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Introduction to Moduli Spaces Associated to Quivers
(with an Appendix by Lieven Le Bruyn and Markus Reineke)

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Abstract. A. King introduced for representation theorists the concept of moduli spaces for representations of quivers. We try to give an as elementary as possible introduction to this material. Our running example is however a problem from systems theory which was studied among others by A. Tannenbaum. Quite a lot can be achieved here elementarily since it is possible to "guess" normal forms. The normal forms we use are different from the classical ones and seem to be new.

This is used in the appendix in order to construct an open subset of an infinite Grassmanian.

It is a classical problem in algebraic geometry to construct moduli spaces for geometric objects with fixed combinatorial invariants like algebraic curves, vector bundles and so on, see the standard reference [**27**]. However, these ideas became available only a decade ago to people working in representation theory: In the situation when a reductive group G acts linearly on a vector space, A. King [**20**] introduced a notion of stability with respect to a character χ of G. The corresponding moduli space is projective over the usual quotient. King moreover spelled out the meaning of this idea in the context of representations of quivers, see also [**34**],[**17**] for further discussion. This concept proved to be quite fruitful and provided connections with areas of research which involve algebraic or symplectic geometry. Just to mention some examples:

- The study of moduli for thin representations provides a connection with toric geometry, [**1**],[**16**].
- Klyachko's solution of Horn's problem [**21**] admits a quite easy interpretation in terms of stability for quiver representations [**9**], [**10**].
- Using classical ideas from the theory of moduli spaces it is possible to calculate Betti numbers for moduli of representations [**33**], compare also [**18**]. On the other hand the rationality of these moduli spaces is reduced to the fundamental problem of simultaneous conjugacy of tuples of square matrices [**36**].

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- Finally we should mention in this context Marsden-Weinstein reductions for representations of quivers [**7**],[**8**]. We specially recommend the introduction of [**7**] where Crawley-Boevey introduces this construction and outlines the connection with similar constructions as Kronheimer's resolution of Kleinian singularities and their deformations [**24**], and Nakajima's quiver varieties [**30**],[**31**].

The above references are far from complete, they should rather be considered as starting points for further reading.

Our aim here is to give an as elementary as possible introduction to this topic by discussing a particular example. We choose a situation, perhaps surprisingly, from systems theory: The state space realization of a time invariant linear system with m inputs $\mathbf{u} \in \mathrm{k}^m$ and p outputs $\mathbf{y} \in \mathrm{k}^p$ with internal states $\mathbf{x} \in \mathrm{k}^n$ is a linear system of differential equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{0.1}$$

$$\mathbf{y} = C\mathbf{x} \tag{0.2}$$

where A, B, C are matrices of the appropriate size. This system is *observable* if the rank of the block matrix $(C, CA, \dots, CA^{n-1})$ is n, which implies that the internal states $\mathbf{x}$ can be recovered from knowledge of $\mathbf{y}(t)$ and $\mathbf{u}(t)$. It is *controllable* (or *completely reachable* in some references) if the rank of the block matrix $(B, AB, \dots, A^{n-1}B)$ is n, which means that the system may be driven to any fixed internal state. The above realization is *minimal* if and only if it is both observable and controllable and in this case n is the *McMillan degree* of the system. Clearly, changing coordinates of the internal variables does not change our system. This apparently simple setup is connected to several interesting geometrical problems: The question about dynamical compensators which stabilize the system leads to an alternative approach to quantum Schubert calculus. Moreover, the space of systems with a given McMillan degree has interesting properties, see [**39**] for an excellent exposition and references for these ideas. If we look for simplicity only at the "input part" of our system just have the affine space $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ with the action of Gl_n given by $g\dot{(}A, B) = (gAg^{-1}, gB)$. This has been studied with different methods by system theorists since quite a while, see for example [**14**],[**40**]. It turns out that the controllable systems are precisely the stable points in the sense of King, and there exists even a fine moduli space. We want to work out in these notes this last aspect, and compare the result with the information which can be obtained in this special situation by elementary methods. In fact, quite a lot can be done here without GIT due to the fact that one can "guess" local normal forms for controllable systems. Finally, by combining this with GIT we find an explicit system of generators for the algebra of relative invariants for our action of Gl_n on $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$. Let me point out that most of the material we discuss here on controllable systems can be found in some form in [**40**]; we think however that with King's notion of stability the situation becomes particularly transparent, our normal forms are different, and we include relative invariants. It is amusing to note that this problem from systems theory is precisely the situation of Nakajima's quiver varieties [**30**] for the case $\tilde{\mathsf{A}}_0$. I thank L. Hille for several helpful discussions on moduli spaces, and in particular for pointing out the connection between the above problem from control theory with framed moduli.

1. Preliminaries

Let k be an algebraically closed field of arbitrary characteristic. We will use the name "variety" for abstract varieties, though in most cases we can think just of quasi projective varieties.

In the following discussion on quotients and reductive groups I took great advantage from the exposition of K. Bongartz in [**6**]. The exposition on families and moduli spaces is essentially a compressed version of [**32**, 1]. We do not discuss the étale topology, see for example [**38**, §1.] for a introduction.

1.1. Quotients. Let an algebraic group G act on a variety X, i.e. the action is given by a morphism $G \times X \longrightarrow X, (g, x) \mapsto g\,.\,x$. And let $\varphi\colon X \longrightarrow M$ be a G-invariant morphism.

1.1.1. Definition. (φ, M) is called a *categorical quotient*, if it is universal in the sense, that each other G-invariant morphism $\varphi'\colon X \longrightarrow M'$ factors over φ. Write in this case also $X//G := M$.

A categorical quotient (φ, M) is called *orbit space* if the fibers of φ are precisely the G-orbits on X.

(φ, M) is called a *geometric quotient*, if φ is open, its fibers are the orbits, and for any open subset $U \subset Y$ the restriction of φ to $\varphi^{-1}(U) \longrightarrow U$ induces an isomorphism between the algebra of G-invariant functions $\mathrm{k}[\varphi^{-1}(U)]^G$ and $\mathrm{k}[U]$.

1.1.2. Remark. (1) A geometric quotient is easily seen to be a categorical quotient [**27**, Proposition 0.1], thus both are unique, if they exist.

(2) Suppose, that a G-invariant morphism $\varphi\colon X \longrightarrow M$ admits a section σ, that meets each G-orbit on M, then (φ, M) is an orbit space. In fact, the hypothesis implies, that the fibers of φ are precisely the orbits, thus $\varphi' = \varphi'\sigma\varphi$ for each G-invariant morphism φ'.

(3) If an orbit space for the action of G on X exists, this action is necessarily *separated*, i.e. the image Γ of the morphism

$$\Psi\colon G \times X \longrightarrow X \times X, (g, x) \mapsto (g\,.\,x, x)$$

is closed in $X \times X$. If moreover Ψ induces an isomorphism of $G \times X$ with Γ, the action is called *free*.

1.2. Reductive Groups. A linear algebraic group is called *reductive* if its radical (the unique maximal connected normal solvable subgroup) is isomorphic to a direct product of copies of k^*. Important examples are the groups $\mathrm{Gl}_n, \mathrm{PGl}_n$ and Sl_n.

We have the following fundamental result, which contains the work of many mathematicians, among them W.J. Haboush, D. Hilbert, D. Mumford, M. Nagata. For a proof (except the result of Haboush [**12**]) and remarks on the history of the theorem consult the textbook [**32**] and the standard reference [**27**, 1§2]. Note moreover, that the result is much easier to prove if $\mathrm{char}\,\mathrm{k} = 0$.

1.2.1. Theorem. *Let X be an affine variety ($X = \mathrm{Spec}(R)$, where $R := \mathrm{k}[X]$ is the ring of regular functions on X) with the action of a reductive group G, and let $R^G \subseteq R$ be the ring of invariant functions on X. Then R^G is a finitely generated* k*-algebra, and the invariant morphism $\varphi\colon X \longrightarrow \mathrm{Spec}(R^G) = X//G$ is a categorical quotient. Moreover, φ sends disjoint G-invariant closed sets to disjoint closed sets in* $\mathrm{Spec}(R^G)$.

There exists an (possibly empty) open subset $U \subseteq X//G$, such that $\varphi^{-1}(U)$ consists of the orbits of maximal dimension which are also closed. The corresponding restriction of φ is a geometric quotient.

1.2.2. Remarks. (1) In the situation of the theorem the (k-rational) points of $\mathrm{Spec}(R^G)$ correspond bijectively to the closed G-orbits on X.

(2) The action of a non-reductive group G even on an affine variety X provides a much less transparent situation: R^G might not be finitely generated as some famous examples by Nagata [**28**] show. Moreover $\mathrm{Spec}(R^G)$ might be different from the categorical quotient $X//G$ (if it exists at all).

For example take the action of $G = B_n$ the subgroup of lower triangular matrices in $X = \mathrm{Gl}_n$ (an affine variety), then the quotient $\mathrm{Gl}_n //B$ exists and can be identified with a complete flag variety (a projective variety). On the other hand $\mathrm{k}[\mathrm{Gl}_n]^{B_n}$ is trivial.

(3) If X is not affine, even for a free action of G reductive on X, a quotient needs not to exist, see [**6**, 6.3] for an elementary example.

If X can be covered by open affine and G-invariant pieces, one can try to glue the local quotients together. This works nicely for the action of the multiplicaticative group k^* on $\mathrm{k}^n \setminus \{0\}$ by coordinate wise multiplication, the result is $\mathrm{Proj}_{\mathrm{k}}^{n-1}$. However, in general the result might be non separated (a very "pre" prescheme) [**27**, p.38]. For example take $X = \mathrm{k} \times \mathrm{k} \setminus \{(0,0)\}$ with the action of k^* by $\lambda . (a,b) := (\lambda a, \lambda^{-1} b)$. The result of the gluing is the affine line with the origin doubled.

1.3. Families. In the situation of 1.1 we can define a family F of G-orbits over a variety S by the following data: An open covering $(U_i)_{i\in I}$ of S and a "compatible" collection of morphisms $\varphi_i \colon U_i \longrightarrow X$, where the "compatibility" is given by morphisms $\gamma_{i,j} \colon U_i \cap U_j \longrightarrow G$, such that $\varphi_i(x) = \gamma_{i,j}(x) . \varphi_j(x)$ for all $x \in U_i \cap U_j$. Moreover the $\gamma_{i,j}$ should fulfill the usual cocycle conditions:

$$\begin{aligned} \gamma_{i,i}(x) &= 1_G \qquad \text{for all } x \in U_i \\ \gamma_{i,j}(x) \cdot \gamma_{j,k}(x) &= \gamma_{i,k}(x) \text{ for all } x \in U_i \cap U_j \cap U_k \end{aligned}$$

Note, that each family F on S defines a map (of sets)

$$\nu_{\mathsf{F}} \colon S \longrightarrow \{G\text{-orbits on } X\}$$

We have several natural choices (which we call (a), (b) and (c) below) to define an equivalence relation on the families over a given variety S. Two families F and F' on S are *equivalent*, if

(a) $\nu_F = \nu_{F'}$.
(b) for each pair $(i,i') \in I \times I'$ there exists a morphism $\delta_{i,i'} \colon U_i \cap U'_{i'} \longrightarrow G$, such that $\varphi_i(x) = \delta_{i,i'}(x) . \varphi'_{i'}(x)$ for $x \in U_i \cap U_{i'}$.
(c) for each pair $(i,i') \in I \times I'$ there exists a morphism $\delta_{i,i'} \colon U_i \cap U'_{i'} \longrightarrow G$, such that $\varphi_i(x) = \delta_{i,i'}(x) . \varphi'_{i'}(x)$ for $x \in U_i \cap U_{i'}$, *and* $\delta_{i,i'}(x)\gamma'_{i',j'}(x) = \gamma_{i,j}(x)\delta_{j,j'}(x)$ for $x \in U_i \cap U_j \cap U_{i'} \cap U_{j'}$.

We have obviously $[\mathsf{F}]_c \subseteq [\mathsf{F}]_b \subseteq [\mathsf{F}]_a$ for the respective equivalence classes of a family F.

This setup fulfills the usual requirements for a moduli problem:

- The equivalence classes of families on the trivial variety $\{\mathrm{pt}\}$ are identified with the G-orbits on X.

- We have an obvious pull back construction: If $\psi\colon T \longrightarrow S$ is a morphism of varieties, the family $\psi^* F$ on T is given by the data: $(\psi^{-1}(U_i))_{i\in I}$ and $\psi \circ \varphi_i\colon \psi^{-1}(U_i) \longrightarrow X$ etc., which is compatible with each of our equivalence relations.

Moreover, the family U on X given by $\mathbb{1}\colon X \longrightarrow X$ has the local universal property: For $x \in S$ we can find $i \in I$ such that $x \in U_i$, and $\mathsf{F} \,|_{U_i} = \varphi_i^* \mathsf{U}$ trivially. We call such a family *tautological*.

Thus, we obtain a contravariant functor $\mathcal{F}_i\colon$ varieties $\longrightarrow$ Sets, $i \in \{a, b, c\}$, which assigns to each variety S the set of equivalence classes of families on S. For $[\mathsf{F}] \in \mathcal{F}(S)$, we get a map $\bar{\nu}_{\mathsf{F}}\colon S \longrightarrow \mathcal{F}(\{\mathrm{pt}\}), s \mapsto [\mathsf{F}_s]$, where F_s is the restriction of F to the point $s \in S$.

1.3.1. Example. (1) If we take $X = \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ as in the introduction, then a family of systems on a variety S can be described as a triple (F, α, β), where

- F is a rank n vector bundle on S,
- α is a (vector bundle) endomorphism of F
- $\beta\colon S \times \mathrm{k}^m \longrightarrow F$ is a homomorphism of vector bundles.

Naturally, we will consider two families (F, α, β) and (F', α', β') *equivalent*, if there is an isomorphism $\delta\colon F \longrightarrow F'$ of vector bundles such that $\delta\alpha = \alpha'\delta$ and $\beta' = \delta\beta$. This is exactly the relation of type (c), discussed above.

(2) We can consider alternatively on X the action of $G = \mathrm{Gl}_n \times \mathrm{Gl}_m$ by $(g_1, g_2)\dot{(}A, B) = (g_1 A g_1^{-1}, g_1 B g_2^{-1})$, thus orbits correspond now to isoclasses of representations of a quiver. The new group action changes the compatibility conditions for a family. Now a family is given by a pair of (possibly non-trivial) vector bundles F_1, F_2 of rank n and m respectively, together with homomorphisms of vector bundles $\alpha\colon F_1 \longrightarrow F_1$ and $\beta\colon F_2 \longrightarrow F_1$.

We have an obvious restriction to families of controllable systems, and we will see in 2.3, that for families of controllable systems in fact the three above defined equivalence relations coincide. For general families of systems however, this is not true, compare [**32**, 2.4].

1.4. Definition. A *fine moduli space* for $\mathcal{F}$, consists of a variety M, and a natural transformation $\Phi\colon \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, M)$, which represent the functor $\mathcal{F}$.

A *coarse moduli space* for $\mathcal{F}$ consists of a variety M, and a natural transformation $\Phi\colon \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, M)$, such that

(i) $\Phi_{\{\mathrm{pt}\}}$ is bijective
(ii) Φ is universal, i.e. if $\Psi\colon \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, N)$ is an other natural transformation, then there exists a *unique* natural transformation $\Omega\colon \mathrm{Hom}(-, M) \longrightarrow \mathrm{Hom}(-, N)$ such that $\Psi = \Omega \circ \Phi$

1.4.1. Remarks. (1) A fine moduli space is also coarse, and both are unique up to isomorphism, if they exist. The existence of a coarse moduli space does *not* depend on the choice of the equivalence relation on families (as long as it fulfills the usual requirements).

(2) A coarse moduli space N is fine, if

(i) it admits a *universal family* $\tilde{\mathsf{U}}$, i.e. $\nu_{\tilde{\mathsf{U}}}\colon N \longrightarrow \mathcal{F}(\{\mathrm{pt}\})$ is bijective,
(ii) For families F, F' on a variety S, we have $\nu_{\mathsf{F}} = \nu_{\mathsf{F}'} \Longrightarrow [\mathsf{F}] = [\mathsf{F}']$.

Obviously, $\tilde{\mathsf{U}}$ should correspond to $\mathbb{1}_N$. Note, that condition (ii) is on the equivalence relation, and not on M; it is obviously necessary for Φ_S be injective.

In our situation the equivalence relation (a) fulfills by construction condition (ii), but this might not always be the most natural choice.

(3) In our situation (where a semi universal family on X exists), it follows basically from abstract nonsense, that a coarse moduli space "is the same" as a orbit space for the action of G on X.

1.5. Stability. We present a version of classical GIT, adapted by A. King [**20**] to the linear action of a reductive group on an *affine* space.

Let $G \times \mathbb{V} \longrightarrow \mathbb{V}, (g, v) \mapsto g\,.\,v$ be a linear representation of G, and denote by K its kernel. Moreover, let $\chi\colon G \longrightarrow \mathrm{k}^*$ be a character of G. We write

$$\mathrm{k}[\mathbb{V}]^{G,\chi} := \{f \in \mathrm{k}[\mathbb{V}] \mid f(g\,.\,v) = \chi(g)f(x)\ \forall g \in G, v \in \mathbb{V}\},$$

the space of relatively invariant functions of weight χ.

1.5.1. Definition. (i) A point $v \in \mathbb{V}$ is χ-*semistable* if there exists a relative invariant $f \in \mathrm{k}[\mathbb{V}]^{G,\chi^n}$ with $n \geq 1$ such that $f(x) \neq 0$. Write $\mathbb{V}^{\mathrm{ss}}$ for the open set of semistable points in $\mathbb{V}$.

(ii) A point $v \in \mathbb{V}$ is χ-*stable*, if there exists a relative invariant $f \in \mathrm{k}[\mathbb{V}]^{G,\chi^n}$ with $n \geq 1$, such that $f(x) \neq 0$, and, $\dim G\,.\,x = \dim G/K$ and the action on the affine variety $\mathbb{V}_f$ is closed. Write $\mathbb{V}^{\mathrm{s}}$ for the open set of stable points.

By a standard construction (compare for example [**32**, 3.14]) together with the fundamental properties of reductive groups 1.2, we obtain an algebraic quotient

$$\phi\colon \mathbb{V}^{\mathrm{ss}} \longrightarrow \mathbb{V}/\!/(G,\chi) := \mathrm{Proj}(\bigoplus_{n\geq 0} \mathrm{k}[\mathbb{V}]^{G,\chi^n})$$

This contains an open set $S \subset \mathbb{V}/\!/(G,\chi)$ with $\phi^{-1}(S) = \mathbb{V}^{\mathrm{s}}$, and the restriction to this is a geometric quotient. Note moreover, that $\mathbb{V}/\!/(G,\chi)$ is by construction projective over the usual algebraic quotient $\mathbb{V}/\!/G = \mathrm{Spec}\,\mathrm{k}[\mathbb{V}]^G$.

Usually, it is difficult, to identify stable and semistable points. However, we have the following version of Mumford's "numerical criterion".

In this context, we need the definition of a *one parameter subgroup* (1-PSG) of G, i.e. an morphism of algebraic groups $\lambda\colon \mathrm{k}^* \longrightarrow G$. Given such a group, we obtain for any $v \in \mathbb{V}$ a morphism $\lambda_v\colon \mathrm{k}^* \longrightarrow \mathbb{V}, t \mapsto \lambda(t)\,.\,v$. If λ_v can be extended to a morphism $\mathrm{k} \longrightarrow \mathbb{V}$, then this is unique, and we write $\lim_{t\to 0} \lambda(t)\,.\,v$ for the corresponding element of $\mathbb{V}$.

Note moreover, that the composition $\chi \circ \lambda$ is an automorphism of the multiplicative group k^*, i.e. it is necessarily of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$; we express this situation by writing $\langle \chi, \lambda \rangle := n$.

1.5.2. Proposition. [**20**, 2.5] *A point $v \in \mathbb{V}$ is χ-semistable if and only if $\chi(K) = \{1\}$ and every 1-PSG λ, for which*

$\lim_{t\to 0} \lambda(t)\,.\,v$ *exists, satisfies* $\langle \chi, \lambda \rangle \geq 0$.

Such a point is χ-stable if and only if the only 1-PSG subgroups λ of G, for which $\lim_{t\to 0} \lambda(t)\,.\,v$ *exists, and $\langle \chi, \lambda \rangle = 0$, are in K.*

Roughly speaking, the proof is based on the following construction: Consider the "lift" of the G-action on $\mathbb{V} \times \mathrm{k}$ by $g\,.\,(v,t) := (g\,.\,v, \chi(g)^{-1}t)$; now, by the result 1.2 on reductive groups, semistability of x is equivalent to saying that the closure of $G\,.\,(x,1)$ contains no point of $\mathbb{V} \times \{0\}$. This can be translated back into the original action by the 'fundamental theorem' [**19**, 1.4], that any closed

G-invariant set, that meets the closure of an orbit contains a point in the closure of some 1-PSG orbit.

2. Basic Results

Define for $(A,B) \in \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m} =: \mathbb{A}^{(n,m)}$ the block matrices $\tilde{R}(A,B) := (B, AB, A^2B, \cdots A^nB) \in \mathrm{k}^{n\times(n+1)m}$ and $R(A,B) := (B, AB, \ldots, A^{n-1}B) \in \mathrm{k}^{n\times nm}$. We will write for example $\tilde{R}(A,B)_i$ for the i-th column vector of $\tilde{R}(A,B)$, and $\tilde{R}(A,B)^j$ for the j-th row vector, and $\tilde{R}(A,B)_{i_1,\ldots,i_n}$ will be the submatrix of $\tilde{R}(A,B)$ formed from the columns $i_1,\ldots,i_n$ in that order. Finally, we denote by $\mathbf{E} \in \mathrm{Gl}_n$ the unit matrix, and consequently $\mathbf{E}_i$ is the i-th unit vector of k^n.

2.1. Remark. (1) If we consider $\mathrm{k}^{n\times(n+1)m}$ as a Gl_n-variety with the natural action by left multiplication of matrices, then $\tilde{R} :: \mathbb{A}^{(n,m)} \longrightarrow \mathrm{k}^{n\times(n+1)m}$ is a Gl_n morphism of varieties. Recall from the introduction that the action of Gl_n on the space of systems $\mathbb{A}^{(n,m)}$ is given by $g\,.\,(A,B) := (gAg^{-1}, gB)$.

(2) If $j_1, j_j, \ldots j_n$ is a sequence in $\{1,\ldots,(n+1)m\}$, then we define

$$d_{j_i,\ldots,j_n}(A,B) := \det(\tilde{R}(g\,.\,(A,B))_{j_1,\ldots,j_n})$$

and clearly $\det(g)d_{j_i,\ldots,j_n}(A,B) = d_{j_i,\ldots,j_n}(g\,.\,(A,B))$, i.e. we have found some relative invariants. We will see later, that we get in this way a system of generators for the algebra of relative invariants.

(3) Let $(A,B) \in \mathbb{A}^{(n,m)}$ then (A,B) is by definition controllable if $\operatorname{rank} R(A,B) = n$, by Cayley-Hamilton this is equivalent to $\operatorname{rank}\tilde{R}(A,B) = n$ or to the fact that the $\mathrm{k}[x]$-module defined by A is generated by the columns of B.

2.2. Corollary. *If $(A,B) \in \mathbb{A}^{(n,m)}$ is controllable, then $\mathrm{Stab}_{\mathrm{Gl}_n}(A,B)$ is (even scheme-theoretically) trivial. In particular, the orbit map $\mathrm{Gl}_n \longrightarrow \mathrm{Gl}_n\,.\,(A,B), g \mapsto g\,.\,(A,B)$ is separated.*

Proof: Note first that for arbitrary $(A,B) \in \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ the stabilizer is defined by *linear* equations, thus it is (scheme-theoretically) reduced. As a consequence the differential of the orbit map is surjective. Now, let $g\,.\,(A,B) = (A,B)$, for (A,B) controllable, thus by the above remark $R(A,B) = R(g\,.\,(A,B)) = gR(A,B)$. By the lemma $R(A,B)$ has rank n, implying $g = \mathbb{1}$. □

2.3. Remark. If (F,α,β) is a family of controllable systems on S, as discussed in the example of 1.3, then by the above lemma

$$(\beta, \alpha\beta, \ldots, \alpha^{n-1}\beta)\colon S \times \mathrm{k}^{nm} \longrightarrow F)$$

is an epimorphism of vector bundles. It follows, that the isomorphism class of F is already determined by the morphisms $\gamma_i\colon U_i \longrightarrow \mathrm{k}^{n\times n}\times\mathrm{k}^{n\times m}$; this implies that the equivalence relations (b) and (c), defined in 1.3, coincide. Since we will see later, that the open subset of $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ of controllable systems is a Gl_n-principal bundle in the Zariski topology, even the equivalence relations (a) and (b) coincide.

2.4. Proposition. *(a) The set $\mathbb{V}^{(n,m)}$ of controllable systems $(A,B) \in \mathbb{A}^{(n,m)}$ is open in the Zariski topology and Gl_n-invariant.*

(b) The Gl_n-equivariant restriction to the open subsets

$$\tilde{R}\colon \mathbb{V}^{(n,m)} \longrightarrow \mathrm{Mat}(n,(n+1)m)_{reg} := \{M \in \mathrm{k}^{n\times(n+1)m} \mid \operatorname{rank} M = n\}$$

is injective.

(c) The action of Gl_n *on* $\mathbb{V}^{(n,m)}$ *is separated.*

Proof: (a) is clear. For (b) suppose $\tilde{R}(A,B) = \tilde{R}(A',B')$, then already $B = B'$ and $R(A,B) = R(A',B')$ has rank n. Now $\tilde{R}(A,B) = (B, AR(A,B))$ implies $A'R(A,B) = AR(A,B)$ and consequently $A = A'$.

c) It is well-known that the action of Gl_n on $\mathrm{Mat}(n,(n+1)m)_{\mathrm{reg}}$ admits as geometric quotient the Grassmanninan $\gamma\colon \mathrm{Mat}(n,(n+1)m)_{\mathrm{reg}} \longrightarrow \mathrm{Gr}_n^{(n+1)m}$, thus $\psi := \gamma\tilde{R}$ is a Gl_n invariant morphism, with fibers the Gl_n-orbits on $\mathbb{V}^{(n,m)}$. Thus $\Gamma := \{(g\,.\,v, v) \in \mathbb{V}^{(n,m)} \times \mathbb{V}^{(n,m)} \mid g \in \mathrm{Gl}_n, v \in \mathbb{V}^{(n,m)}\}$ coincides with $(\psi\times\psi)^{-1}(\Delta)$, where Δ is the diagonal in $\mathrm{Gr}_n^{(n+1)m}$, thus it is closed. □

2.4.1. Remark. If the differential of $\tilde{R}\colon \mathbb{V}^{(n,m)} \longrightarrow \mathrm{Mat}(n,(n+1)m)_{\mathrm{reg}}$ were injective we could conclude by the étale version of the implicit function theorem that the image of $\tilde{R}$ is locally closed. Unfortunately, this differential is quite complicated, due to the powers of A appearing in the definition of $\tilde{R}$.

3. Elementary Methods

The aim of this section is to show how far one can proceed on the construction of a fine moduli space without GIT if it is possible to “guess” nice normal forms, as it is the case in our example. Consider $X_{n,m} := \{1,\dots,m\}\times\{1,\dots,n\}$ with the usual lexicographical order. We say, that a sequence $I = ((i_1(1), i_2(1)), \dots, (i_1(n), i_2(n)))$ in $X_{n,m}$ is *nice* if it is strictly monotonous ascending, and if $i_2(j) > 1$, then $(i_1(j-1), i_2(j-1)) = (i_1(j), i_2(j)-1)$.

Given a nice sequence I, we find two sequences $j_I(1),\dots,j_I(k)$ and $p_I(1),\dots,p_I(k)$ such that

$$I = ((j_I(1),1),\dots,(j_I(1),p_I(1)),(j_I(2),1),\dots(j_I(k),p_I(k))),$$

and we define

$$h_I(t) := \sum_{s=1}^{t} p_I(s);\quad h_I(0) := 0$$

For $\mathbf{i} := (i_1,i_2) \in X_{n,m}$, we define $R(A,B)_{\mathbf{i}} := R(A,B)_{i_1+m(i_2-1)}$. Finally, for a sequence $I = \mathbf{i}(1),\dots,\mathbf{i}(n) \in X_{n,m}$ we write $d_I(A,B) := d_{i_1(1)+m(i_2(1)-1),\dots,i_1(n)+m(i_2(n)-1)}(A,B)$, as defined in 2.1 (2), thus the principal open sets $(\mathrm{k}^{n\times n}\times\mathrm{k}^{n\times m})_{d_I} := \{(A,B) \in \mathrm{k}^{n\times n}\times\mathrm{k}^{n\times m} \mid d_I(A,B) \neq 0\}$ are Gl_n-invariant.

3.1. Lemma (Kalman). $(A,B) \in \mathbb{V}^{(n,m)}$*, if and only if there exists a* <u>*nice*</u> *sequence* I *in* $X_{n,m}$*, such that* $d_I(A,B) \neq 0$*. Thus* $\mathbb{V}^{(n,m)}$ *admits an open covering by* Gl_n*-invariant affine sets*

$$\mathbb{V}^{(n,m)} = \bigcup_{I\subset X_{n,m}\,nice} (\mathrm{k}^{n\times n}\times\mathrm{k}^{n\times m})_{d_I}$$

3.2. Lemma. *(a) For* (A,B) *in the open set* $(\mathrm{k}^{n\times n}\times\mathrm{k}^{n\times m})_{d_I} = \mathbb{V}^{(n,m)}_{d_I}$ *with* I *nice, there exists a unique* $g \in \mathrm{Gl}_n(\mathrm{k})$*, such that* $g\,.\,(A,B) := (A',B')$ *is of the form:*

$$\begin{aligned} A'_i &= \mathbf{E}_{i+1} && \text{if } i \notin \{h_I(1),h_I(2),\dots,h_I(k)\},\\ B'_{j_I(i)} &= \mathbf{E}_{h_I(i-1)+1} && \text{for } i = 1,\dots,k \end{aligned}$$

In particular, $\mathbb{V}_{d_I}^{(n,m)}$ is a trivial $\mathrm{k}^{n\times m}$ bundle.

(b) The non-constant components of (A', B') can be expressed in terms of invariant functions:

$$\begin{aligned}{A'}^i_{h(t)} &= \det R(A,B)_{\mathbf{i}(1),\dots,\mathbf{i}(t-1),(j_I(t),p_I(t)),\mathbf{i}(t+1),\dots,\mathbf{i}(n)}/d_I(A,B)\\ {B'}^i_t &= \det R(A,B)_{\mathbf{i}(1),\dots,\mathbf{i}(t-1),(t,0),\mathbf{i}(t+1),\dots,\mathbf{i}(n)}/d_I(A,B)\end{aligned}$$

Proof: (a) Note first, that since I is nice, $(A', B') \in \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ is of the form described in the Lemma if and only if $R(A', B') = \mathbf{E}$. In particular, all these matrices lie in $\mathbb{V}_{d_I}^{(n,m)}$.

For $(A, B) \in \mathbb{V}_{d_I}^{(n,m)}$ we have by definition $g(A, B) := \tilde{R}(A, B)_I \in \mathrm{Gl}_n(\mathrm{k})$, thus uniquely $g(A, B)^{-1}\tilde{R}(A, B)_I = \mathbf{E}$. Since in general $\tilde{R}(g\,.\,(A, B)) = g\tilde{R}(A, B)$, we conclude, that $g(A, B)^{-1}\,.\,(A, B)$ is of the desired form.

Next, note, that the space of matrices in that form, ${\mathbb{V}'}_{d_I}^{(n,m)}$, is naturally isomorphic to $\mathrm{k}^{n\times m}$, and that

$$\phi\colon \mathbb{V}_{d_I}^{(n,m)} \to \mathrm{Gl}_n \times {\mathbb{V}'}_{d_I}^{(n,m)}, (A,B) \mapsto (g(A,B)^{-1}, g(A,B)^{-1}\,.\,(A,B))$$

is a Gl_n-morphism of varieties with an obvious inverse.

(b) The functions in question, being quotients of relative invariants of weight det are clearly Gl_n-invariant on $\mathbb{V}_{d_I}^{(n,m)}$, thus we have to verify the equalities only on matrices in ${\mathbb{V}'}_{d_I}^{(n,m)}$, where the result is immediate from the fact that $\tilde{R}(A', B')_{(j_I(t),p_I(t))} = A'_{h_I(t)}$ for $t = 1, \dots, k$. □

3.3. Theorem. *There exists a geometric quotient $\varphi\colon \mathbb{V}^{(n,m)} \longrightarrow Y$ with an open covering $Y = \cup_{I nice} Y_I$ such that $Y_I \cong \mathrm{k}^{n\times m}$ and $\phi^{-1}(Y_I) = \mathbb{V}_I^{(n,m)}$. In particular, ϕ admits local sections and Y is a smooth rational variety.*

Proof: Clearly, we can glue together the quotients for the $\mathbb{V}_I^{(n,m)}$ to obtain a *pre*-variety Y with the desired properties [**13**, II, exercise 2.12]. However, we have to make sure that this is a variety indeed [**27**], i.e. that the diagonal Δ is closed. This follows easily from the fact, that the action of Gl_n on $\mathbb{V}^{(n,m)}$ is closed 2.4, and the fact, that each of the local quotients admits a section [**6**, Lemma 5.5]. □

3.4. Corollary. *Y is a fine moduli space for (algebraic) families of completely observable systems.*

Proof: By construction, the tautological family on $\mathbb{V}^{(n,m)}$ as the local semi-universal property for families of completely observable systems. Thus, the geometric quotient Y is a coarse moduli space for this problem. Since $\varphi\colon \mathbb{V}^{(n,m)} \longrightarrow Y$ is a Gl_n-principal bundle in the Zariski topology (see construction in Theorem 3.3 and Lemma 3.2), we obtain from the local sections the required (1.4, remark (2)) universal family on Y. Finally, we need not worry about the equivalence relation by 2.3. □

4. Geometric Invariant Theory

Let $\mathrm{Gl}_n \longrightarrow \mathrm{k}^*, g \mapsto \det(g)$ be the natural character of Gl_n. We want to interpret the controllable tuples $(A, B) \in \mathbb{A}^{(n,m)}$ as det-stable points, using a version of classical GIT, adapted by A. King [**20**] to the linear action of a reductive group G on a *affine* space $\mathbb{V}$, see 1.5.

4.1. Proposition. *Let* $\det\colon \mathrm{Gl} \longrightarrow \mathrm{k}^*$ *be the "standard" character of* Gl_n. *With the action* $\mathrm{Gl}_n \times \mathbb{A}^{(n,m)} \longrightarrow \mathbb{A}^{(n,m)}, (g,(A,B)) \mapsto (gAg^{-1}, gB)$ *a point* $(A,B) \in \mathbb{A}^{(n,m)}$ *is controllable if and only if it is* $\det$*-semistable if and only if it is* $\det$*-stable.*

Proof: Recall, that (A,B) controllable implies that $\det R(A,B)_I \neq 0$ for some nice selection I of columns of $R(A,B)$ and that the stabilizer of (A,B) is trivial 2.2. Thus, (A,B) is stable (just use the definition), since $d_I\colon \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m} \longrightarrow \mathrm{k}, (A',B') \mapsto \det R(A',B')$ is a relative invariant of weight det.

Thus, it remains to show that if $(A,B)\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ is not controllable, then it is not semistable. In this situation, $r := \operatorname{rank} R(A,B) < n$, and we can suppose (up to the action of Gl_n, that

$$
\begin{aligned}
A^i_j &= 0 \text{ if } i \leq n-r \text{ and } j > n-r\\
B^i_j &= 0 \text{ if } i \leq n-r.
\end{aligned}
$$

Now consider the 1-PSG $\lambda(t) := \operatorname{diag}(t^{-1}, t^{-1}, \ldots, t^{-1}, 1, \ldots, 1)$ of Gl_n ($n-r$-times t^{-1}), then clearly $\lim_{t\to 0} \lambda(t) \,.\, (A,B)$ exists, and $\langle \lambda, \det\rangle = r-n < 0$; thus (A,B) is not det-semistable by the numerical criterion 1.5.2. $\square$

Thus, we obtain with

$$
\mathcal{M}_{n,m} := \operatorname{Proj}(\bigoplus_{i\geq 0} \mathrm{k}[\mathbb{A}^{(n,m)}]^{\mathrm{Gl}_n, \det^i})
$$

a geometric quotient $\phi\colon \mathbb{V}^{(n,m)} \longrightarrow \mathcal{M}_{n,m}$. By remark (3) in 1.4 this is also a coarse moduli space for families of controllable systems. Moreover, we get directly from a Corollary of Luna's 'slice theorem', that this is a Gl_n-principal bundle in the étale topology if $\operatorname{char}\mathrm{k} = 0$ (see for example [**38**, §5, Korollar 1]). Since by 2.2 the orbit map $\mathrm{Gl}_n \longrightarrow \mathbb{V}^{(n,m)}, g \mapsto g\,.\,(A,B)$ is separated for each $(A,B) \in \mathbb{V}^{(n,m)}$, we get the same result also for $\operatorname{char}\mathrm{k} \neq 0$ by the characteristic-p version of the slice theorem [**2**]. Finally, by an old result of Serre [**37**], each Gl_n-principal bundle in the étale topology is also locally trivial in the Zariski topology (Warning: This is not true for PGl-bundles, see for [**6**] for examples). Thus we get without invoking the results from our explicit calculations in section 3:

4.2. Theorem. $\phi\colon \mathbb{V}^{(n,m)} \longrightarrow \mathcal{M}_{n,m}$ *is a* Gl_n *principal bundle in the Zariski topology, thus* $\mathcal{M}_{n,m}$ *is a fine moduli space for controllable systems of type* (n,m). *Moreover,* $\mathcal{M}_{n,m}$ *is for* m *greater or equal to* 2 *a stably rational quasi projective variety which is not affine.*

4.2.1. Remark. For the open, affine Gl_n-invariant subsets $\mathbb{V}^{(n,m)}_{d_I}$ there exists by 1.2 an affine geometric quotient $\psi\colon \mathbb{V}^{(n,m)}_{d_I} \to \mathbb{V}^{(n,m)}_{d_I} /\!/ \mathrm{Gl}_n$, which is by similar arguments as above, a Gl_n-principal bundle even in the Zariski topology. Clearly, $\mathbb{V}^{(n,m)}_{d_I} /\!/ \mathrm{Gl}_n$ is a smooth variety, but we will need the explicit calculations from 3.2, to identify it with $\mathrm{k}^{n\times m}$ in case I is nice; compare with [**40**, p.52].

4.3. Relative invariants. Let $R := \oplus_{i\geq 0}\mathrm{k}[\mathbb{A}^{(n,m)}]^{\mathrm{Gl}_n,\det^i}$ be the graded ring of relative invariants.

$R_0 = \mathrm{k}[s_1,\dots,s_n]$ is a polynomial ring in n-variables, with the functions s_i given by $s_i(A,B) = s_i(A)$ and $\det(t\,.\,E - A) = t^n + s_1(A)t^{n-1} + \cdots + s_n$ the usual invariant functions of $n\times b$-matrices under conjugation. Indeed, R_0 contains clearly this ring. On the the other hand, since Gl_n is reductive and $\mathbb{A}^{(n,m)}$ is affine, $\mathrm{Spec}(R_0)$ is the categorical quotient for the action of Gl_n on $\mathbb{A}^{(n,m)}$, thus $f(A,B) = f(A',B')$ for $f \in R_0$ and (A',B') in the Zariski closure of $\mathrm{Gl}_n\,.\,(A,B)$. In particular $(A,0) = \lim_{t\to 0} tE\,.\,(A,B)$, i.e. we are reduced to the well-known case $m = 0$, in particular $\mathrm{Spec}\,R_0$ is the affine n-space; see for example [**32**].

Next, we claim that R is generated as R_0 algebra by the elements d_I where $d_I(A,B) := \det R(A,B)_I$, with $I = (i_1, i_2, \dots i_n)$ is a strictly monotonous sequence of integers in $\{1,2,\dots,nm\}$. In fact, as we already observed, $d_I(g\,.\,(A,B)) = \det(g)d_I(A,B)$, i.e. $d_I \in R_1$, thus we can consider the graded R_0-subalgebra of R, generated by these d_I. Note first, that by Cayley-Hamilton $\tilde{d}_I \in R'_1$, where $\tilde{d}_I(A,B) := \det \tilde{R}(A,B)_I$ for some $I = (i_1,\dots i_n)$ in $\{1,\dots,(n+1)m\}$. Thus for I nice, $(R'_{d_I})_0$ contains the invariant functions on $\mathbb{A}^{(n,m)}_{d_I}$ constructed in Lemma 3.2 (b). Since we had obtained in this way a geometric quotient for $\mathbb{A}^{(n,m)}_{d_I}$ we conclude

$$\mathrm{k}[\mathbb{A}^{(n,m)}_{d_I}]^{\mathrm{Gl}_n} \subseteq (R'_{d_I})_0 \subseteq (R_{d_I})_0 = \mathrm{k}[\mathbb{A}^{(n,m)}_{d_I}]^{\mathrm{Gl}_n}$$

i.e. we have equality everywhere. Since the $(d_I \mid I \text{ nice}) = R_+$ we conclude $R' = R$.

4.4. The projection $\mathcal{M}_{n,m} \longrightarrow \mathbb{A}^n$. The inclusion $\mathrm{k}[X_1,\dots,X_n] \cong R_0 \subset R$ induces the projection $\pi\colon \mathrm{Proj}(R) = \mathcal{M}_{n,m} \longrightarrow \mathrm{Spec}\,R_0 \cong \mathbb{A}^n$.

4.4.1. Lemma. *The projection π is a flat morphism. Moreover, for general $a = (a_1,\dots,a_n) \in \mathbb{A}^n(\mathrm{k})$, we have $\pi^{-1}(a) \cong (\mathbb{P}^{(m-1)})^n$*

Proof: Since both varieties are smooth, we have to show for the flatness of π only $\dim \pi^{-1}(v) = n(m-1)$ for all $a \in \mathbb{A}^n$, see [**13**, III, Ex. 10.9] or [EGA IV 6.1.5]. Clearly, the dimension of the fiber is always at least $n(m-1)$, since π is projective, thus in particular closed and surjective. Now, we have

$$\pi^{-1}(a_1,\dots,a_n) = \varphi(\{(A,B) \in \mathbb{V}^{(n,m)} \mid A^n + a_1A^{n-1} + \cdots + a_n\mathbf{E} = 0\}),$$

thus we can stratify $\pi^{-1}(a_1,\dots,a_n)$ along the possible Jordan normal forms $A_{(1)},\dots,A_{(t)}$ for the given characteristic polynomial, i.e.

$$\pi^{-1}(a) = \dot{\bigcup_{i=1,\dots,t}} \{B \in \mathrm{k}^{n\times m} \mid \mathrm{rank}\,R(A_{(i)},B) = n\}/\!/\,\mathrm{Stab}_{\mathrm{Gl}_n}(A_{(i)})$$

via the corresponding associated fiber bundles. Now the dimension of each of these stabilizers is at least n, acting on an open subset of $\mathrm{k}^{n\times m}$, and thus leading to a space of dimension at most $n(m-1)$.

Observe, that in the generic case, when the discriminant of $(a_1,\dots,a_n)$ does not vanish, we have only one normal form $A_{(1)} = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with stabilizer $\mathrm{k}^* \times \cdots \times \mathrm{k}^*$ and

$$\{B \in \mathrm{k}^{n\times m} \mid \mathrm{rank}\,R(A_{(1)},B) = n\} = \{B \in \mathrm{k}^{n\times m} \mid B^i \neq 0 \text{ for } i = 1,\dots n\}$$

thus we obtain $\pi^{-1}(a) = (\mathbb{P}^{(m-1)})^n$ generically. $\square$

Let us recall finally, that the flatness of $\pi\colon \operatorname{Proj}(R) \longrightarrow \operatorname{Spec} R_0 \cong \mathbb{A}^n$ implies, that the Hilbert polynomial of $\pi^{(-1)}(a)$ for $a \in \mathbb{A}^n$ is constant for every closed embedding $\operatorname{Proj}(R) \subset \mathbb{P}^s_{\mathbb{A}^n}$, see [**13**, III 9.9].

4.5. Representations of Quivers. In representation theory one would like to classify representations of a quiver $Q = (Q_0, Q_1, s, t)$ for a fixed dimension vector $\mathbf{v} \in \mathbb{N}^{Q_0}$. Here, it is quite obvious to find a locally universal family of such representations:

$$\operatorname{Rep}_Q^{\mathbf{v}} := \prod_{\alpha \in Q_1} \operatorname{Hom}_{\mathrm{k}}(\mathrm{k}^{\mathbf{v}(s\alpha)}, \mathrm{k}^{\mathbf{v}(t\alpha)})$$

with the action of $\mathrm{Gl}_{\mathbf{v}} := \prod_{i \in Q_0} \mathrm{Gl}_{\mathbf{v}(i)}$ by conjugation:

$$g \cdot (f_\alpha)_{\alpha \in Q_1} := (g_{t\alpha} f_\alpha g_{s\alpha}^{-1})_{\alpha \in Q_1}$$

The $\mathrm{Gl}_{\mathbf{v}}$-orbits are precisely the isoclasses of representations of Q of dimension $\mathbf{v}$. In this situation the characters of $\mathrm{Gl}_{\mathbf{v}}$ are of the form

$$\chi_\theta\colon \mathrm{Gl}_{\mathbf{v}} \longrightarrow \mathrm{k}^*, g \mapsto \textstyle\prod_{i \in Q_0} (\det g_i)^{\theta(i)}$$

for $\theta \in \mathbb{Z}^{Q_0}$. Since $\mathrm{Gl}_{\mathbf{v}}$ acts on $\operatorname{Rep}_Q^{\mathbf{v}}$ with kernel $\Delta = \{(\lambda e_i)_{i \in Q_0} \mid \lambda \in \mathrm{k}^*\}$, we can expect χ_θ-semistable points only for $\sum_{i \in Q_0} \theta(i) = 0$, see 1.5. By the work of Schofield [**35**], one finds even stable points if $\mathbf{v}$ is a Schur root (i.e. if there exists a representation of dimension $\mathbf{v}$ with trivial endomorphism ring). The corresponding geometric quotient for χ_θ-stable points

$$\varphi\colon (\operatorname{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \longrightarrow \mathcal{M}(Q, \mathbf{v}, \chi)$$

is by the same argument as in the proof of 4.2 a $\mathrm{PGl}_{\mathbf{v}} := \mathrm{Gl}_{\mathbf{v}} / \Delta$-principal bundle in the étale topology. In order to find local sections for φ, (which would define a universal family on $\mathcal{M}(Q, \mathbf{v}, \chi)$), we should know, that this is locally trivial in the Zariski topology. This is the case if $\mathbf{v}$ is moreover indivisible, i.e. $\gcd(\mathbf{v}(i)_{i \in Q_0}) = 1$, since then the action of $\mathrm{Gl}_{\mathbf{v}}$ on $(\operatorname{Rep}_Q^{\mathbf{v}})^{\mathrm{s}}$ can be lifted to a action on $(\operatorname{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \times \mathrm{k}^*$ with trivial stabilizer, and we obtain now a $\mathrm{Gl}_{\mathbf{v}}$-principal bundle

$$\tilde{\varphi}\colon (\operatorname{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \times \mathrm{k}^* \longrightarrow \mathcal{M}(Q, \mathbf{v}, \chi)$$

by the result of Serre in [**37**]. This is our interpretation of [**20**, 5.3].

The notion of a family of representations of Q on a variety S is just a representation of Q in the category of vector bundles on S, i.e. a family F of representations of dimension vector E on S consists of a family of vector bundles $(F_i)_{i \in Q_0}$ on S with $\operatorname{rank} F_i = E(i)$, and a family of morphism of vector bundles $(f_\alpha\colon F_{s\alpha} \longrightarrow F_{t\alpha})_{\alpha \in Q_1}$. This can be obviously restricted to families of χ-stable representations. Unfortunately, the natural notion of equivalence for such families, induced by isomorphisms of vector bundles, will never admit a fine moduli space for the same reasons as in [**32**, 2.1]. Thus, we have to stick to a equivalence relation of type (b) as in 1.3, in order to obtain a fine moduli space in case of a indivisible Schur root $\mathbf{v}$.

4.6. Problems. (1) Are there, besides the controllable system problem and the linear quiver discussed in [**29**], other quivers, where one can “guess” normal forms for the framed moduli? One will need a version of the Kalman Lemma 3.1.

(2) Understand the special fibers of the projection π discussed above 4.4. The normal forms from 3.2 suggest, that the singularities appearing there, should be similar to the singularities found in [**23**].

(3) Understand the type of varieties, which can appear as fine moduli spaces for stable representations of dimension $\mathbf{v}$, where $\mathbf{v}$ is a indivisible Schur root. From [**36**] it is known, that these spaces are always rational, and in the case $\mathbf{v} = (1, 1, \ldots, 1)$ it is known, that the moduli are toric varieties [**15**].

5. Appendix by L. Le Bruyn and M. Reineke

Non-commutative geometry, as outlined by M. Kontsevich in [**22**], offers a possibility to glue together closely related moduli spaces into an infinite dimensional variety controlled by a non-commutative algebra. The individual moduli spaces are then recovered as moduli spaces of simple representations (of specific dimension vectors) of the non-commutative algebra. An illustrative example is contained in the recent work by G. Wilson and Yu. Berest [**41**] [**3**] relating Calogero-Moser spaces to the adelic Grassmannian (see also [**5**] and [**11**] for the connection with non-commutative geometry). The main aim of this appendix is to offer another (and more elementary) example: moduli spaces of canonical systems can be glued to a specific open subset of an infinite Grassmannian.

5.1. The setting. As before a linear control system Σ of type $(m, n, p) \in \mathbb{N}^3$ is determined by the system of linear differential equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{5.1}$$

$$\mathbf{y} = C\mathbf{x} \tag{5.2}$$

that is, Σ is described by a triple of matrices $\Sigma = (A, B, C) \in M_n(\mathrm{k}) \times M_{n\times m}(\mathrm{k}) \times M_{p\times n}(\mathrm{k}) = V_{m,n,p}$ and is said to be equivalent to a system $\Sigma' = (A', B', C') \in V_{m,n,p}$ if and only if there is a basechange matrix $g \in GL_n$ such that

$$\Sigma \sim \Sigma' \quad \Leftrightarrow \quad A' = gAg^{-1}, \quad B' = gB \quad \text{and} \quad C' = Cg^{-1}.$$

The controllable (resp. observable) systems define GL_n-open subsets $V^{cc}_{m,n,p}$, resp. $V^{co}_{m,n,p}$, consisting of systems with trivial GL_n-stabilizer, whence we have corresponding orbit spaces

$$\mathtt{sys}^{cc}_{m,n,p} = V^{cc}_{m,n,p}/GL_n \qquad \text{and} \qquad \mathtt{sys}^{co}_{m,n,p} = V^{co}_{m,n,p}/GL_n,$$

which are known to be smooth quasi-projective varieties of dimension $(m+p)n$, see for example [**40**, Part IV]. A system $\Sigma = (A, B, C) \in V_{m,n,p}$ is said to be *canonical* if it is both completely controllable and completely observable. The corresponding moduli space

$$\mathtt{sys}^{c}_{m,n,p} = (V^{cc}_{m,n,p} \cap V^{co}_{m,n,p})/GL_n$$

classifies canonical systems having the same input-output behavior, that is, such that all the $p\times m$ matrices CA^iB for $i \in \mathbb{N}$ are equal [**40**, Part VI - VII]. Conversely, if $F = \{F_j \ : \ j \in \mathbb{N}_+\}$ is a sequence of $p \times m$ matrices such that the corresponding *Hankel matrices*

$$H_{ij}(F) = \begin{bmatrix} F_1 & F_2 & \ldots & F_j \\ F_2 & F_3 & \ldots & F_{j+1} \\ \vdots & \vdots & & \vdots \\ F_i & F_{i+1} & \ldots & F_{i+j-1} \end{bmatrix}$$

are such that there exist integers r and s such that $rk\ H_{rs}(F) = rk\ H_{r+1,s+j}(F)$ for all $j \in \mathbb{N}_+$, then F is *realizable* by a canonical system $\Sigma = (A, B, C) \in V^c_{m,n,p}$

(for some n which is equal to $rk\ H_{rs}(F)$), that is,

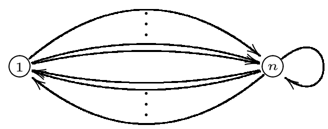
$$F_j = CA^{j-1}B \qquad \text{for all } j \in \mathbb{N}_+,$$

see for example [**40**, Part VI - VII] for connections between this realization problem and classical problems in analysis. These problems would be facilitated if there was an infinite dimensional manifold X together with a natural stratification

$$X = \bigsqcup_n \mathtt{sys}^c_{m,n,p}$$

by the moduli spaces of canonical systems (for fixed m and p and varying n).

Consider the quiver setting $(Q, \mathbf{v})$ where the dimension vector is $\mathbf{v} = (1, n)$ and the quiver Q



has m arrows $\{b_1, \ldots, b_m\}$ from left to right and p arrows $\{c_1, \ldots, c_p\}$ from right to left. We can identify $V_{m,n,p}$ with $\mathrm{Rep}^{\mathbf{v}}_Q$, where we associate to a system $\Sigma = (A, B, C)$ the representation V_Σ which assigns to the arrow b_i (resp. c_j) the i-th column B_i of B (resp. the j-th row C^j of C) and the matrix A to the loop. The basechange action of $(\lambda, g) \in GL(\alpha) = \mathrm{k}^* \times GL_n$ on the representation $V_\Sigma = (A, B_1, \ldots, B_m, C^1, \ldots, C^p)$ is as follows:

$$(\lambda.g).V_\Sigma = (gAg^{-1}, gB_1\lambda^{-1}, \ldots, gB_m\lambda^{-1}, \lambda C^1 g^{-1}, \ldots, \lambda C^p g^{-1}),$$

and as the central subgroup $\mathrm{k}^*(1, \mathbf{1}_n)$ acts trivially on $\mathrm{Rep}^{\mathbf{v}}_Q$, there is a natural one-to-one correspondence between equivalence classes of systems in $V_{m,n,p}$ and isomorphism classes of $\mathbf{v}$-dimensional representations in $\mathrm{Rep}^{\mathbf{v}}_Q$.

5.2. Simple representations. It is perhaps surprising that the system theoretic notion of canonical system corresponds under these identifications to the algebraic notion of simple representation.

5.2.1. LEMMA. *The following are equivalent:*

(1) $\Sigma = (A, B, C) \in V_{m,n,p}$ *is a canonical system,*
(2) $V_\Sigma = (A, B_1, \ldots, B_m, C^1, \ldots, C^p) \in \mathrm{Rep}^{\mathbf{v}}_Q$ *is a simple representation.*

Proof: $1 \Rightarrow 2$: If V_Σ has a proper subrepresentation of dimension vector $(1, l)$ for some $l < n$, then the rank of the control-matrix $c(\Sigma)$ is at most l, contradicting complete controllability. If V_Σ has a proper subrepresentation of dimension vector $(0, l)$ with $l \neq 0$, then the observation-matrix $o(\Sigma)$ has rank at most $n - l$, contradicting complete observability. $2 \Rightarrow 1$: If $rk\ c(\Sigma) = l < n$ then there is a proper subrepresentation of dimension vector $(1, l)$ of V_Σ. If $rk\ o(\Sigma) = n - l$ with $l > 0$, then there is a proper subrepresentation of dimension vector $(0, l)$ of V_Σ. □

From [**26**] we recall that for a general quiver setting $(Q, \mathbf{v})$ the isomorphism classes of $\mathbf{v}$-dimensional semi-simple representations are classified by the *affine* algebraic quotient variety

$$\mathrm{Rep}^{\mathbf{v}}_Q // GL(\mathbf{v}) = \mathtt{iss}_{\mathbf{v}}\ Q$$

whose coordinate ring is generated by all traces along oriented cycles in the quiver Q. If $\mathbf{v}$ is the dimension vector of a simple representation, this affine quotient has dimension $1 - \chi_Q(\mathbf{v}, \mathbf{v})$ where χ_Q is the Euler form of Q. Moreover, the isomorphism classes of simple representations form a Zariski open smooth subvariety of $\mathtt{iss}_{\mathbf{v}}\ Q$. Specializing these general results from [**26**] to the case of interest, we recover Hazewinkels theorem.

5.3. THEOREM (Hazewinkel). *The moduli space* $\mathtt{sys}^c_{m,n,p}$ *of canonical systems is a smooth quasi-affine variety of dimension* $(m+p)n$.

In fact, combining the theory of local quivers (see for example [**25**]) with the classification of all quiver settings having a smooth quotient variety due to Raf Bocklandt [**4**], it follows that (unless $m = p = 1$) $\mathtt{sys}^c_{m,n,p}$ is precisely the smooth locus of the affine quotient variety $\mathtt{iss}_{\mathbf{v}}\ Q$.

5.4. Stable representations. In the special case when $\mathbf{v} = (1, n)$ and Q is the quiver introduced before, there are essentially two different stability structures on $\mathrm{Rep}^{\mathbf{v}}_Q$ determined by the integral vectors

$$\theta_+ = (-n, 1) \qquad \text{and} \qquad \theta_- = (n, -1)$$

By the identification of $\mathrm{Rep}^{\mathbf{v}}_Q$ with $V_{m,n,p}$ and the proof of lemma 5.2.1 we have

5.4.1. LEMMA. *For* $\theta_+ = (-n, 1)$ *the following are equivalent:*

1. $\Sigma \in V_{m,n,p}$ *is controllable,*
2. $V_\Sigma \in \mathrm{Rep}^{\mathbf{v}}_Q$ *is* θ_+*-stable.*

For $\theta_- = (n, -1)$ *the following are equivalent:*

1. $\Sigma \in V_{m,n,p}$ *is observable,*
2. $V_\Sigma \in \mathrm{Rep}^{\mathbf{v}}_Q$ *is* θ_-*-stable.*

Therefore, we have the isomorphisms

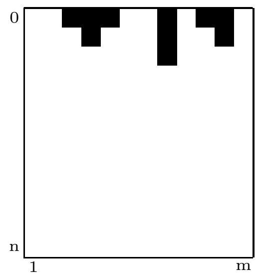
$$\mathtt{sys}^{cc}_{m,n,p} = \mathcal{M}(Q, \mathbf{v}, \chi_{\theta_+}) \qquad \text{and} \qquad \mathtt{sys}^{co}_{m,n,p} = \mathcal{M}(Q, \mathbf{v}, \chi_{\theta_-}).$$

In [**33**] the Harder-Narasimhan filtration associated to a stability structure was used to compute the cohomology of the moduli spaces $\mathcal{M}(Q, \mathbf{v}, \chi)$ (at least if the quiver Q has no oriented cycles). For general quivers the same methods can be applied to compute the number of $\mathbb{F}_q$-points of these moduli spaces, where $\mathbb{F}_q$ is the finite field of $q = p^l$ elements. In the case of interest to us, we get the rational functions

$$\begin{cases} \#\ \mathcal{M}(Q, \mathbf{v}, \chi_{\theta_+})\ (\mathbb{F}_q) & = q^{n(p+1)} \prod_{i=1}^n \frac{q^{m+i-1}-1}{q^i-1} \\ \\ \#\ \mathcal{M}(Q, \mathbf{v}, \chi_{\theta_-})\ (\mathbb{F}_q) & = q^{n(m+1)} \prod_{i=1}^n \frac{q^{p+i-1}-1}{q^i-1}, \end{cases}$$

These formulas motivate the main result of the next paragraph.

5.5. Kalman codes. To a completely controllable $\Sigma = (A, B, C)$ one associates its *Kalman code* K_Σ, which is an array of $n \times m$ boxes $\{(i, j) \mid 0 \leq i < n, 1 \leq j \leq\}$, ordered lexicographically, with exactly n boxes painted black. If the column $A^i B_j$ is linearly independent of all column vectors $A^k B_l$ with $(k, l) < (i, j)$ we paint box (i, j) black. From this rule it is clear that if (i, j) is a black box so are (i', j) for all $i' \leq i$. That is, the Kalman code K_Σ (which only depends on the GL_n-orbit of Σ) looks like



Assume $\kappa = K_\Sigma$ has k black boxes on its first row at places $(0, i_1), \ldots, (0, i_k)$. Then we assign to κ the strictly increasing sequence

$$1 \leq j_\kappa(1) = i_1 < j_\kappa(2) = i_2 < \ldots < j_\kappa(k) = i_k \leq m$$

and another sequence $p_\kappa(1), \ldots, p_\kappa(k)$, where $p_\kappa(j)$ is the total number of black boxes in the i_j-th column of κ, that is,

$$p_\kappa(1) + p_\kappa(2) + \ldots + p_\kappa(k) = n.$$

It is clear that there is a one-to-one correspondence between Kalman codes and pairs of functions satisfying these conditions. Further, define the strictly increasing sequence

$$h_\kappa(0) = 0 < h_\kappa(1) = p_\kappa(1) < \ldots < h_\kappa(j) = \sum_{i=1}^{j} p_\kappa(i) < \ldots < h_\kappa(k) = n.$$

With these notations we have the following canonical form for $\Sigma = (A, B, C) \in V^{cc}_{m,n,p}$ which is essentially Lemma 3.2.

5.5.1. Lemma. *For a completely reachable system $\Sigma = (A, B, C)$ with Kalman code $\kappa = K_\Sigma$, there is a unique $g \in GL_n$ such that $g.(A, B, C) = (A', B', C')$ with*

- $B'_{j_\kappa(i)} = \mathbf{1}_{h_\kappa(i-1)+1}$ *for all $1 \leq i \leq k$.*
- $A'_i = \mathbf{1}_{i+1}$ *for all $i \notin \{h_\kappa(1), h_\kappa(2), \ldots, h_\kappa(k)\}$.*
- *All entries in the remaining columns of A' and B' are determined as the quotient of two specific $n \times n$ minors of $c(\Sigma)$.*
- $C' = Cg^{-1}$.

5.6. Theorem. *The moduli space $\mathtt{sys}^{cc}_{m,n,p}$ of completely controllable systems has a cell decomposition identical to the natural cell decomposition of a vectorbundle of rank $n(p+1)$ over the Grassmann manifold $\mathtt{Gras}_n(m+n-1)$.*

Proof: Define a map $V^{cc}_{m,n,p} \xrightarrow{\phi} \mathtt{Gras}_n(m+n-1)$ by sending a completely reachable system $\Sigma = (A, B, C)$ to the point in $\mathtt{Gras}_n(m+n-1)$ determined by the $n \times (m+n-1)$ matrix

$$M_\Sigma = \begin{bmatrix} B'_1 & \ldots & B'_m & A'_1 & \ldots & A'_{n-1} \end{bmatrix},$$

where (A', B', C') is the canonical form of Σ given by the previous lemma. By construction, M_Σ has rank n with invertible $n \times n$ matrix determined by the columns

$$I_\kappa = \{j_\kappa(1) < \ldots < j_\kappa(k) < m + c_1 < \ldots < m + c_{n-k}\} \subset \{1, \ldots, m+n-1\},$$

where $\{c_1,\ldots,c_{n-k}\} = \{1,\ldots,n\} - \{h_\kappa(1),\ldots,h_\kappa(k)\}$. As all remaining entries of (A',B') are determined by $c(\Sigma)$ it follows that $\phi(\Sigma)$ depends only on the GL_n-orbit of Σ, whence the map factorizes through

$$\mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\psi} \mathtt{Gras}_n(m+n-1),$$

and we claim that ψ is surjective. To begin, all multi-indices $I = \{1 \leq d_1 < d_2 < \ldots < d_n \leq n+m-1\}$ are of the form I_κ for some Kalman code κ. Define

$$\{d_1,\ldots,d_n\} = \{i_1,\ldots,i_k\} \cup \{m+c_1,\ldots,m+c_{n-k}\}$$

with $i_j \leq m$ and $1 \leq c_j < n$, and let $\{e_1 < \ldots < e_k\} = \{1,\ldots,n\} - \{c_1,\ldots,c_{n-k}\}$, and set $e_0 = 0$. Construct the Kalman code κ having $e_j - e_{j-1}$ black boxes in the i_j-th column and verify that I is indeed I_κ.

$\mathtt{Gras}_n(m+n-1)$ is covered by modified Schubert cells S_I (isomorphic to some affine space) consisting of points such that the I-minor is invertible, where I is a multi-index $\{d_1,\ldots,d_n\}$, and the dimension of the subspace spanned by the first k columns is i iff $k < d_{i+1}$. A point in S_I can be taken such that the d_i-th column is equal to

$$\begin{cases} \mathbf{1}_{h_\kappa(i-1)+1} & \text{for } d_i \leq m \\ \mathbf{1}_{j+1} & \text{for } d_i = m+i, \end{cases}$$

where $I = I_\kappa$. This determines a $n \times (n+m-1)$ matrix of shape $\begin{bmatrix} B_1 & \ldots & B_m & A_1 & \ldots & A_{n-1}\end{bmatrix}$, and choosing any last column A_n and any $p \times n$ matrix C we obtain a system $\Sigma = (A,B,C)$ which is completely controllable, and which is mapped to the given point under ψ. This finishes the proof. $\square$

Because the map $(A,B,C) \longrightarrow (A^{tr},C^{tr},B^{tr})$ defines a duality between $V^{co}_{m,n,p}$ and $V^{cc}_{p,n,m}$, we have a similar result for the moduli spaces of completely observable systems.

5.7. THEOREM. *The moduli space of completely observable systems $\mathtt{sys}^{co}_{m,n,p}$ has a cell decomposition identical to that of a vectorbundle of rank $n(p+1)$ over the Grassmann manifold $\mathtt{Gras}_n(p+n-1)$.*

The counting argument of the previous section gives us also a conjectural description of the infinite dimensional variety admitting a stratification by the moduli spaces $\mathtt{sys}^{cc}_{m,n,p}$. It follows from the explicit rational form of $\#\ \mathtt{sys}^{cc}_{m,n,p}\ (\mathbb{F}_q)$ and the *q-binomial theorem* that

$$\sum_{n=0}^{\infty} \#\ \mathtt{sys}^{cc}_{m,n,p}\ (\mathbb{F}_q)\ t^n = \prod_{i=1}^{m} \frac{1}{1-q^{p+i}t}$$

In the special case when $p = 0$ we recover the cohomology of the infinite Grassmannian $\mathtt{Gras}_m(\infty)$ of m-dimensional subspaces of a countably infinite dimensional vectorspace. For $p \geq 1$ we only get a factor of the cohomology of $\mathtt{Gras}_{m+p}(\infty)$, which led to the following result.

5.8. THEOREM. *The disjoint union $\bigsqcup_n\ \mathtt{sys}^{cc}_{m,n,p}$ is the open subset of the infinite dimensional Grassmann manifold $\mathtt{Gras}_{m+p}(\infty)$ which is the union of all standard affine open sets corresponding to a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p}\}$ such that*

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I.$$

Proof: Let $\Sigma = (A, B, C)$ be a completely controllable system in canonical form represented by the point $p_\Sigma \in \mathtt{sys}^{cc}_{m,n,p}$. Consider the $n \times (m + p + n)$ matrix

$$L_\Sigma = \begin{bmatrix} B & C^{tr} & A \end{bmatrix}.$$

The submatrix $M_\Sigma = \begin{bmatrix} B_1 & \dots & B_m & A_1 & \dots & A_{n-1} \end{bmatrix}$ has rank n, whence so has L_Σ, and p_Σ determines a point in $\mathtt{Gras}_n(n + m + p)$. Under the natural duality

$$\mathtt{Gras}_n(m + p + n) \xrightarrow{D} \mathtt{Gras}_{m+p}(m + p + n),$$

the point p_Σ is mapped to the point determined by the $(m+p)\times(m+p+n)$ matrix N_Σ whose rows give a basis for the linear relations holding among the columns of L_Σ. Because M_Σ has rank n it follows that the columns of C^{tr} and the last column A_n of A are linearly dependent of those of M_Σ. As a consequence the matrix

$$N_\Sigma = \begin{bmatrix} U_1 & \dots & U_m & V_1 & \dots & V_p & W_1 & \dots & W_n \end{bmatrix}$$

has the property that the submatrix $\begin{bmatrix} V_1 & \dots & V_p & W_n \end{bmatrix}$ has rank $p + 1$. This procedure defines a morphism

$$\mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\gamma_n} \mathtt{Gras}_{m+p}(m + p + n),$$

the image of which is the open union of all standard affine opens determined by a multi-index set $I = \{1 \le d_1 < d_2 < \ldots < d_{m+p} \le m + p + n\}$ satisfying

$$\{m + 1, m + 2, \ldots, m + p, m + p + n\} \subset I.$$

Therefore, the image of the morphism

$$\bigsqcup_n \mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\sqcup\gamma_n} \mathtt{Gras}_{m+p}(\infty)$$

is the one of the statement of the theorem. The dimension n of the system corresponding to a point in this open set of $\mathtt{Gras}_{m+p}(\infty)$ is determined by $d_{m+p} = m + p + n$. $\square$

By the duality between $V^{cc}_{m,n,p}$ and $V^{co}_{p,n,m}$ used in the previous section we deduce:

5.9. Theorem. *The disjoint union* $\bigsqcup_n \mathtt{sys}^{co}_{m,n,p}$ *is the open subset of* $\mathtt{Gras}_{m+p}(\infty)$ *which is the union of all standard affine opens corresponding to a multi-index set* $I = \{1 \le d_1 < d_2 < \ldots < d_{m+p}\}$ *such that*

$$\{1, 2, \ldots, m, m + p + n\} \subset I.$$

This, in turn, proves our main theorem.

5.10. Theorem. *The disjoint union* $\bigsqcup_n \mathtt{sys}^{c}_{m,n,p}$ *of all moduli spaces of canonical systems with fixed input- and output-dimension* m *and* p *is the open subset of the infinite Grassmannian* $\mathtt{Gras}_{m+p}(\infty)$ *of* $m+p$*-dimensional subspaces of a countably infinite dimensional vectorspace which is the intersection of all possible standard open subsets* X_I *and* X_J*, where* I *and* J *are multi-index sets satisfying the conditions*

$$\{m + 1, m + 2, \ldots, m + p, m + p + n\} \subset I \qquad \text{and} \qquad \{1, 2, \ldots, m, m + p + n\} \subset J.$$

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