces a map between the corresponding sets of letters: $G_{\mathfrak{w}}^* \colon H_*^{\mathfrak{w}} \to Q_*$

$$\begin{aligned}
\underline{\nu} \quad &\mapsto \begin{cases} \underline{G_{\mathfrak{w}}(\nu)} & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \underline{G_{\mathfrak{w}}(\nu)}^* & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \\
\underline{\nu}^{-1} &\mapsto \begin{cases} \underline{G_{\mathfrak{w}}(\nu)}^{-1} & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \underline{G_{\mathfrak{w}}(\nu)}^* & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \\
\underline{\eta}^* &\mapsto \quad \underline{G_{\mathfrak{w}}(\eta)}^*
\end{aligned}$$

$G_{\mathfrak{w}}^*$ extends naturally to a map between the corresponding words.

Observe, that for any $\mathfrak{w} \in \mathfrak{X}$ and $\mathfrak{h}$ a word in $H^{\mathfrak{w}}$, there is at most one letter $h \in H^{\mathfrak{w}}_*$ such that $\mathfrak{h}\,h$ is again a word. Thus, given $(j, \rho) \in H^{\mathfrak{w}}_0 \times \{-1, 1\}$ there is a unique, right inextensible word $\mathfrak{h}^{\mathfrak{w}}(j, \rho)$ with $t(\mathfrak{h}^{\mathfrak{w}}(j, \rho)) = j$ and $\tau(\mathfrak{h}^{\mathfrak{w}}(j, \rho)) = \rho$. Note, in case $\mathfrak{w}$ is a band, $\mathfrak{h}^{\mathfrak{w}}(j, \rho)$ is right infinite.

Remark. For $s \xrightarrow{\nu} t$ an arrow in $H^{\mathfrak{w}}_1$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ let $\rho := \sigma(\nu)$, then trivially also $\rho = \tau(\nu)$. By the construction of $H^{\mathfrak{w}}$ we have

$$G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t, -\rho)) > G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s, -\rho)).$$

See 3.2 and 7.2 for related constructions.

2.4. **How to interpret the quivers.** For each letter $l \in Q_*$ we define

$$\mathbf{t}(\mathfrak{w}, l) := \{t(x) \in H^{\mathfrak{w}}_0 \mid x \in H^{\mathfrak{w}}_*,\ G^*_{\mathfrak{w}}(x) = l\}$$
$$\mathbf{s}(\mathfrak{w}, l) := \{s(x) \in H^{\mathfrak{w}}_0 \mid x \in H^{\mathfrak{w}}_*,\ G^*_{\mathfrak{w}}(x) = l\},$$

thus $\mathbf{t}(\mathfrak{w}, l) = \mathbf{s}(\mathfrak{w}, l^{-1})$.

If $W \in A_w\text{-}\mathsf{mod}$, we define the representation $S_{\mathfrak{w}}(W)$ by the following data:
For every $p \in Q_0$ let

$$S_{\mathfrak{w}}(W)_p := \bigoplus_{j \in G^{-1}_{\mathfrak{w}}(p)} W_j, \text{ where each } W_j = W.$$

For each letter $l \in Q_*$

$$l^+(S_{\mathfrak{w}}(W)) := \bigoplus_{l' \leq l} \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l')} W_j$$
$$l^-(S_{\mathfrak{w}}(W)) := \bigoplus_{l' < l} \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l')} W_j$$

and identify

$$l^+(S_{\mathfrak{w}}(W))/l^-(S_{\mathfrak{w}}(W)) \text{ with } \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l)} W_j = \bigoplus_{j' \in \mathbf{s}(\mathfrak{w}, l^{-1})} W_j$$

For each (ordinary) arrow $\alpha \in Q_1$ we define

$$S_{\mathfrak{w}}(W)_\alpha \colon \bigoplus_{j \in \mathbf{s}(\mathfrak{w}, \underline{a})} W_j \to \bigoplus_{j' \in \mathbf{t}(\mathfrak{w}, \underline{a})} W_{j'}$$

by

$$W_{s(\nu)} \ni w \mapsto U_{\mathfrak{w}}(\nu) w \in W_{t(\nu)} \text{ for all } \nu \in G^{-1}_{\mathfrak{w}}(\alpha)$$

For each special loop $\varepsilon \in Q_2$ we define

$$S_{\mathfrak{w}}(W)_\varepsilon \colon \bigoplus_{j \in \mathbf{s}(\mathfrak{w}, \underline{\varepsilon}^*)} W_j \to \bigoplus_{j' \in \mathbf{t}(\mathfrak{w}, \underline{\varepsilon}^*)} W_{j'}$$

by

$$W_{s(\nu)} \ni w \mapsto U_{\mathfrak{w}}(\nu)w \in W_{t(h)} \text{ for all } h \in G_{\mathfrak{w}}^{-1}(\beta)$$

and

$$W_{t(\nu)} \ni w \mapsto w \in W_{t(\nu)} \text{ for all } \nu \in G_{\mathfrak{w}}^{-1}(\beta) \text{ with } s(\nu) \neq t(\nu)$$

2.5. By the work of Bondarenko [Bd] and Deng [De] we have the following result:

Theorem 1. *If $\mathcal{C} = (\mathrm{k}, Q, (\sigma, \tau), \geq)$ is a clan, the representations $S_{\mathfrak{w}}(W)$ with $\mathfrak{w}$ running through $\mathfrak{X}$, and W running through a complete set of non-isomorphic indecomposable $A_{\mathfrak{w}}$-modules, provide a complete set of non-isomorphic indecomposable representations of $\mathcal{C}$.*

Our exposition is essentially that of Crawley-Boevey [CB2], rather than that of Deng, the transfer is straightforward, though somewhat cumbersome.

3. Reduction to Combinatorics

3.1. **The system of equations.** Our aim is essentially to find a basis of the linear space $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ for $v, w \in \mathfrak{X}$ and V (resp. W) in $A_{\mathfrak{v}}$-mod resp. ($A_{\mathfrak{w}}$-mod). To this end we write down explicitly the corresponding system of linear equations to solve.

Proposition 1. *Consider for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{X}$ the representations $S_{\mathfrak{v}}(V)$ and $S_{\mathfrak{w}}(W)$ of $\mathcal{C}$ constructed in 2.2 and 2.4. Let $\mathbf{f} = (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$, with $\mathbf{f}_{(j,i)} \in \mathsf{Hom}_{\mathrm{k}}(V_i, W_j)$ and*

$$H_0^{\mathfrak{w},\mathfrak{v}} := \{(j,i) \in H_0^{\mathfrak{w}} \times H_0^{\mathfrak{v}} \mid G_{\mathfrak{w}}(j) = G_{\mathfrak{v}}(i)\}.$$

We have: $\mathbf{f} \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ if and only if $\mathbf{f}$ fulfills the following set of equations:

$$\mathbf{f}_{(t(\nu),t(\mu))} = 0 \tag{1}$$

for all $(\nu, \mu) \in H_*^{\mathfrak{w}} \times H_*^{\mathfrak{v}}$ with $G_{\mathfrak{w}}^*(\nu) > G_{\mathfrak{v}}^*(\mu)$.

$$\mathbf{f}_{(t(\nu),t(\mu))}U_{\mathfrak{v}}(\mu) - U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),s(\mu))} = 0 \tag{2}$$

for all $(\nu, \mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu) \in Q_1$.

$$\mathbf{f}_{(t(\nu),t(\mu))}U_{\mathfrak{v}}(\mu) - U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),s(\mu))} = \mathbf{f}_{(t(\nu),s(\mu))} \text{ and} \tag{3a}$$

$$\mathbf{f}_{(s(\nu),t(\mu))} = 0 \tag{3a'}$$

for all $(\nu, \mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu) \in Q_2$.

$$\mathbf{f}_{(j,t(\mu))}U_{\mathfrak{v}}(\mu) - U_{\mathfrak{w}}(\eta)\mathbf{f}_{(j,s(\mu))} = 0 \tag{3b}$$

for all $(\eta, \mu) \in H_2^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\eta) = G_{\mathfrak{v}}(\mu) \in Q_2$ ($j := s(\eta) = t(\eta)$).

$$\mathbf{f}_{(t(\nu),i)}(U_{\mathfrak{v}}(\zeta) - 1) - U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),i)} = 0 \tag{3c}$$

for all $(\nu,\zeta)\in H_1^{\mathfrak{w}}\times H_2^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\zeta)\in Q_2$ $(i:=s(\zeta)=t(\zeta))$.

$$\mathbf{f}_{(j,i)}U_{\mathfrak{v}}(\zeta)-U_{\mathfrak{w}}(\eta)\mathbf{f}_{(j,i)}=0 \tag{3d}$$

for all $(\eta,\zeta)\in H_2^{\mathfrak{w}}\times H_2^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\mu)\in Q_2$, $(j:=s(\nu),i:=s(g))$.

Remarks. (1) We understand here $U_{\mathfrak{w}}(\nu)$ as an k-endomorphism of W and $U_{\mathfrak{v}}(\mu)$ as a k-endomorphism of V.

(2) Observe, that $U_{\mathfrak{w}}(\nu)$ (resp. $U_{\mathfrak{v}}(\mu)$) is *invertible* while ν (resp. μ) is *not* a loop; $U_{\mathfrak{w}}(\eta)$ (resp. $U_{\mathfrak{v}}(\zeta)$) is idempotent, while η (resp. ζ) is a loop.

The proof of this proposition will be given in section 6.

3.2. **A device.** For technical reasons we need the following: Let $\mathcal{C}=(\mathrm{k},Q,(\sigma,\tau),>)$ be a clan. In this situation we define the quiver $\widehat{Q}=(\widehat{Q}_0,\widehat{Q}_1,\widehat{Q}_2,\hat{s},\hat{t})$ by $\widehat{Q}_0:=Q_0,\widehat{Q}_1:=Q_1\cup Q_2,\widehat{Q}_2:=\emptyset,\hat{s}=s,\hat{t}=t$, i.e. we just convert all special loops into ordinary loops. Moreover $\hat{Q}$ obtains "the same" polarization as Q.

There is an obvious quiver homomorphism $\gamma:\widehat{Q}\to Q$, which induces a map $\gamma_*:\widehat{Q}_*\to Q_*$ between the corresponding sets of letters. Observe, $\gamma_*^{-1}(\underline{\varepsilon}^*)=\{\underline{\varepsilon},\underline{\varepsilon}^{-1}\}$ for all special letters $\underline{\varepsilon}^*\in Q_*$. Moreover we define a linear order $\succ$ on the sets of letters $\widehat{Q}_*(p,\rho)$ by

$$l\succ l'\Longleftrightarrow\begin{cases}\gamma_*(l) & >\ \gamma_*(l') \qquad \text{or}\\ l=\underline{\varepsilon}^{-1} & \text{and}\ \ l'=\underline{\varepsilon},\ \varepsilon\in Q_2\end{cases}$$

We write also $\succ$ for the induced order on the words in $\widehat{Q}_*$, see 1.3. Compare 2.3 and see also 7.2.

If $\mathfrak{w}\in\mathfrak{X}$, the quiver homomorphism $G_{\mathfrak{w}}$ induces two maps $\widehat{G}^{\rho}_{\mathfrak{w}}\colon H_*^{\mathfrak{w}}\to\widehat{Q}_*$, $\rho\in\{-,+\}$ defined by

$$\begin{aligned}\underline{\nu}\ &\mapsto \underline{G_{\mathfrak{w}}(\nu)}\\ \underline{\nu}^{-1} &\mapsto \underline{G_{\mathfrak{w}}(\nu)}^{-1}\\ \underline{\eta}^*\ &\mapsto \underline{G_{\mathfrak{w}}(\eta)}^{\rho\,1}\end{aligned}$$

Here, by abuse of notation we understand $G_{\mathfrak{w}}\colon H^{\mathfrak{w}}\to\widehat{Q}$.

Remark. Trivially, $\widehat{G}^+_{\mathfrak{w}}=\widehat{G}^-_{\mathfrak{w}}$ if and only if $\mathfrak{w}$ is asymmetric. Also these maps extend to maps between the corresponding words, thus $\widehat{G}^-_{\mathfrak{w}}(\mathfrak{h})\succeq\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h})$ always.

3.3. **The quiver $H^{\mathfrak{w},\mathfrak{v}}$.** We visualize the equations of Proposition 1 in a (bi)-quiver $H^{\mathfrak{w},\mathfrak{v}}$ and a function $U_{\mathfrak{w},\mathfrak{v}}\to A_{\mathfrak{w}}\times A_{\mathfrak{v}}$:

$$H_0^{\mathfrak{w},\mathfrak{v}}:=\qquad\{(j,i)\in H_0^{\mathfrak{w}}\times H_0^{\mathfrak{v}}\quad|\ G_{\mathfrak{w}}(j)=G_{\mathfrak{v}}(i)\}$$

$$
\begin{aligned}
H_1^{\mathfrak{w},\mathfrak{v}} :=& \{ \ (s(\nu), s(\mu)) \xrightarrow{(\nu,\mu)} (t(\nu), t(\mu)) \ \mid (\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}},\ G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
& \cup \{ \ (j_0, i_0) \xrightarrow{(\eta,\zeta)_o} (j_0, i_0) \qquad \mid (\eta,\zeta) \in H_2^{\mathfrak{w}} \times H_2^{\mathfrak{v}},\ G_{\mathfrak{w}}(\eta) = G_{\mathfrak{v}}(\zeta), \\
& \qquad\qquad j_0 := s(\nu), i_0 := s(g)\} \\
H_{2,-}^{\mathfrak{w},\mathfrak{v}} :=& \{ \ (t(\nu), s(\mu)) \overset{(\nu,\mu)_+}{\dashrightarrow} (t(\nu), t(\mu)) \ \mid (\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}},\ G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
& \cup\{ \ (t(\nu), s(\mu)) \overset{(\nu,\mu)_-}{\dashrightarrow} (s(\nu), s(\mu)) \ \mid (\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}},\ G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
H_{2,o}^{\mathfrak{w},\mathfrak{v}} :=& \ \{ \ (j_0, s(\mu)) \overset{(\eta,\mu)_\oplus}{\dashrightarrow} (i_0, s(\mu)) \qquad \mid (\eta,\mu) \in H_2^{\mathfrak{w}} \times H_1^{\mathfrak{v}},\ G_{\mathfrak{w}}(\eta) = G_{\mathfrak{v}}(\mu), \\
& \qquad\qquad j_0 := s(\eta) = t(\eta)\} \\
& \cup \{ \ (t(\nu), i_0) \overset{(\nu,\zeta)_\ominus}{\dashrightarrow} (s(\nu), i_0) \qquad \mid (\nu,\zeta) \in H_1^{\mathfrak{w}} \times H_2^{\mathfrak{v}},\ G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\zeta), \\
& \qquad\qquad i_0 := s(\zeta) = t(\zeta)\} \\
H_2^{\mathfrak{w},\mathfrak{v}} :=& \qquad H_{2,-}^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}}
\end{aligned}
$$

The vertices

$$\{(t(y), t(x)) \in H_0^{\mathfrak{w},\mathfrak{v}} \mid (y,x) \in H_*^{\mathfrak{w}} \times H_*^{\mathfrak{v}}, \widehat{G}_{\mathfrak{w}}^+(y) \succ \widehat{G}_{\mathfrak{v}}^-(x)\}$$

are called *0-points*. These vertices are labeled by a "0" in $H^{\mathfrak{w},\mathfrak{v}}$. This "codifies" the equations (1) and (3a').

The function $U_{\mathfrak{w},\mathfrak{v}} = (U^{(1)}_{\mathfrak{w},\mathfrak{v}}, U^{(2)}_{\mathfrak{w},\mathfrak{v}})$ is defined as follows:

$$
\begin{aligned}
(\nu,\mu) &\mapsto (U_{\mathfrak{w}}(\nu), U_{\mathfrak{v}}(\mu)) && \text{(compare (1))} \\
(\eta,\zeta)_o &\mapsto (U_{\mathfrak{w}}(\eta), U_{\mathfrak{v}}(\zeta)) && \text{(compare (3d))} \\
(\nu,\mu)_+ &\mapsto (1, U_{\mathfrak{v}}(\mu)) && \text{(compare (3a))} \\
(\nu,\mu)_- &\mapsto (U_{\mathfrak{w}}(\nu), -1) && \text{(compare (3a))} \\
(\eta,\mu)_\oplus &\mapsto (U_{\mathfrak{w}}(\eta), U_{\mathfrak{v}}(\mu)) && \text{(compare (3b))} \\
(\nu,\zeta)_\ominus &\mapsto (U_{\mathfrak{w}}(\nu), U_{\mathfrak{v}}(\zeta) - 1) && \text{(compare (3c))}
\end{aligned}
$$

Finally we put on $H^{\mathfrak{w},\mathfrak{v}}$ the polarization induced by $H^{\mathfrak{w}}$ and $H^{\mathfrak{v}}$.

Definition. (1) The connected components of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}})$ are called *lines*. We denote the set of lines by $\mathcal{L}(\mathfrak{w},\mathfrak{v})$,
(2) A line L is a *h-line*, if for some vertex $(j,i) \in L_0$ we have:

$$\widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \preceq \widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ for } \rho \in \{-1,1\}$$

(see 3.2 and 2.3 for details).
(3) A line Z is a *z-line*, if for some vertex $(j,i) \in Z_0$ we have:

$$
\begin{aligned}
&\widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \succ \widehat{G}_{\mathfrak{v}}^+(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ or} \\
&\widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(i,\rho)) \succ \widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ for some } \rho \in \{-1,1\}
\end{aligned}
$$

(4) A h-line L is a *real h-line* if it is *not* a z-line, otherwise it is a *virtual h-line*. We denote the set of real h-lines by $\mathcal{H}(\mathfrak{w},\mathfrak{v})$. Similarly, a z-line is a *real z-line* if it is *not* a h-line. We denote the set of real z-lines by $\mathcal{Z}(\mathfrak{w},\mathfrak{v})$.

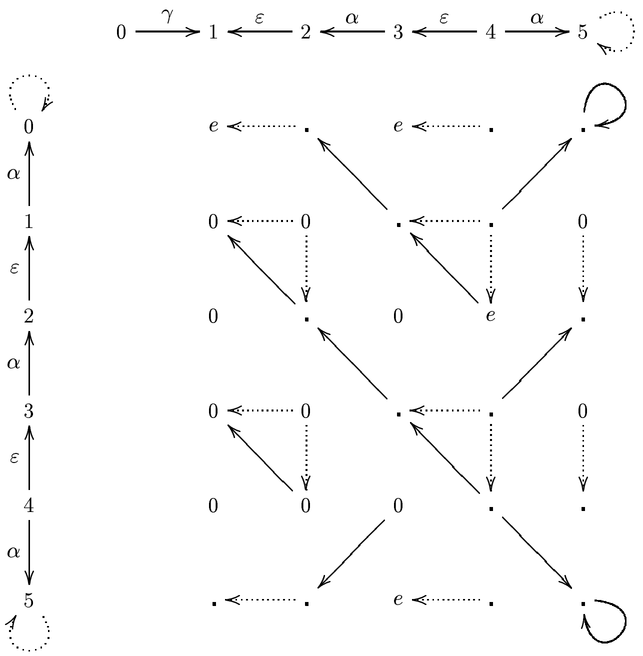
3.4. **Example.** Let $\mathcal{C}$ be the clan given by

$$\begin{tikzcd} 0 \arrow[r, "\gamma"] & 1 \arrow[loop above, "\alpha"] \arrow[loop below, dotted, "\varepsilon"] \end{tikzcd} \quad \text{and } Q_*(1,-1) := \{\underline{\alpha}^{-1} > \underline{\gamma} > \underline{\alpha}\}.$$

This clan represents the "problem of a an idempotent and a nilpotent matrix" [CB1]. Look at the following two words:

$$\mathfrak{v} := \underline{\gamma}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\gamma} \text{ and } \mathfrak{w} := \underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}^{-1}$$

The first one is a symmetric string, the second a symmetric band. We write down the quivers $H^{\mathfrak{w}}, H^{\mathfrak{v}}$ and $H^{\mathfrak{w},\mathfrak{v}}$.

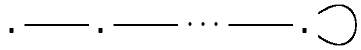


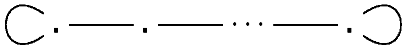
The connected components of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}})$, containing the vertices $(0,4),(0,5),(1,3)$ and $((5,4)$ are precisely the real h-lines; the components containing the vertices $(0,1),(0,3)$ and $(5,3)$ are the virtual h-lines (in our case case these are just points); the remaining components are real z-lines. Compare also the remarks below.

3.5. **Remarks.** (1) The underlying graph of a line L is exactly one of the following types:

$\mathbb{A}$:

$\widetilde{\mathbb{A}}$: 

$\mathbb{D}$: 

$\widetilde{\mathbb{D}}$: 

Moreover, a line L is of type $\widetilde{\mathbb{A}}$ (resp. $\widetilde{\mathbb{D}}$) if and only if $\mathfrak{w} = \mathfrak{v}$ is an asymmetric (resp. symmetric) band and L is the diagonal.

(2) $\widehat{G}_{\mathfrak{v}}^{-/+}(\mathfrak{h}^{\mathfrak{v}}(p,\rho))$ and $\widehat{G}_{\mathfrak{w}}^{+/-}(\mathfrak{h}^{\mathfrak{w}}(q,\rho))$ are comparable for $\rho \in \{-1,+1\}$, thus every line is a h-line or a z-line. Moreover it is easy to see, that the definition is independent of the choice of the point on a line. Obviously, virtual h-lines appear only if $\mathfrak{v}$ or $\mathfrak{w}$ is symmetric.

(3) A line Z is a z-line if and only if Z contains a 0-point or if it contains the endpoint of an arrow from $H_{2,o}^{\mathfrak{w},\mathfrak{v}}$. Thus, also real h-lines are also easy to recognize.

(4) A line L is a h-line if and only if the connected component of L in $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ contains no 0-point, otherwise it is a real z-line. This is a direct consequence of Corollary 1 and Proposition 3 (a) below.

3.6. **The combinatorial results.** The quiver $H^{\mathfrak{w},\mathfrak{v}}$ as defined above has some interesting properties: Consider on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$ the following relation:

$$L' >' L \iff \text{there is an arrow } s \dashrightarrow t \text{ in } H_2^{\mathfrak{w},\mathfrak{v}} \text{ with } s \in L'_0, t \in L_0$$

Write $>$ for the transitive relation generated by $>'$.

Proposition 2. *Suppose, that $\mathfrak{v}$ and $\mathfrak{w}$ are both not asymmetric bands, then $>$ is a partial order on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$.*

The proof is quite easy, see 7.1, but it does not work in general, if asymmetric bands are involved.

We conjecture, that the proposition is true in general. A partial result in this direction is Proposition 3 (c) below.

Corollary 1. *Consider on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$ the following weaker relation:*

$$L' >'' L \iff \textit{there is an arrow } s \dashrightarrow t \textit{ in } H_{2,o}^{\mathfrak{w},\mathfrak{v}} \textit{ with } s \in L'_0, t \in L_0$$

The transitive relation generated by $>''$ is always a partial order on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$.

This is a trivial consequence of Proposition 2 if no asymmetric bands are involved. The remaining cases are easy to control (for example, if $\mathfrak{v}$ and $\mathfrak{w}$ are both asymmetric, then $>''$ is empty).

Proposition 3. *We have the following:*

(a) *Let $L' >' L \in \mathcal{L}(\mathfrak{w},\mathfrak{v})$. If L' is a h-line, then L is also a h-line of type $\mathbb{A}$ or $\mathbb{D}$.*

(b) *If L is a h-line, then there is at most one arrow from $s \xrightarrow{(h,g)_\rho} t$ in $H_{2,o}^{\mathfrak{w},\mathfrak{v}}$ with $t \in L_0$; if L is additionally of type $\mathbb{D}$, there is no such arrow.*

(c) *The set of real z-lines $(\mathcal{Z}(\mathfrak{w},\mathfrak{v}),>)$ is a poset.*

(d) *If $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$ lies on a h-line, holds*
$$G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \leq G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho))$$
for $\rho = -1, 1$.

The proof of this result will be given in section 7.

Corollary 2. *Let $\mathfrak{v},\mathfrak{w} \in \mathfrak{X}$ and suppose, that $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$ lies on a h-line. If $\mathfrak{v} \neq \mathfrak{w}$ or $(i \neq j)$ then $(i,j) \in H_0^{\mathfrak{v},\mathfrak{w}}$ lies on a real z-line.*

This follows at once from Proposition 3 (d) above. This looks quite innocent, but together with part (a) of our Theorem 2 this is fundamental for the description of irreducible homomorphisms in 5.1. We call a vertex $q \in H^{\mathfrak{w},\mathfrak{v}}$ an *e-point* if

- p is *not* a zero-point,
- p is not start-point of an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$.
- p is endpoint of a connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}}, H_{2,o}^{\mathfrak{w},\mathfrak{v}})$.

Corollary 3. *Suppose, that $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$ is a poset. Let $p \in H_0^{\mathfrak{w},\mathfrak{v}}$ be a vertex lying on a h-line not of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$ and $\rho \in \{-1,1\}$. Then there is a word $\mathfrak{h}$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $s(\mathfrak{h}) = p$, $\sigma(\mathfrak{h}) = \rho$ and $t(\mathfrak{h})$ an e-point.*

This is a direct consequence of the above proposition. Observe that, usually, there are many choices for these words.

4. Description of the Homomorphisms

Theorem 2. *Let $\mathcal{C}$ be a clan, and $\mathfrak{v},\mathfrak{w} \in \mathfrak{X}$, $V \in A_{\mathfrak{v}}$-mod and $W \in A_{\mathfrak{w}}$-mod.*

(a) *For $\mathbf{f} = (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}} \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ we have $\mathbf{f}_{(j,i)} = 0$ if (j,i) lies on a real z-line in $H^{\mathfrak{w},\mathfrak{v}}$.*

(b) *If $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$ is a poset (see Proposition 2), there is a k-linear isomorphism*
$$\mathbf{\Phi}\colon \oplus_{L\in\mathcal{H}(\mathfrak{w},\mathfrak{v})} M_L \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$
with
$$M_L := \begin{cases} \mathsf{Hom}_{\mathrm{k}}(V,W) & \text{if } L \text{ is of type } \mathbb{A} \\ \mathsf{Hom}_{\mathrm{k}[z]/(z^2-z)}(V,W) & \text{if } L \text{ is of type } \mathbb{D} \\ \mathsf{Hom}_{A_{\mathfrak{v}}}(V,W) & \text{if } L \text{ is of type } \widetilde{\mathbb{A}} \text{ or } \widetilde{\mathbb{D}} \end{cases}$$

Remark. If L is of type $\mathbb{D}$, let $(\eta,\zeta)_o$ be the unique loop of L, then the algebra homomorphisms $\mathrm{k}[z]/(z^2-z)\to A_{\mathfrak{v}}$ and $\mathrm{k}[z]/(z^2-z)\to A_{\mathfrak{w}}$ defined by $z\mapsto U_{\mathfrak{v}}(\zeta)$ and $z\mapsto U_{\mathfrak{w}}(\eta)$ induce a natural $\mathrm{k}[z]/(z^2-z)$-module structure on V and W, thus M_L is well defined in this case.

If L is of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$, we have (see 3.5 (1)) $\mathfrak{w}=\mathfrak{v}$ a band, thus also in this situation M_L is well-defined.

Of course, in order to prove (b), we have to define explicitly a basis of $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V),S_{\mathfrak{w}}(W))$; this will be done in 4.1, essentially in terms of the quiver $H^{\mathfrak{w},\mathfrak{v}}$. The proof of the theorem is then given in 4.2.

4.1. **Construction of the basis.** For every $L\in\mathcal{L}(\mathfrak{w},\mathfrak{v})$ we construct a k-linear map

$$\boldsymbol{\varphi}_L\colon M_L\to\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V),S_{\mathfrak{w}}(W))$$

with $(\boldsymbol{\varphi}_L(f))_{(j,i)_L}=f$ for a previously chosen vertex $(j,i)_L$ of L.

4.1.1. If L is a (real) h-line of type $\widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{A}}$, recall that $\mathfrak{w}=\mathfrak{v}$ is a band, and L the diagonal. Define for $f\in\mathsf{Hom}_{\mathrm{k}}(V,W)$ and $q\in H_0^{\mathfrak{w},\mathfrak{v}}$

$$\mathbf{f}_{(j,i)}:=\begin{cases} f & \text{if } (j,i)\in L_0\\ 0 & \text{else}\end{cases}$$
$$\boldsymbol{\varphi}_L(f):=(\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$$

It is an easy exercise to check, that this actually defines an injective k-linear map

$$\boldsymbol{\varphi}_L\colon\mathsf{Hom}_{A_{\mathfrak{v}}}(V,W)\to\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V),S_{\mathfrak{w}}(W))$$

In fact, $\boldsymbol{\varphi}_L(f)=S_{\mathfrak{w}}(f)$ and trivially $(\boldsymbol{\varphi}_L(f))_{(j,i)_L}=f$ for all vertices $(j,i)_L$ of L.

4.1.2. If L is a real h-line of type $\mathbb{D}$, we take $(i,j)_L=:s((\eta,\zeta)_o)$, where $(\eta,\zeta)_o$ is the unique loop in L. By Corollary 3 we find a word $\mathfrak{x}=x_m\cdots x_1$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $x_1\neq(\eta,\zeta)_o$ and $t(\mathfrak{x})$ an e-point. Set $H(\mathfrak{x}):=\{(j_n,i_n),\cdots,(j_1,i_1),(j_0,i_0)\}$ with $(j_k,i_k):=t(x_k)$ and $(j_0,i_0)=(i,j)_L$. For $f\in\mathsf{Hom}_{\mathrm{k}[z]/(z^2-z)}(V,W)$ (see Theorem 2) we define inductively

$$\mathbf{f}_{(j_0,i_0)}:=f$$
$$\mathbf{f}_{(j_k,i_k)}:=\begin{cases} U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k)\,\mathbf{f}_{(j_{k-1},i_{k-1})}\,(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1} & \begin{array}{l}\text{if } x_k \text{ is a direct letter or}\\ \text{if } x_k=\underline{(g,h)}^*_\rho \text{ is special}\\ \text{with } \rho\in\{+,\oplus\}\end{array}\\[2ex] (U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1}\,\mathbf{f}_{(i_{k-1},j_{k-1})}\,U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k) & \begin{array}{l}\text{if } x_k \text{ is a inverse letter or}\\ \text{if } x_i=\underline{(\nu,\mu)}^*_\rho \text{ is special}\\ \text{with } \rho\in\{-,\ominus\}\end{array}\end{cases}$$
$$\mathbf{f}_{(j,i)}:=0 \text{ for } (j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}\setminus H(\mathfrak{x})$$

and finally

$$\boldsymbol{\varphi}_L(f) := (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}},$$

see 3.3 for the definition of the functions $U^{(i)}_{\mathfrak{w},\mathfrak{v}}$. In order to see, that actually $\boldsymbol{\varphi}_L(f) \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$, we observe first $L_0 \subset H(\mathfrak{x})$ and $(i_k, j_k) = (i_l, j_l)$ if and only if $k = l$. Moreover, the elements of $\mathbb{M}(\mathfrak{x})$ lie all on h-lines (Proposition 3 (b)), thus they are all no 0-points. Finally note, that if there is an arrow $(i_k, j_k) \overset{x}{\dashrightarrow} (i_l, j_l)$ in $H_2^{\mathfrak{w},\mathfrak{v}}$ then $x = h_l$ and $k = j-1$; this makes it very easy to verify the equations (1)–(3d). Trivially, $\boldsymbol{\varphi}_L$ is injective.

4.1.3. If L is a real h-line of type $\mathbb{A}$, we choose an arbitrary vertex $(j_0, i_0) = (j,i)_L$ in L and two words $\mathfrak{y} = y_n \cdots y_1$, $\mathfrak{x} = x_m \cdots x_1$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $s(\mathfrak{y}) = (j_0, i_0) = s(\mathfrak{x})$, $\sigma(\mathfrak{y}) = 1 = -\sigma(\mathfrak{x})$, both ending in an e-point as in Corollary 3. Set $H(\mathfrak{x}) := \{(j_m, i_m) \cdots, (j_0, i_0)\}$ with $(j_k, i_k) := t(x_k)$ as before.

For $f \in \mathsf{Hom}_{\mathrm{k}}(V, W)$ (see Theorem 2) we define (inductively)

$$\mathbf{f}^-_{(j_0,i_0)} := f$$

$$\mathbf{f}^-_{(j_k,i_k)} := \begin{cases} U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k)\,\mathbf{f}^-_{(j_{k-1},i_{k-1})}\,(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1} & \begin{array}{l}\text{if } x_k \text{ is a direct letter or}\\ \text{if } x_k = \underline{(\nu,\mu)}^*_\rho \text{ is special}\\ \text{with } \rho \in \{+,\oplus\}\end{array} \\ (U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_i))^{-1}\,\mathbf{f}^-_{(j_{k-1},i_{k-1})}\,U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k) & \begin{array}{l}\text{if } x_k \text{ is a inverse letter or}\\ \text{if } x_i = \underline{(g,h)}^*_\rho \text{ is special}\\ \text{with } \rho \in \{-,\ominus\}\end{array} \end{cases}$$

$$f^-_{(j,i)} := 0 \text{ for } (j,i) \in H_0^{\mathfrak{w},\mathfrak{v}} \setminus H(\mathfrak{x})$$

and similarly, starting with $\mathfrak{y}$ we define $\mathbf{f}^+_{(j,i)}$ for $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$,then

$$\mathbf{f}_{(j,i)} := \begin{cases} f & \text{if } q = p \\ f^+_{(j,i)} + f^-_{(j,i)} & \text{else} \end{cases}$$

$$\phi_L(f) := (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$$

It is essentially the same argument as in the case $\mathbb{D}$ to see that this defines a injective k-linear map

$$\boldsymbol{\varphi}_L \colon \mathsf{Hom}_{\mathrm{k}}(V, W) \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

From Proposition 3 (a) and from the construction of the $\boldsymbol{\varphi}_L$ is is clear, that

$$\boldsymbol{\Phi} := \oplus_{L\in\mathcal{H}}\, \boldsymbol{\varphi}_L \colon \oplus_{L\in\mathcal{H}} M_L \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

is injective (the point is, that words starting in $(j,i)_L$ can not end in a line L' with $L' > L$).

4.2. **Proof of the theorem.** Both statements of the theorem follow now from the following lemma by induction on the *posets* $(\mathcal{Z}(\mathfrak{w},\mathfrak{v}),>)$ resp. $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$.

Lemma. *Let* $f=(\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}\in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V),S_{\mathfrak{w}}(W))$ *and* $L\in\mathcal{L}(\mathfrak{w},\mathfrak{v})$. *Suppose, that* $L' >' L$ *implies* $\mathbf{f}_{(j',i')}=0$ *for* $(j',i')\in L'_0$. *Then we have:*

(a) *If* L *is a z-line, then we have already* $\mathbf{f}_{(j,i)}=0$ *for all* $(j,i)\in L_0$.
(b) *Otherwise,* L *is a real h-line (by definition),* $\mathbf{f}_{(j,i)_L}\in M_L$, *and* $\big(\mathbf{f}-\boldsymbol{\varphi}_L(\mathbf{f}_{(j,i)_L})\big)_{(j,i)}=0$ *for all* $(j,i)\in L'_0$ *with* $L'\geq L$.

Proof. (1) Under the hypothesis of the lemma we have: $\mathbf{f}_{(j,i)}=0$ for some $(j,i)\in L_0$ implies $\mathbf{f}_{q'}=0$ for all $(j',i')\in L_0$. In fact, we may suppose, that there is an arrow $(j,i)\xrightarrow{h}(j',i')$ in $H_1^{\mathfrak{w},\mathfrak{v}}$, then we have $\mathbf{f}_{(j',i')}=U^{(1)}_{\mathfrak{w},\mathfrak{v}}(h)\mathbf{f}_{(j,i)}(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(h))^{-1}$ either directly by equation (2), or by equation (3a), using the hypothesis .

(2) If L is a z-line, there is a 0-point $(j,i)\in L_0$, or there is an arrow $(j',i')\xrightarrow{h'}(j,i)$ with $h'\in H_{2,o}^{\mathfrak{w},\mathfrak{v}}$ and $(j,i)\in L_0$ (remark (3) in 3.5). In the first case we are immediately done by (1); in the second case (j',i') belongs to a line $L'>L$, and since $\mathbf{f}_{(j,i)}=U^{(1)}_{\mathfrak{w},\mathfrak{v}}(h')\mathbf{f}_{(j',i')}(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(h'))^{-1}=0$ by hypothesis, we are also done.

(3) If L is a real h-line, we have only to show that $\mathbf{f}_{(j,i)_L}\in M_L$, then we can apply (1) to $(\mathbf{f}-\boldsymbol{\varphi}_L(\mathbf{f}_{(j,i)_L}))$. This is clear if L is of type $\mathbb{A}$ or $\mathbb{D}$ (!). In the remaining two cases the hypothesis imply $\mathbf{f}_{(j,i)}=\mathbf{f}_{(j,i)_L}$ for all $(j,i)\in L_0$, the remaining equalities force $\mathbf{f}_{(j,i)_L}\in M_L$ □

5. Auslander-Reiten theory

Let $\mathcal{C}=(\mathrm{k},Q,(\sigma,\tau),>)$ be a clan.

5.1. **Irreducible maps.** We start with two observations.

(1) Let $\mathfrak{w}\in\mathfrak{X}$ be a band, and $\mathfrak{v}\in\mathfrak{X}$ arbitrary. Consider the following composition of morphisms in $\mathsf{R}(\mathcal{C})$

$$S_{\mathfrak{w}}(W)\xleftarrow{\mathbf{f}}S_{\mathfrak{v}}(V)\xleftarrow{\mathbf{g}}S_{\mathfrak{w}}(W')$$

with $W,W'\in A_{\mathfrak{w}}\text{-}\mathsf{mod}$ and $V\in A_{\mathfrak{v}}\text{-}\mathsf{mod}$. If $\mathfrak{v}\neq\mathfrak{w}$ or $\mathbf{f}_{(i,i)}=0$, we have $(\mathbf{fg})_{(i,i)}=0$ for all $i\in H_0^{\mathfrak{w}}$. In fact,

$$(\mathbf{fg})_{(i,i)}=\sum_{j\in H_0^{\mathfrak{v}},G_{\mathfrak{v}}(j)=G_{\mathfrak{w}}(i)}\mathbf{f}_{(i,j)}\mathbf{g}_{(j,i)},$$

and $\mathbf{g}_{(j,i)}\neq 0$ implies $\mathbf{f}_{(i,j)}=0$ by Corollary 2 and Theorem 2 (a).

(2) Let $\mathfrak{w}<\mathfrak{v}$ be both inextensible, $\mathfrak{x}\in\mathfrak{X}\setminus\{\mathfrak{v},\mathfrak{w}\}$ (we may suppose, $\mathfrak{v},\mathfrak{w}\in\mathfrak{X}$) and consider the following composition of morphisms in $\mathsf{R}(\mathcal{C})$

$$S_{\mathfrak{w}}(W)\xleftarrow{\mathbf{f}}S_{\mathfrak{x}}(X)\xleftarrow{\mathbf{g}}S_{\mathfrak{v}}(V)$$

with $W \in A_{\mathfrak{w}}\text{-mod}$, $V \in A_{\mathfrak{v}}\text{-mod}$, $X \in A_{\mathfrak{x}}\text{-mod}$. If $(\mathbf{fg})_{(0,0)} \neq 0$, then $\mathfrak{x}$ is a string, and we may suppose $\mathfrak{v} > \mathfrak{x} > \mathfrak{w}$. In fact,

$$(\mathbf{fg})_{(0,0)} = \sum_{j \in H_0^{\mathfrak{x}}, G_{\mathfrak{x}}(j) = G_{\mathfrak{v}}(0)} \mathbf{f}_{(0,j)} \mathbf{g}_{(j,0)},$$

and, since $\mathfrak{v}$ is inextensible, $G_{\mathfrak{x}}(j) = G_{\mathfrak{v}}(0)$ for some $j \in H_0^{\mathfrak{x}}$ implies that $\mathfrak{x}$ is also inextensible, and we may suppose that it is comparable with $\mathfrak{v}$ and $\mathfrak{w}$. Finally, $\mathfrak{v} > \mathfrak{x}$ implies $\mathbf{g}_{(0,0)} = 0$ and $\mathfrak{x} > \mathfrak{w}$ implies $\mathbf{f}_{(0,0)}$ by a similar argument as in (1).

Thus, we have shown:

Proposition 4. *(1) For $\mathfrak{w}$ a band, the functor*

$$S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-mod} \to \mathsf{R}(\mathcal{C})$$

preserves irreducible maps.

(2) If $\mathfrak{w} < \mathfrak{v}$ are inextensible words, there is a real h-line L of type $\mathbb{A}$ that contains $(0,0) \in H_0^{\mathfrak{w},\mathfrak{v}}$. (Use the Lemma from 7.2!) Construct

$$\varphi_L : \mathsf{Hom}_{\mathrm{k}}(V, W) \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

as in 4.1. In case there is no inextensible word $\mathfrak{x}$ with $\mathfrak{v} > \mathfrak{x} > \mathfrak{w}$, the morphisms $\varphi_L(f) \neq 0$ are irreducible in $\mathsf{R}(\mathcal{C})$.

Remark. Suppose there is no *symmetric* string $\mathfrak{x}$ with $\mathfrak{w} < \mathfrak{x} < \mathfrak{v}$, denote by L the (real) h-line containing $(0,0)$ and let $\tilde{L}$ be the connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ that that contains L, (compare remark 3.5 (3) and Corollary 1). Than it is easy to see, that $\tilde{L}$ contains *two* e-points (one of them is just $(0,0)$). Thus, the construction of φ_L involves no choices in that case.

Notation. In case $\mathfrak{w} < \mathfrak{v}$ we denote by $L(\mathfrak{w} <_r \mathfrak{v})$ the real h-line in $H^{\mathfrak{w},\mathfrak{v}}$ that contains the vertex $(0,0)$. Similarly, in case $\mathfrak{w}^{-1} < \mathfrak{v}^{-1}$ there is a real h-line in $H^{\mathfrak{w},\mathfrak{v}}$ naturally associated with that relation; we denote this line by $L(\mathfrak{w} <_l \mathfrak{v})$.

5.2. **Exact structures.** We shall define on $\mathsf{R}(\mathcal{C})$ an exact structure ([GRo, 9.1]). Consider the class $\mathcal{E}$ of pairs $(\mathbf{i}, \mathbf{d})$ of composable morphisms

$$M \xrightarrow{\mathbf{i}} E \xrightarrow{\mathbf{d}} N$$

in $\mathsf{R}(\mathcal{C})$ such, that the induced k-linear maps

$$(*) \qquad 0 \to l^+(M)/l^-(M) \xrightarrow{\mathbf{i}_l} l^+(E)/l^-(E) \xrightarrow{\mathbf{d}_l} l^+(N)/l^-(N) \to 0$$

(see 1.2) form short exact sequences for all letters $l \in Q_*$.

Remark. A morphism $\mathbf{d} : E \to N$ is a deflation with respect to $\mathcal{E}$ if (and only if) the induced k-linear maps $\mathbf{d}_l : l^+(E)/l^-(E) \to l^+(N)/l^-(N)$ are surjective for each letter $l \in Q_*$.

In fact, define $M_p := \ker \mathbf{d}_p \subset E_p$ for all $p \in Q_0$, and $\mathbf{i}_p$ the inclusion.

$$l^{+/-}(M) := l^{+/-}(E) \cap \ker \mathbf{d}_{t(l)}$$

for all $l \in Q_*$. Now, it is easy to construct the induced structure maps for M to make it a representation of $\mathcal{C}$, and our hypothesis on $\mathbf{d}$ implies $(*)$ for all letters $l \in Q_*$.

Dually, a morphism $\mathbf{i}\colon M \to E$ is an inflation if (and only if) the induced maps $\mathbf{i}_l$ are injective for each letter $l \in Q_*$.

With this observation it is straightforward to verify for $\mathcal{E}$ the axioms of an exact structure [GRo, 9.1] directly.

The following lemma seems to be well-known, but we could not find an explicit reference.

Lemma. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, $(\mathbf{i}, \mathbf{d}) \in \mathcal{E}$ with indecomposable end terms and $\mathbf{i}, \mathbf{d}$ irreducible, then $(\mathbf{i}, \mathbf{d})$ is an Auslander-Reiten sequence.*

Proof. Let $\mathbf{f}\colon X \to M$ be not a retraction and consider the pull-back diagram

$$\begin{tikzcd} M \arrow[r, "\mathbf{i}'"] \arrow[d, equal] & E' \arrow[r, "\mathbf{d}'"] \arrow[d, "\mathbf{f}'"] & X \arrow[d, "\mathbf{f}"] \\ M \arrow[r, "\mathbf{i}"] & E \arrow[r, "\mathbf{d}"] & N \end{tikzcd}$$

If the first row splits, $\mathbf{f}$ factors trivially over $\mathbf{d}$, otherwise, since $\mathbf{i}$ is irreducible, $\mathbf{f}'$ admits a section $\mathbf{g}'$, thus $\mathbf{d} = \mathbf{f}(\mathbf{d}'\mathbf{g}')$. Since $\mathbf{d}$ is irreducible and $\mathbf{f}$ admits no section, $\mathbf{d}'\mathbf{g}'$ admits a retraction, and $\mathbf{f}$ factors again over $\mathbf{d}$. □

5.3. **Finite-dimensional clans.** We start with some combinatorial considerations.

Definition. We say, the clan $\mathcal{C} = (\mathrm{k}, Q, (\sigma, \tau), >)$ is *finite dimensional*, if for each $(q, \rho) \in Q_0 \times \{-1, 1\}$ the linearly ordered set $\mathfrak{W}(q, \rho)$ (see 1.3) has a (unique) minimal element $\mathfrak{i}_{(q,\rho)}$ and a (maximal) element $\mathfrak{p}_{(q,\rho)}$.

Remarks. Let $\mathcal{C}$ as above be finite-dimensional.

(1) In this situation each $\mathfrak{w} \in \mathfrak{W}(q, \rho) \setminus \{\mathfrak{i}_{(q,\rho)}\}$ admits a unique successor $\mathfrak{w}[1]$, in fact, we have a unique composition

$$\mathfrak{w} = \mathfrak{w}' l \mathfrak{i}_{(s(l),\sigma(l))}$$

for some letter $l \in Q_*(s(l), -\sigma(l))$ which is not minimal in that set; with l' the successor of l in $Q_*(s(l), -\sigma(l))$ we have

$$\mathfrak{w}[1] = \mathfrak{w}' l' \mathfrak{p}_{(s(l'),-\sigma(l'))}$$

Dually, each $\mathfrak{w} \in \mathfrak{W}(q, \rho) \setminus \{\mathfrak{p}_{(q,\rho)}\}$ has a unique predecessor $\mathfrak{w}[-1]$.

In case $\mathfrak{w}$ is left inextensible we set $[i]\mathfrak{w} := (\mathfrak{w}^{-1}[i])^{-1}$ whenever this makes sense.

(2) Consider the following sets of inextensible words:

$$\mathfrak{I} := \{\mathfrak{i}_l := \mathfrak{i}^{-1}_{(t(l),-\tau(l))}\, l\, \mathfrak{i}_{(s(l),-\sigma(l)} \mid l \in Q_*\}$$
$$\mathfrak{P} := \{\mathfrak{p}_l := \mathfrak{p}^{-1}_{(t(l),-\tau(l))}\, l\, \mathfrak{p}_{(s(l),-\sigma(l)} \mid l \in Q_*\}$$

If $\mathfrak{w}$ is an inextensible word, *not* in $\mathfrak{I}$, then $[1](\mathfrak{w}[1])$ and $([1]\mathfrak{w})[1]$ are defined and equal, thus we write just $[1]\mathfrak{w}[1] \notin \mathfrak{P}$ in this situation. Dually, if $\mathfrak{w}$ is *not* in $\mathfrak{P}$, we can write just $[-1]\mathfrak{w}[-1] \notin \mathfrak{I}$.

Observe, $[-1]([1]\mathfrak{w}[1])[-1] = \mathfrak{w}$, thus $[1]\mathfrak{w}[1]$ is symmetric if and only if $\mathfrak{w}$ is symmetric.

(3) Let $\mathfrak{i}_l \in \mathfrak{I}$ and $V \in A_{\mathfrak{i}_l}\text{-}\mathsf{mod}$, then $S_{\mathfrak{i}_l}(V)$ is injective in $(\mathsf{R}(\mathcal{C}), \mathcal{E})$. In fact, take $\mathfrak{w} \in \mathfrak{X} \setminus \{\mathfrak{i}_l\}$ (we may suppose $\mathfrak{i}_l \in \mathfrak{X}$) and consider $f \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{i}_l}(V), S_{\mathfrak{w}}(W))$. Then follows easily from our previous results that $f_l(V_{t(x)}) = 0$ if x is the letter in $H_*^{\mathfrak{i}_l}$ corresponding to l. Thus every inflation starting in $S_{\mathfrak{i}_l}(V)$ splits. By a dual argument $S_{\mathfrak{p}}(W)$ is projective for each $\mathfrak{p} \in \mathfrak{P}$.

5.4. **Auslander-Reiten sequences.**

Let $\mathcal{C}$ be a finite-dimensional clan.

5.4.1. If $\mathfrak{w}$ is an *asymmetric string*, not in $\mathfrak{I}$, then $[1]\mathfrak{w}[1]$ is also asymmetric, and we consider the following diagram:

$$\begin{tikzcd}
 & S_{\mathfrak{w}[1]}(V_r) \arrow[dr, "\mathbf{g}_l"] & \\
S_{\mathfrak{w}}(\mathrm{k}) \arrow[ur, "\mathbf{f}_r"] \arrow[dr, "\mathbf{f}_l"'] & & S_{[1]\mathfrak{w}[1]}(\mathrm{k}) \\
 & S_{[1]\mathfrak{w}}(V_l) \arrow[ur, "\mathbf{g}_r"'] &
\end{tikzcd}$$

In case $\mathfrak{w}[1]$ is asymmetric, we take $V_r := \mathrm{k}$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}(1)$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(-1)$.
Otherwise $\mathfrak{w}[1]$ is symmetric (thus $A_{\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2 - z)$), we take $V_r = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{1}{0}$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(0, -1)$.

In case $[1]\mathfrak{w}$ is asymmetric, we take $V_l := \mathrm{k}$, $\mathbf{f}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}<_l\mathfrak{w})}(1)$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_r[1]\mathfrak{w})}(1)$.
Otherwise $[1]\mathfrak{w}$ is symmetric (thus $A_{[1]\mathfrak{w}} = \mathrm{k}[z]/(z^2 - z)$), we take $V_l = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\mathbf{f}_l := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{0}{1}$ and $\mathbf{g}_r := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(1, 0)$.

By Proposition 4 the morphisms $\binom{\mathbf{f}_l}{\mathbf{f}_r}$ and $(\mathbf{g}_r, \mathbf{g}_l)$ are irreducible. It is a direct calculation to verify that $(\binom{\mathbf{f}_l}{\mathbf{f}_r}, (\mathbf{g}_r, \mathbf{g}_l)) \in \mathcal{E}$, the point is to use the remark in 5.1. Thus, by the lemma from 5.2 we have constructed an Auslander-Reiten sequence starting at $S_{\mathfrak{w}}(\mathrm{k})$ for every asymmetric string not in $\mathfrak{I}$. Similarly, we can construct an Auslander-Reiten sequence ending in $S_{\mathfrak{w}}(\mathrm{k})$ for every asymmetric string not in $\mathfrak{P}$.

5.4.2. If $\mathfrak{w}$ is a *symmetric string* not in $\mathfrak{I}$, $[1]\mathfrak{w}[1]$ is also symmetric and $A_{\mathfrak{w}} = A_{[1]\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2 - z)$. Consider the following diagram:

$$S_{\mathfrak{w}}(\mathrm{k}_a) \xrightarrow{\mathbf{f}_r} S_{\mathfrak{w}[1]}(V_r) \xrightarrow{\mathbf{g}_l} S_{[1]\mathfrak{w}[1]}(\mathrm{k}_{(1-a)})$$

Here, we denote by k_a the one-dimensional $\mathrm{k}[z]/(z^2-z)$-module with $\mathrm{k}_a(z) = a$ for $a \in \{0,1\}$.

In case $\mathfrak{w}[1]$ is asymmetric, we take $V_r := \mathrm{k}$, $\mathbf{f}_r := \varphi_{L(\mathfrak{w}[1]<_r\mathfrak{w})}(1)$ and $\mathbf{g}_l := \varphi_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(1)$.
Otherwise $\mathfrak{w}[1]$ is symmetric (thus $A_{\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2-z)$), we take $V_r = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $\mathbf{f}_r := \varphi_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{1}{0}$ and $\mathbf{g}_l := \varphi_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(0,1)$.

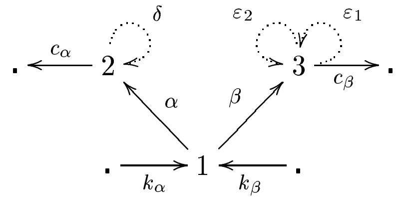
Now, we find similarly that $(\mathbf{f}_l, \mathbf{g}_r) \in \mathcal{E}$ with $\mathbf{f}_l, \mathbf{g}_r$ irreducible. Thus, we have constructed an Auslander-Reiten sequence starting at $S_{\mathfrak{w}}(\mathrm{k}_a)$ (with $a \in \{0,1\}$) for every symmetric string $\mathfrak{w}$ not in $\mathfrak{I}$. Similarly, we can construct an Auslander-Reiten sequence ending in $S_{\mathfrak{w}}(\mathrm{k}_a)$ for every symmetric string $\mathfrak{w}$ not in $\mathfrak{P}$.

5.4.3. If $\mathfrak{w}$ is a *band*, the functor $S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$ is easily seen to be exact, and since it preserves irreducible maps (Proposition 4), it also preserves Auslander-Reiten sequences by the above lemma. Thus, we obtain in the asymmetric case a $\mathsf{Spec}(\mathrm{k}[T,T^{-1}])$-family of homogeneous tubes, and in the symmetric case a $\mathsf{Spec}(\mathrm{k}[T])$-family of tubular type $(2,2)$.

Altogether, we established for a finite-dimensional clan the global existence of Auslander-Reiten sequences. The sequences for bands exist for arbitrary clans.

We should point out that the exact structure $\mathcal{E}$ tends to be too restrictive if the clan $\mathcal{C}$ is obtained by "translating" for example a subspace-problem. Typically, one gets "too many" projectives and injectives and "looses" thus a finite number of Auslander-Reiten sequences from the original problem.

5.5. **Example.** Consider the following clan $\mathcal{C}$, given by



$$\begin{aligned} Q_*(2,-1) &= \{\underline{c_\alpha}^{-1} > \underline{\alpha}\} & Q_*(3,-1) &= \{\underline{c_\beta}^{-1} > \underline{\beta}\} \\ Q_*(2,\ \ 1) &= \{\underline{\delta}^*\} & Q_*(3,\ \ 1) &= \{\underline{\varepsilon}_2^* > \underline{\varepsilon}_1^*\} \\ Q_*(1,-1) &= \{\underline{\alpha}^{-1} > \underline{k_\alpha}\} & & \\ Q_*(1,\ \ 1) &= \{\underline{\beta}^{-1} > \underline{k_\beta}\} & & \end{aligned}$$

This is the "clannish version" of the Zavadskij-poset.

Compare [CB2, 2.8] and [ZN]. Obviously $\mathcal{C}$ is finite-dimensional. We exhibit a piece of the following particular component of the Auslander-Reiten quiver of $\mathsf{R}(\mathcal{C})$:

$$\begin{tikzcd}[column sep=small]
 & & S_{[-1]\mathfrak{w}[-3]}(\mathrm{k}) \arrow[dl] \arrow[dr] \arrow[drr] & & \\
 & S_{[-1]\mathfrak{w}[-2]}(\mathrm{k}) \arrow[dl] \arrow[dr] & & S_{\mathfrak{w}[-3]}(\mathrm{k}_1) \arrow[dl] & S_{\mathfrak{w}[-3]}(\mathrm{k}_0) \arrow[dll] \\
S_{[-1]\mathfrak{w}[-1]}(\mathrm{k}) \arrow[dr] & & S_{\mathfrak{w}[-2]}(\mathrm{k}) \arrow[dl] & & \\
 & S_{\mathfrak{w}[-1]}(\mathrm{k}) \arrow[dl] \arrow[dr] & & & \\
S_{\mathfrak{w}}(\mathrm{k}) \arrow[dr] & & S_{[1]\mathfrak{w}[-1]}(\mathrm{k}) \arrow[dl] & & \\
 & S_{[1]\mathfrak{w}}(\mathrm{k}) \arrow[dl] \arrow[dr] & & & \\
S_{[1]\mathfrak{w}[1]}(\mathrm{k}) \arrow[dr] & & S_{[2]\mathfrak{w}}(\mathrm{k}) \arrow[dl] \arrow[dr] \arrow[drr] & & \\
 & S_{[2]\mathfrak{w}[1]}(\mathrm{k}) \arrow[dl] \arrow[dr] & & S_{[3]\mathfrak{w}}(\mathrm{k}_0) \arrow[dl] & S_{[3]\mathfrak{w}}(\mathrm{k}_1) \arrow[dll] \\
S_{[2]\mathfrak{w}[2]}(\mathrm{k}) \arrow[dr] & & S_{[3]\mathfrak{w}[1]}(\mathrm{k}) \arrow[dl] \arrow[dr] \arrow[drr] & & \\
 & S_{[3]\mathfrak{w}[2]}(\mathrm{k}) & & S_{[4]\mathfrak{w}[1]}(\mathrm{k}_1) & S_{[4]\mathfrak{w}[1]}(\mathrm{k}_0)
\end{tikzcd}$$

Where $\mathfrak{w} = \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{\beta}\ \underline{\alpha}^{-1}\underline{\delta}^*\ \underline{\alpha}\ \underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1}$. For the convenience of the reader we give some more strings explicitly:

$$\begin{aligned}
\mathfrak{w}[-3] &= \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{\beta}\ \underline{k}_\alpha & \underline{c}_\beta\underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1} &= [3]\mathfrak{w} \\
\mathfrak{w}[-2] &= \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1} & \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1} &= [2]\mathfrak{w} \\
\mathfrak{w}[-1] &= \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{\beta}\ \underline{\alpha}^{-1}\underline{\delta}^*\ \underline{c}_\alpha^{-1} & \underline{k}_\beta^{-1}\underline{\alpha}^{-1}\underline{\delta}^*\ \underline{\alpha}\ \underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1} &= [1]\mathfrak{w}
\end{aligned}$$

and

$$[1]\mathfrak{w}[-1] = \underline{k}_\beta^{-1}\underline{\alpha}^{-1}\underline{\delta}^*\ \underline{c}_\alpha^{-1}$$

Moreover we have:

$$\mathfrak{i}_{\underline{c}_\beta^{-1}} = \mathfrak{w}[-2], \quad \mathfrak{i}_{\underline{c}_\alpha^{-1}} = [1]\mathfrak{w}[-1] = \mathfrak{p}_{\underline{k}_\beta^{-1}}, \quad [2]\mathfrak{w} = \mathfrak{p}_{\underline{k}_\alpha^{-1}}$$

The postprojective and the preinjective component are well-known; the remaining components containing strings are of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\mathbb{D}_\infty$, and they are parameterized by the asymmetric resp. symmetric strings of the form $\underline{c}_\beta\underline{\varepsilon}_1^*\ ?\ \underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1}$ (use an argument as in [Ge]).

5.6. **Clannish algebras.** Let Q' be an ordinary (connected) quiver and Sp a subset of the loops of Q'. Call the elements of Sp "special loops" and the remaining arrows "ordinary arrows". Let $R = R^{\mathrm{Sp}} \cup Z$ be a set of relations for Q' consisting of $R^{\mathrm{Sp}} := \{\varepsilon^2 - \varepsilon \mid \varepsilon \in \mathrm{Sp}\}$ and Z a set of zero-relations.

Definition. [CB2] $\Lambda := \mathrm{k}Q'/(R)$ is a *clannish* algebra provided that the following holds:

(C0) None of the relations in Z do start or end with a special loop, or involve the square of a special loop.

(C1) At most two arrows terminate at any vertex, and at most two arrows start at any vertex.

(C2) For any *ordinary* arrow β in Q' there is at most one (*arbitrary*) arrow α with $\beta\alpha$ a path not in Z, and there is at most one (*arbitrary*) arrow γ in Q with $\gamma\beta$ a path not in Z.

In [CB2] is indicated, how Λ-mod is embedded in $\mathsf{R}(\mathcal{C})$ for an adequate clan $\mathcal{C}$. Since we want to calculate Auslander-Reiten sequences in Λ-mod for Λ a finite-dimensional clannish algebra, we have to make that explicitly:

Write $Z = Z' \cup Z''$, where Z'' is minimal with respect to condition (C2) above. Thus we can define two functions $\sigma, \tau \colon Q'_1 \to \{-1, 1\}$ subject to the following conditions: – $\sigma(\varepsilon) = \tau(\varepsilon)$ for $\varepsilon \in \mathrm{Sp}$
– $s(\alpha) = s(\beta)$ implies $\sigma(\beta) = -\sigma(\alpha)$
– $t(\alpha) = t(\beta)$ implies $\tau(\beta) = -\tau(\alpha)$
– $\beta\alpha$ a path not in Z'' implies $\sigma(\beta) = -\tau(\alpha)$
Now, we consider the biquiver $Q'' := (Q'_0, (Q'_1 \setminus \mathrm{Sp}), \mathrm{Sp})$ with the above polarization (σ, τ). Observe, that the sets $Q''_*(q, \rho)$ are of one of the following forms:

$$\emptyset,\ \{\underline{\alpha}\},\ \{\underline{\beta}^{-1}\},\ \{\underline{\beta}^{-1}, \underline{\alpha}\},\ \{\underline{\varepsilon}^*\}$$

for certain ordinary arrows α, β special loops ε. We shall extend Q'' to a biquiver Q by "adding kernels and co-kernels" to the ordinary arrows and determine by the way a linear order $>$ on the sets of $Q_*(q, \rho)$ with more than one element. This data defines our clan $\mathcal{C}$.

- In case $Q''(q, \rho) = \{\underline{\alpha}\}$ add an arrow $q \xrightarrow{c_\alpha} t(c_\alpha)$ with and set $Q_*(q, \rho) := \{\underline{c}_\alpha^{-1} > \underline{a}\}$.

- In case $Q''(q, \rho) = \{\underline{\beta}^{-1}\}$ add an arrow $s(k_\beta) \xrightarrow{k_\beta} q$ and set $Q_*(q, \rho) := \underline{\beta}^{-1} > \underline{k}_\beta$.

- In case $Q''(q, \rho) = \{\underline{b}^{-1}, \underline{a}\}$ add an arrow $s(\underline{z}_{\beta\alpha}) \xrightarrow{z_{\beta\alpha}} q$ and set $Q(q, \rho) := \underline{\beta}^{-1} > \underline{z}_{\beta\alpha} > \underline{\alpha}$. Moreover let in this situation formally $\underline{z}_{\beta\alpha} = \underline{c}_\beta^{-1} = \underline{k}_\alpha$.

We get an equivalence of categories $F \colon \mathsf{R}(\mathcal{C}) \xrightarrow{\sim} \mathsf{rep}_{\mathrm{k}}(Q', R^{\mathrm{Sp}} \cup Z'')$ defined by

$$\begin{aligned} (FM)_q &:= M_q \text{ for all } q \in Q'_0 \\ (FM)(\varepsilon) &:= M_\varepsilon \text{ for all special loops } e \\ (FM)(\alpha) &:= \iota_\alpha M_\alpha \pi_\alpha \text{ for all ordinary arrows } \alpha \end{aligned}$$

Note: For a special loop ε we have $(\underline{\varepsilon}^*)^+(M) = M_{s(\varepsilon)}$ and $(\underline{\varepsilon}^*)^- = 0$. While for an ordinary arrow α we have $(\underline{\alpha}^{-1})^+(M) = M_{s(\alpha)}$ and $(\underline{\alpha})^-(M) = 0$, thus $\pi_\alpha \colon M_{s(\alpha)} \to (\underline{\alpha}^{-1})^+(M)/(\underline{\alpha}^{-1})^-(M)$ and $\iota_\alpha \colon (\underline{\alpha})^+(M)/(\underline{\alpha})^-(M) \to M_{t(\alpha)}$ are well-defined.

Let us point out, that F is exact in the sense, that it sends exact sequences from the standard exact structure $\mathcal{E}$ on $\mathsf{R}(\mathcal{C})$ to exact sequences in $\mathsf{rep}_{\mathrm{k}}(Q', R^{\mathrm{Sp}} \cup Z'')$, but usually in the latter category there are "more" exact sequences.

If $\mathfrak{w}$ is a string, and $0 \neq W \in A_{\mathfrak{w}}\text{-}\mathsf{mod}$, then $F(S_{\mathfrak{w}}(W))$ fulfills the relations from Z' if and only if $\mathfrak{w}$ is not of the form $\mathfrak{w}_l \mathfrak{z}^{\omega} \mathfrak{w}_r$ for some $\mathfrak{z} \in Z'$, some $\omega \in \{-1, 1\}$ and $\mathfrak{w}_l, \mathfrak{w}_r$ words.
Similarly, if $\mathfrak{w}$ is a band, the same is true if $\mathfrak{w}^n$ is not of the form $\mathfrak{w}_l \mathfrak{z}^{\omega} \mathfrak{w}_r$. Thus, we call a string (resp. band) $\mathfrak{w}$ *admissible* if $\mathfrak{w}$ (resp. $\mathfrak{w}^n$) is not of the above form.

In case the clannish algebra $\Lambda = \mathrm{k}Q/(R)$ is finite dimensional, the sets

$$\mathfrak{W}'(q, \rho) := \{\mathfrak{w} \in \mathfrak{W}(q, \rho) \text{ admissable } \}$$

contain maximal and minimal elements $\mathfrak{p}'_{(q,\rho)}$ (which contain no inverse letter from Q''_*) and $\mathfrak{i}'_{(q,\rho)}$ (which contain no direct letter from Q''_*). Consequently an element $\mathfrak{w} \in \mathfrak{W}'(q, \rho)$ has a predecessor $\mathfrak{w}[-1]'$ and a successor $\mathfrak{w}[1]'$ in $\mathfrak{W}'(q, \rho)$! After defining sets $\mathfrak{I}'$ (resp. $\mathfrak{P}'$ of injective (resp. projective) strings relative to Z' we can define exact sequences

$$\begin{tikzcd}
 & S_{\mathfrak{w}[1]'}(V_r) \arrow[dr, "\mathbf{g}_l"] & \\
S_{\mathfrak{w}}(\mathrm{k}) \arrow[ur, "\mathbf{f}_r"] \arrow[dr, "\mathbf{f}_l"'] & & S_{[1]'\mathfrak{w}[1]'}(\mathrm{k}) \\
 & S_{[1]'\mathfrak{w}}(V_l) \arrow[ur, "\mathbf{g}_r"'] &
\end{tikzcd}$$

for asymmetric admissible strings not in $\mathfrak{I}'$ and

$$S_{\mathfrak{w}}(\mathrm{k}_a) \xrightarrow{\mathbf{f}_r} S_{\mathfrak{w}[1]'}(V_r) \xrightarrow{\mathbf{g}_l} S_{[1]'\mathfrak{w}[1]'}(\mathrm{k}_{(1-a)})$$

for symmetric admissible strings not in $\mathfrak{I}'$, which become Auslander-Reiten sequences in $\mathsf{rep}_{\mathrm{k}}(Q', R)$. (The exactness follows basically as in 5.4, using the remark at the end of 5.1). Additionally we obtain for every *ordinary* arrow α in Q' an Auslander-Reiten sequence corresponding to the following sequence of asymmetric strings:

$$\mathfrak{i}'^{-1}_{(t(\alpha),-\tau(\alpha))} \underline{c}_{\alpha}^{-1} \to \mathfrak{i}'^{-1}_{(t(\alpha),-\tau(\alpha))} \underline{\alpha}\, \mathfrak{p}'_{(s(\alpha),-\sigma(\alpha))} \to \underline{k}_{\alpha}^{-1}\, \mathfrak{p}'_{(s(\alpha),-\sigma(\alpha))},$$

which come *not* from exact sequences in $\mathsf{R}(\mathcal{C})$, in fact, the end-terms of these sequences come precisely from the strings in $\mathfrak{I}'$ resp. $\mathfrak{P}'$ which give *not* rise to injective (resp. projective) modules. The admissible bands yield families of tubes as in the case of clans.

All together, we obtained just a description of the Auslander-Reiten sequences, as it could be expected (in case $\operatorname{char} \mathrm{k} \neq 2$) by transferring the well-known results for string-algebras via [ReRd] to clannish algebras – despite the fact, that the description of the indecomposables obtained in that way is not directly compatible with our description.

6. The system of equations

Condition (H1) is for f trivially equivalent to equation (1).

Condition (H2) is equivalent to equation (2). Let us explain this: For each arrow $\alpha \in Q_1$ we have a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{j\in\mathbf{s}(\mathfrak{w},\underline{\alpha})} W_j \arrow[d, "S_{\mathfrak{w}}(W)_\alpha"'] & \bigoplus\limits_{i\in\mathbf{s}(\mathfrak{v},\underline{\alpha})} V_i \arrow[l, "\mathbf{f}_{\underline{\alpha}^{-1}}"'] \arrow[d, "S_{\mathfrak{v}}(V)_\alpha"] \\
\bigoplus\limits_{j'\in\mathbf{t}(\mathfrak{w},\underline{\alpha})} W_{j'} & \bigoplus\limits_{i'\in\mathbf{t}(\mathfrak{v},\underline{\alpha})} V_{i'} \arrow[l, "\mathbf{f}_{\underline{\alpha}}"]
\end{tikzcd}$$

which decomposes into

$$\begin{tikzcd}
W_{s(\nu)} \arrow[d, "U_{\mathfrak{w}}(\nu)"'] & V_{s(\nu)} \arrow[l, "\mathbf{f}_{(s(\nu),s(\mu))}"'] \arrow[d, "U_{\mathfrak{v}}(\mu)"] \\
W_{t(\nu)} & V_{t(\nu)} \arrow[l, "\mathbf{f}_{(t(\nu),t(\mu))}"]
\end{tikzcd}$$

for all $(\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \alpha = G_{\mathfrak{v}}(\mu)$ Similarly, (H3) is equivalent to the equations (3a)–(3d): For each special loop $\varepsilon \in Q_2$ we have a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{j\in\mathbf{s}(\mathfrak{w},\underline{\varepsilon}^*)} W_j \arrow[d, "S_{\mathfrak{w}}(W)_\varepsilon"'] & \bigoplus\limits_{i\in\mathbf{s}(\mathfrak{v},\underline{\varepsilon}^*)} V_i \arrow[l, "\mathbf{f}_{\underline{\varepsilon}^*}"'] \arrow[d, "S_{\mathfrak{v}}(V)_\varepsilon"] \\
\bigoplus\limits_{j'\in\mathbf{t}(\mathfrak{w},\underline{\varepsilon}^*)} W_{j'} & \bigoplus\limits_{i'\in\mathbf{t}(\mathfrak{v},\underline{\varepsilon}^*)} V_{i'} \arrow[l, "\mathbf{f}_{\underline{\varepsilon}^*}"]
\end{tikzcd}$$

which decomposes into the following four types of diagrams:

$$\begin{tikzcd}
W_{s(\nu)} \oplus W_{t(\nu)} \arrow[d, "(U_{\mathfrak{w}}(\nu){,}1)"'] & V_{s(\mu)} \oplus V_{t(\mu)} \arrow[l, "\begin{pmatrix} \mathbf{f}_{(s(\nu),s(\mu))} & \mathbf{f}_{(s(\nu),t(\mu))} \\ \mathbf{f}_{(t(\nu),s(\mu))} & \mathbf{f}_{(t(\nu),t(\mu))} \end{pmatrix}"'] \arrow[d, "(U_{\mathfrak{v}}(\mu){,}1)"] \\
W_{t(\nu)} & V_{t(\mu)} \arrow[l, "\mathbf{f}_{(t(\nu),t(\mu))}"']
\end{tikzcd}$$

for each pair of arrows $(\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \varepsilon = G_{\mathfrak{v}}(\mu)$. From this we can read off the equations (3a) and (3a') (use that $U_{\mathfrak{w}}(\nu)$ is invertible in this situation).

$$\begin{tikzcd}
W_j \arrow[d, "U_{\mathfrak{w}}(\eta)"'] & & V_{s(\mu)} \oplus V_{t(\mu)} \arrow[ll, "(\mathbf{f}_{(j,s(\mu))},\mathbf{f}_{(j,t(\mu))})"'] \arrow[d, "(U_{\mathfrak{v}}(\mu){,}1)"] \\
W_j & & V_{t(\mu)} \arrow[ll, "\mathbf{f}_{(j,t(\mu))}"']
\end{tikzcd}$$

for each pair of arrows $(\eta, \mu) \in H_2^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\eta) = \varepsilon = G_{\mathfrak{v}}(\mu)$ and $s(\eta) = j = t(\eta)$ From this we can read off the equations (3b) (use, that $U_{\mathfrak{w}}(\eta)$ is idempotent and $U_{\mathfrak{v}}\zeta$ is invertible in this situation).

$$\begin{tikzcd}
W_{s(\nu)} \oplus W_{t(\nu)} \arrow[d, "(U_{\mathfrak{w}}(\nu){,}1)"'] & & V_i \arrow[ll, "\left(\begin{smallmatrix}\mathbf{f}_{(s(\nu),i)} \\ \mathbf{f}_{(t(\nu),i)}\end{smallmatrix}\right)"'] \arrow[d, "U_{\mathfrak{v}}(\zeta)"] \\
W_{t(\nu)} & & V_i \arrow[ll, "\mathbf{f}_{(t(\nu),i)}"']
\end{tikzcd}$$

for each pair of arrows $(\nu, \zeta) \in H_1^{\mathfrak{w}} \times H_2^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \varepsilon = G_{\mathfrak{v}}(\zeta)$, $s(\zeta) = i = t(\zeta)$ is a loop. From this we can read off the equations (3c) (observe, that $1 - U_{\mathfrak{v}}(\zeta)$ is idempotent in this situation).

$$\begin{tikzcd}
W_j \arrow[d, "U_{\mathfrak{w}}(\eta)"'] & V_i \arrow[l, "\mathbf{f}_{(j,i)}"'] \arrow[d, "U_{\mathfrak{v}}(\zeta)"] \\
W_j & V_i \arrow[l, "\mathbf{f}_{(j,i)}"']
\end{tikzcd}$$

for each pair of arrows $(\eta, \zeta) \in H_2^{\mathfrak{w}} \times H_2^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\eta) = \varepsilon = G_{\mathfrak{v}}(\zeta)$ and $s(\eta) = j = t(\eta)$ and $s(\zeta) = i = t(\zeta)$. From this we can read off the equations (3d).

7. Proof of the combinatorial results

7.1. **Proof of Proposition 2.** Let $\mathfrak{w} \in \mathfrak{X}$ not an asymmetric band, and $H_0^{\mathfrak{w}} = \{0, 1, \cdots, n\}$ as in 2.2. We define a function $\omega_{\mathfrak{w}} \colon H_0^{\mathfrak{w}} \to \mathbb{Z}$ uniquely by the following prescription:

$$\omega_{\mathfrak{w}}(0) = 0 \text{ and } \omega_{\mathfrak{w}}(t(\nu)) = \begin{cases} \omega_{\mathfrak{w}}(s(\nu)) & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \omega_{\mathfrak{w}}(s(\nu)) + 1 & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \text{ for all } \nu \in H_1^{\mathfrak{w}}$$

Now, if $\mathfrak{w}, \mathfrak{v} \in \mathfrak{X}$ are not asymmetric bands, we set

$$\Omega_{\mathfrak{w},\mathfrak{v}} \colon H_0^{\mathfrak{w},\mathfrak{v}} \to \mathbb{Z},\ (j, i) \mapsto \omega_{\mathfrak{w}}(j) - \omega_{\mathfrak{v}}(i),$$

and observe, that $\Omega_{\mathfrak{w},\mathfrak{v}}$ is constant on (the vertices of) any line, while $L >' L'$ implies $\Omega_{\mathfrak{w},\mathfrak{v}}(L_0) = \Omega_{\mathfrak{w},\mathfrak{v}}(L'_0) + 1$. Thus, in this situation the transitive relation $>$ on $\mathcal{L}(\mathfrak{w}, \mathfrak{v})$ generated by $>'$ is an order relation.

Remark. If $\mathfrak{w}$ is an asymmetric band, we can define the function $\omega_{\mathfrak{w}}$ in this way only, if the number c of clockwise arrows $\nu \in H_1^{\mathfrak{w}}$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ coincides with the number a of anti-clockwise arrows ν' with $G_{\mathfrak{w}}(\nu') \in Q_2$. In all examples of asymmetric bands $\mathfrak{w}$ I know, it is possible to define slight variations of this ω which work similarly if $\mathfrak{v}$ is not an asymmetric band.

7.2. For the proof of Proposition 3 we need the following technical result:

Lemma. *Let $\mathfrak{w} \in \mathfrak{X}$ and $H^{\mathfrak{w}}$ the corresponding quiver constructed in 2.2. If $s \xrightarrow{h} t$ is an arrow in $H_1^{\mathfrak{w}}$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ and thus $\sigma(\nu) = \tau(\nu) =: -\rho$, we have*

$$(*) \qquad \begin{aligned} &G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) > G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) \text{ and} \\ &\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) \succ \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) \end{aligned}$$

Moreover, $\Delta\big(G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho), G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))\big) \geq \Delta\big(\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))\big)$.

Proof. The first inequality holds, as already observed, by the construction of $H^{\mathfrak{w}}$. Anyway, $\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho))$ and $\widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))$ are comparable, and by construction we have

$$d := \Delta(G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))) \geq \Delta(\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))).$$

We will show $(*)$ only for the most complicated case of $\mathfrak{w}$ a symmetric band, the other cases are up to obvious changes the same argument. We may suppose moreover $\mathfrak{w} = \mathfrak{z}\,\underline{f}^*\mathfrak{z}^{-1}\underline{g}^*$ with $\mathfrak{w} = w_1 \cdots w_{2n+2}$ i.e. $\mathfrak{z} = w_1 \cdots w_n$. We define three (infinite) words, $\widetilde{\mathfrak{w}}$ in Q and $\hat{\mathfrak{w}}^\rho$, $\rho \in \{+,-\}$ in $\widehat{Q}$:

$$\widetilde{\mathfrak{w}} = (\tilde{w}_i)_{i\in\mathbb{Z}}, \quad \tilde{w}_{k(2n+2)+i} := w_i$$

$$\hat{\mathfrak{w}}^\rho = (\hat{w}_i^\rho)_{i\in\mathbb{Z}}, \quad \hat{w}_i^\rho := \begin{cases} \tilde{w}_i & \text{if } \tilde{w}_i \text{ is a direct or or inverse letter} \\ \underline{\varepsilon} & \text{if } \tilde{w}_i = \underline{\varepsilon}^* \text{ special}, \widetilde{\mathfrak{w}}_{[i-1]} > \widetilde{\mathfrak{w}}^{[i]} \\ & \text{or } \tilde{w}_i = \underline{\varepsilon}^* \text{ special}, \widetilde{\mathfrak{w}}_{[i-1]} = \widetilde{\mathfrak{w}}^{[i]} \text{ and } \rho\, i > 0. \\ \underline{\varepsilon}^{-1} & \text{if } \tilde{w}_i = \underline{\varepsilon}^* \text{ special}, \widetilde{\mathfrak{w}}_{[i-1]} < \widetilde{\mathfrak{w}}^{[i]} \\ & \text{or } \tilde{w}_i = \underline{\varepsilon}^* \text{ special}, \widetilde{\mathfrak{w}}_{[i-1]} = \widetilde{\mathfrak{w}}^{[i]} \text{ and } \rho\, i \leq 0. \end{cases}$$

If, say $t = i-1$ and $s = i$ in $H_0^{\mathfrak{w}}$, we have by construction of the objects

$$\begin{aligned} G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) &= \widetilde{\mathfrak{w}}^{[i]} & \qquad G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) &= \widetilde{\mathfrak{w}}^{-1}_{[i-1]} \\ \widehat{G}^\rho_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) &= (\hat{\mathfrak{w}}^\rho)^{[i]} & \qquad \widehat{G}^\rho_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) &= (\hat{\mathfrak{w}}^\rho)^{-1}_{[i-1]} \end{aligned}$$

Thus, by hypothesis we have

$$\text{(i)} \qquad \widetilde{\mathfrak{w}}^{-1}_{[i-1]} > \widetilde{\mathfrak{w}}^{[i]} \text{ and}$$

$$(\widetilde{\mathfrak{w}}^{[i-d-1]}_{[i-1]})^{-1} = \widetilde{\mathfrak{w}}^{[i]}_{[i+d]}, \text{ in particular}$$

$$\text{(ii)} \qquad (\widetilde{\mathfrak{w}}^{[i-k-1]}_{[i-1]})^{-1} = \widetilde{\mathfrak{w}}^{[i]}_{[i+k]} \quad (0 < k \leq d)$$

and we have to show:

$$(\hat{\mathfrak{w}}^{+}_{[i-1]})^{-1} \succ (\hat{\mathfrak{w}}^{-})^{[i]}$$

To this end it is enough, to see for $k \in \{1,\ldots,d\}$, that if $\tilde{w}_{i+k} = \tilde{w}_{i-k}$ is special and $\hat{w}^{-}_{i+k}$ is inverse, then $\hat{w}^{+}_{i-k}$ is a direct letter. Now, recall that

$$\text{(iii)} \qquad \hat{w}^{-}_{i+k} \text{ inverse} \Longleftrightarrow \widetilde{\mathfrak{w}}^{-1}_{[i+k-1]} \leq \widetilde{\mathfrak{w}}^{[i+k]}$$

$$\text{(!)} \qquad \hat{w}^{+}_{i-k} \text{ direct} \Longleftrightarrow \widetilde{\mathfrak{w}}^{-1}_{[i-k-1]} > \widetilde{\mathfrak{w}}^{[i-k]}$$

From (i) and (ii) we get $\widetilde{\mathfrak{w}}^{-1}_{[i-k-1]} > \widetilde{\mathfrak{w}}^{[i+k]}$ and then, using this $\widetilde{\mathfrak{w}}^{-1}_{[i+k-1]} > \widetilde{\mathfrak{w}}^{[i-k]}$. These two inequalities, together with (iii) give (!). □

7.3. **Proof of Proposition 3 (a).** Let L' be a h-line in $H^{\mathfrak{w},\mathfrak{v}}$, and say $(t(\nu), s(\mu)) \xrightarrow{(\nu,g)_+} (t(\nu), t(\mu))$ an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$ starting in L'. Thus we have $\sigma(\nu) = \tau(\nu) = \sigma(\mu) = \tau(\mu) =: -\rho$, and

$$\begin{aligned}
&\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)) \succeq \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho)) \text{ (L' is h-line)}\\
&\widehat{G}^{+}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)) \succ \widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)) \text{ and}\\
&\widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho)) \succ \widehat{G}^{-}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu),\rho)) \text{ by the lemma from 7.2.}
\end{aligned}$$

Thus, in particular

$$\begin{aligned}
&\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)) \succeq \widehat{G}^{+}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)) \succ \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho)) \text{ and}\\
&\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)) \succ \widehat{G}^{-}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu),\rho)) \succeq \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu),\rho))
\end{aligned}$$

This means, that the line L, which contains $(s(\nu), s(\mu)) \xrightarrow{(\nu,\mu)} (t(\nu), t(\mu))$ is a h-line, but it can not be of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$, because then we would have for example $\widehat{G}^{+}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)) = \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho))$. We leave it as an exercise, to show that L neither is of type $\mathbb{D}$.

7.4. **Proof of Proposition 3 (b).** Clearly, no line contains more then two endpoints of arrows from $H^{\mathfrak{w},\mathfrak{v}}_{2,o}$. Thus, suppose that L is a h-line, that contains the endpoints of the arrows from $H^{\mathfrak{w},\mathfrak{v}}_{2,o}$, say $(j, s(\mu)) \xrightarrow{(\eta,\mu)_\oplus} (j, t(\mu))$ and $(t(\nu), i) \xrightarrow{(\nu,\zeta)_\ominus} (s(\nu), i)$. Let $\rho := -\sigma(\mu)$ and $\rho' := -\sigma(\zeta)$. It is easy to see, that we have in this situation

$$\begin{aligned}
&\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)) \succeq \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)), \qquad d_1 := \Delta(\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)), \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(l,\rho)))\\
&\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho')) \succeq \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho)), \quad d_2 := \Delta(\widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho')), \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu),\rho'))).
\end{aligned}$$

Moreover, we have by the lemma from 7.2

$$\begin{aligned}
&\widehat{G}^{+}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)) \succ \widehat{G}^{-}_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu),\rho)) \text{and}\\
&\widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu'),\rho)) \succ \widehat{G}^{-}_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu'),\rho)),
\end{aligned}$$

thus in particular

$$d_1 \geq c_1 := \Delta(\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)), \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)))$$
$$d_2 \geq c_2 := \Delta(\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho')), \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu'),\rho))).$$

On the other hand we have by construction $c_1 > d_2$ and $c_2 > d_1$, a contradiction. Essentially the same argument shows, that for a h-line of type $\mathbb{D}$ there is no such arrow.

7.5. **Proof of Proposition 3 (c).** A real z-line Z contains at most one vertex $(j,i) \in H^{\mathfrak{w},\mathfrak{v}}$, which is starting point of an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$. In case there is no such vertex, trivially Z can not be contained in a cycle $Z_1 >' Z_2 >' \cdots >' Z_n >' Z_1$. Otherwise, there is a unique $\rho \in \{-1,1\}$ with $\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \prec \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho))$, and

$$\delta(Z) := \Delta(\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho)), \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho))) \in \mathbb{N}_0$$

is well defined. Now, if Z' is an other line with $Z' >' Z$ (thus a real z-line), then $\delta(Z') < \delta(Z)$. In fact, this is trivial if the arrow from Z' to Z lies in $H_{2,o}^{\mathfrak{w},\mathfrak{v}}$, otherwise this follows easily from the lemma in 7.2.

7.6. **Proof of Proposition 3 (d).** Let (j,i) lie on the h-line L, and $\tilde{L}$ the connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ that contains L, (compare remark 3.5 (3) and Corollary 1). If there is at most one line L' with $\tilde{L} >' L'$, our affirmation is directly to see. Thus suppose, there are two such lines, and $G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) > G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho))$ for say $\rho = 1$, this implies the same for $\rho = -1$. But now, we find a nontrivial word $\mathfrak{x}$ such that

$$\mathfrak{x}G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,1)) > G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,1)) > G^*_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,1)) > \mathfrak{x}G^*_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,1)),$$

which is absurd.

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