ON THE NOTION OF DERIVED TAMENESS

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Introduction

Let Λ be a finite dimensional algebra over an algebraically closed field. Drozd's definition of tame type for the category $\operatorname{mod}\Lambda$ of finite dimensional Λ-modules is the base of his celebrated Tame and Wild Theorem [6]. It states that Λ is either *tame*, so that for all $n \in \mathbb{N}$ the indecomposable Λ-modules of dimension n can be parametrized using only one continuous parameter, or Λ is *wild*, and it has families of indecomposable Λ-modules depending on arbitrarily many continuous parameters.

Let $\mathcal{D}^b(\operatorname{mod}\Lambda)$ denote the derived category of bounded complexes of finite dimensional Λ-modules. The main aim of this paper is to establish standard properties of two natural definitions for the notion of derived tameness. The first definition (see Definition 1.1) is the analogue of Drozd's tameness definition. Roughly speaking, we say that Λ is *derived tame* if for each vector $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of natural numbers the indecomposable objects in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ of homology dimension $\mathbf{n}$ can be parametrized using only one continuous parameter.

The second definition (see Definition 1.3) involves generic complexes and is the analogue of Crawley-Boevey's definition of generic tameness [4]. Here we use the bounded derived category $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ of all Λ-modules. We say that Λ is *generically derived tame* if for each vector $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of natural numbers there are only finitely many isomorphism classes of generic complexes in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ which have homology endolength $\mathbf{n}$.

Both definitions are motivated by recent work of Vossieck [22] who classified the algebras having a discrete derived category, and by work of de la Peña [18] who introduced the notion of derived tameness via the repetitive algebra.

Our first result shows that an equivalence $\mathcal{D}^b(\operatorname{mod}\Lambda) \to \mathcal{D}^b(\operatorname{mod}\Gamma)$ of triangulated categories preserves derived tameness.

Theorem A. *Let Λ and Γ be finite dimensional algebras over an algebraically closed field and suppose that Λ and Γ are derived equivalent.*

(1) *Λ is derived tame if and only if Γ is derived tame.*
(2) *Λ is generically derived tame if and only if Γ is generically derived tame.*

It is beyond the scope of this paper to prove a derived Tame and Wild Theorem. However, we are able to show that an algebra is not derived tame if there are families of indecomposable complexes depending on at least two continuous parameters, see Theorem 5.2 for a precise statement.

If Λ is of finite global dimension, then $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is equivalent as triangulated category to the stable module category $\underline{\operatorname{mod}}\,\widehat{\Lambda}$ of the repetitive algebra $\widehat{\Lambda}$ by Happel's theorem [10]. It is therefore important to have the following result.

Theorem B. *Let Λ be a finite dimensional algebra over an algebraically closed field and suppose that Λ has finite global dimension. Then the following are equivalent:*

(1) *Λ is derived tame;*

(2) Λ *is generically derived tame;*
(3) $\widehat{\Lambda}$ *is tame.*

The hypothesis of finite global dimension is not needed to prove (1) and (2) if (3) holds. It is remarkable, that for the discrete case also (1) implies (3) without the hypothesis of finite global dimension, but this is based on a complete classification of the algebras of that type [22].

For an algebra of infinite global dimension, not much seems to be known about the derived category. For instance, we have no example showing that the assumption on the global dimension is needed for the other implications in the preceding theorem. However, we feel that in any case the definition of the derived representation type should be formulated in terms of the derived category.

Let us mention some examples of derived tame algebras. The first theorem implies that a piecewise hereditary algebra [11] is derived tame if and only if the Euler-form is non-negative. Both theorems together imply that algebras, which are derived equivalent to skewed-gentle algebras [9], are derived tame. This class of algebras is still not well explored, but it contains all tree-algebras which have non-negative Euler-form and are not piecewise hereditary, see [1] and [8].

We give now a brief outline of the contents of this paper. Section 1 – 2 contain definitions and background material. In Section 3 we collect various estimates for (homology) endolength and dimension of complexes and modules. Note that [22] has similar estimates for dimensions. However, our endolength versions require different proofs. In Section 4 we study the behavior of families of complexes under (standard) derived equivalences and Happel's functor. It is shown that these functors preserve families provided one restricts a given family to some appropriate open subset. Moreover, a new description of the image of Happel's functor is given. In Section 5 we discuss the analogue of Drozd's tameness definition via bimodule complexes, and Section 6 – 7 are devoted to the analogue of Crawley-Boevey's tameness definition via generic complexes. In the final Section 8 we discuss domestic type and polynomial growth which are refinements of derived tame representation type. Note that the essential work is done Section 3 – 4, whereas the remaining Section 5 – 8 are of more formal nature.

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1. Derived Tameness

Notation. Let k be a field and let Λ be a locally bounded k-category (see [3]). We are interested in the study of modules over finite dimensional k-algebras and observe that this is the same as the study of Λ-modules where Λ is finite. A Λ-*module* is a k-linear functor $\Lambda^{\mathrm{op}} \to \operatorname{Mod} k$ into the category of k-vector spaces. We denote by $\operatorname{Mod}\Lambda$ the category of all Λ-modules and $\operatorname{mod}\Lambda$ denotes the full subcategory of finitely presented Λ-modules. For every object $i \in \Lambda$, we denote by $P_i = \operatorname{Hom}_\Lambda(-, i)$ the corresponding indecomposable projective Λ-module, and S_i denotes the unique simple Λ-module with $\operatorname{Hom}(P_i, S_i) \neq 0$. By abuse of notation, we sometimes write Λ for $\coprod_{i\in\Lambda} P_i$. Given a Λ-module X, we denote by

$$\mathbf{dim}\, X = (\dim_k X(i))_{i\in\Lambda}$$

its *dimension* and

$$\mathbf{endol}\, X = (\operatorname{length}_{\operatorname{End}(X)} X(i))_{i\in\Lambda}$$

is called the *endolength* of X. Here, $\operatorname{length}_R M$ denotes the composition length of an R-module M. The Λ-module X is *endofinite* if each entry of the vector $\mathbf{endol}\, X$ is finite.

The derived category of bounded complexes of arbitrary Λ-modules is denoted by $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$ denotes the bounded derived category of finitely presented Λ-modules. We identify the homotopy category $\mathcal{K}^b(\operatorname{proj}\Lambda)$ of finitely generated projective Λ-modules with the full subcategory of perfect complexes. Recall that a complex is *perfect* if it is isomorphic to a bounded complex of finitely generated projective Λ-modules.

A derived category is a triangulated category in the sense of Verdier [21]. We denote for a triangulated category $\mathcal{T}$ by Σ the translation functor $\mathcal{T} \to \mathcal{T}$ and the distinguished triangles in $\mathcal{T}$ are sequences of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

We shall often identify a Λ-module with the corresponding complex concentrated in degree 0. For instance, we have for all $i \in \Lambda$ and $n \in \mathbb{Z}$ the isomorphism

$$H^n(X)(i) \cong \operatorname{Hom}(P_i, \Sigma^n X).$$

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ for the set of natural numbers and define for every set I the set of vectors

$$\mathbb{N}^{(I)} = \{\mathbf{n} = (n_i)_{i \in I} \mid n_i \in \mathbb{N} \text{ and } n_i = 0 \text{ for almost all } i\}.$$

For example, $\mathbf{dim}\, X \in \mathbb{N}^{(\Lambda)}$ for every finitely presented Λ-module X.

Tameness. Our first definition of derived tameness is the analogue of Drozd's tameness definition [6]. We use the *homology dimension* of a complex $X \in \mathcal{D}^b(\operatorname{mod}\Lambda)$ which is by definition the vector

$$\textbf{h-dim}\, X = (\mathbf{dim}\, H^i(X))_{i \in \mathbb{Z}}.$$

Definition 1.1. A locally bounded k-category Λ is called *derived tame* if for every vector $\mathbf{n}$ there exist a localization $R = k[t]_f$ with respect to some $f \in k[t]$ and a finite number of bounded complexes of R-Λ bimodules $C_1, \dots, C_n$ such that each C_i^j is finitely generated free over R and (up to isomorphism) all but finitely many indecomposable objects of homology dimension $\mathbf{n}$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ are of the form $S \otimes_R C_i$ for some $i = 1, \dots, n$ and some simple R-module S.

Generic tameness. Our second definition of derived tameness is the analogue of Crawley-Boevey's definition of generic tameness [4]. This definition involves generic complexes; they also arise in work of Lenzing on tubular algebras [17].

Definition 1.2. A complex X in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ is called *endofinite* if the Λ-module $H^i(X)$ is endofinite for all $i \in \mathbb{Z}$. An endofinite complex X is called *generic* if X is indecomposable and not isomorphic to a bounded complex of finitely presented Λ-modules.

The *homology endolength* of a complex X is by definition the vector

$$\textbf{h-endol}\, X = (\mathbf{endol}\, H^i(X))_{i \in \mathbb{Z}}.$$

Note that homology endolength and homology dimension of X coincide provided that X is an indecomposable complex of finite dimensional modules and the ground field k is algebraically closed.

Definition 1.3. A locally bounded k-category Λ is called *generically derived tame* if for every vector $\mathbf{n}$ there are only finitely many isomorphism classes of generic objects X in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ such that $\textbf{h-endol}\, X = \mathbf{n}$.

Remark 1.4. The preceding Definition 1.3 could be formulated for any noetherian algebra, to include, for example, the polynomial ring $k[t]$. In fact, all basic properties of this concept (see Section 6) remain true for noetherian algebras.

Remark 1.5. Recall that Λ has a *discrete* derived category if for every vector $\mathbf{n}$ there are (up to isomorphism) only finitely many indecomposable objects of homology dimension $\mathbf{n}$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. It can be derived from Vossieck's work [22] that $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is discrete if and only if there is no generic complex in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$

2. The repetitive category

The repetitive category. Let Λ be a locally bounded category. The *repetitive category* $\widehat{\Lambda}$ of Λ is a locally bounded category which is defined as follows. The objects of $\widehat{\Lambda}$ are the pairs (i,n) with $i \in \Lambda$ and $n \in \mathbb{Z}$. The space of maps $(i,n) \to (j,m)$ is $\operatorname{Hom}_\Lambda(i,j)$ if $n = m$, and $\operatorname{Hom}_k(\operatorname{Hom}_\Lambda(j,i),k)$ if $m = n-1$; it is zero otherwise. The composition in $\widehat{\Lambda}$ is induced by that in Λ. We view every $\widehat{\Lambda}$-module X as a family $X = (X_n)_{n\in\mathbb{Z}}$ of Λ-modules where $X_n(i) = X(i,n)$. In particular,

$$\mathbf{dim}\, X = (\mathbf{dim}\, X_n)_{n\in\mathbb{Z}} \quad \text{and} \quad \mathbf{endol}\, X = (\mathbf{endol}\, X_n)_{n\in\mathbb{Z}}$$

are families of vectors in $\mathbb{N}^{(\Lambda)}$. Note that $\widehat{\Lambda}$ is selfinjective. Therefore projective and injective $\widehat{\Lambda}$-modules coincide.

The stable category. The stable module category of $\widehat{\Lambda}$ modulo projectives is denoted by $\underline{\operatorname{Mod}}\,\widehat{\Lambda}$. It coincides with the stable category modulo injectives $\overline{\operatorname{Mod}}\,\widehat{\Lambda}$ since $\widehat{\Lambda}$ is selfinjective. Every $\widehat{\Lambda}$-module X has a maximal injective submodule X^{inj} which exists by Zorn's lemma since direct limits of injective modules are injective. We define $X^{\mathrm{red}} = X/X^{\mathrm{inj}}$ and call X *reduced* if $X = X^{\mathrm{red}}$. Note that two $\widehat{\Lambda}$-modules X and Y are isomorphic as objects in $\underline{\operatorname{Mod}}\,\widehat{\Lambda}$ if and only if X^{red} and Y^{red} are isomorphic in $\operatorname{Mod}\widehat{\Lambda}$. We define

$$\mathbf{r\text{-}dim}\, X = \mathbf{dim}\, X^{\mathrm{red}} \quad \text{and} \quad \mathbf{r\text{-}endol}\, X = \mathbf{endol}\, X^{\mathrm{red}}.$$

Given $i \in \mathbb{Z}$, we denote for every $\widehat{\Lambda}$-module X by $\Omega^i X$ the i-th syzygy. The assignment $X \mapsto \Omega^i X$ induces an equivalence $\Omega^i\colon \underline{\operatorname{Mod}}\,\widehat{\Lambda} \to \underline{\operatorname{Mod}}\,\widehat{\Lambda}$. Note that $\underline{\operatorname{Mod}}\,\widehat{\Lambda}$ is a triangulated category; see [13] for details. The functor Ω^{-1} serves as translation functor so that the distinguished triangles in $\underline{\operatorname{Mod}}\,\widehat{\Lambda}$ are of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Omega^{-1} X.$$

Next we formulate some estimates for the syzygies in $\operatorname{Mod}\widehat{\Lambda}$. We denote for a Λ-module X by $\operatorname{pd} X$ its projective dimension and by $\operatorname{id} X$ its injective dimension.

Lemma 2.1. *Let $X = (X_i)_{i\in\mathbb{Z}} \in \operatorname{Mod}\widehat{\Lambda}$ and $X_i = 0$ for $i < r$ and $i > s$.*

1. $(\Omega X)_i = 0$ *for* $i < r$ *and* $i > s+1$.
2. $(\Omega^{-1} X)_i = 0$ *for* $i < r-1$ *and* $i > s$.
3. $\operatorname{pd} X_r = n$ *implies* $(\Omega^{n+1} X)_r = 0$.
4. $\operatorname{id} X_s = n$ *implies* $(\Omega^{-(n+1)} X)_s = 0$.

Proof. The assertions follow easily from the construction of the syzygies $\Omega^i X$. $\square$

Happel's functor. We fix a finite locally bounded k-category Λ. Consider the Happel functor

$$F\colon \mathcal{D}^b(\operatorname{Mod}\Lambda) \longrightarrow \underline{\operatorname{Mod}}\,\widehat{\Lambda}$$

which restricts to a functor

$$\mathcal{D}^b(\operatorname{mod}\Lambda) \longrightarrow \underline{\operatorname{mod}}\,\widehat{\Lambda}.$$

It is defined in [10] (see also [13]) and we list the essential properties of this functor: F is fully faithful, exact, and sends a complex X concentrated in degree zero to the $\widehat{\Lambda}$-module Y which is concentrated in degree zero with $Y_0 = X^0$. Moreover, if the global dimension of Λ is finite, F induces an equivalence between $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ and the full subcategory $\underline{\operatorname{Mod}}_f\,\widehat{\Lambda}$ of modules $X = (X_i)_{i\in\mathbb{Z}}$ in $\underline{\operatorname{Mod}}\,\widehat{\Lambda}$ such that $X_i = 0$ for almost all i. In particular, F induces an equivalence $\mathcal{D}^b(\operatorname{mod}\Lambda) \longrightarrow \underline{\operatorname{mod}}\,\widehat{\Lambda}$.

3. Estimates

In this section we shall produce various estimates for (homology) dimension and endolength of modules and complexes. Throughout this section we fix two finite locally bounded k-categories Λ and Γ. Given a family $\mathbf{n} = (\mathbf{n}_i)_{i\in\mathbb{Z}}$ of vectors $\mathbf{n}_i \in \mathbb{N}^{(\Lambda)}$ with $\mathbf{n}_i = \mathbf{0}$ for almost all i, we define

$$|\mathbf{n}| = \max\{|i| \mid \mathbf{n}_i \neq \mathbf{0}\} \quad \text{and} \quad \|\mathbf{n}\| = \max\{\sum_{j\in\Lambda}(\mathbf{n}_i)_j \mid i \in \mathbb{Z}\}.$$

Devissage. Let $\mathcal{T}$ be any triangulated category and fix an object T in $\mathcal{T}$. We define inductively

$$\langle T\rangle_0 = \{X \in \mathcal{T} \mid X \text{ is a direct factor of } \Sigma^i T \text{ for some } i \in \mathbb{Z}\}$$

and $X \in \mathcal{T}$ belongs to $\langle T\rangle_n$ for $n > 0$ if X is a direct factor of Y for some distinguished triangle $Y' \to Y \to Y'' \to \Sigma Y'$ with $Y', Y'' \in \langle T\rangle_{n-1}$. Note that $\langle T\rangle = \bigcup_{n\geq 0}\langle T\rangle_n$ is the smallest thick subcategory of $\mathcal{T}$ containing T.

Given an object X in $\langle T\rangle$, one defines the *distance* of X from T as follows:

$$d(X,T) = \min\{n \in \mathbb{N} \mid X \in \langle T\rangle_n\}.$$

By induction on $d(X,T)$, one defines the *width* $w(X,T)$ of X with respect to T as follows. If $d(X,T) = 0$, let

$$w(X,T) = \min\{n \in \mathbb{N} \mid X \text{ is a direct factor of } \Sigma^n T \text{ or } \Sigma^{-n} T\}.$$

Otherwise, let $w(X,T)$ be the smallest $n \in \mathbb{N}$ such that X is a direct factor of Y for some distinguished triangle $Y' \to Y \to Y'' \to \Sigma Y'$ with $d(Y',T), d(Y'',T) < d(X,T)$ and $w(Y',T), w(Y'',T) \leq n$. Finally, we define for two objects X, Y in $\mathcal{T}$

$$[X,Y] = \operatorname{length}_{\operatorname{End}(Y)} \operatorname{Hom}(X,Y).$$

Lemma 3.1. *Let $S \in \langle T\rangle$ and $X \in \mathcal{T}$.*

(1) $[\Sigma^i T, X] = 0$ *for* $|i| > n$ *implies* $[\Sigma^i S, X] = 0$ *for* $|i| > n + w(S,T)$.
(2) $[S,X] \leq 2^{d(S,T)} \max\{[\Sigma^i T, X] \mid |i| \leq w(S,T)\}$.

Proof. Every distinguished triangle $A \to B \to C \to \Sigma A$ in $\mathcal{T}$ induces an exact sequence

$$\operatorname{Hom}(\Sigma A, X) \longrightarrow \operatorname{Hom}(C,X) \longrightarrow \operatorname{Hom}(B,X) \longrightarrow \operatorname{Hom}(A,X)$$

of $\operatorname{End}(X)$-modules. The assertions are immediate consequences of this basic fact and the definitions. □

Derived equivalence. We show that a derived equivalence controls homology endolength and dimension.

Lemma 3.2. *Let* $G\colon \mathcal{D}^b(\operatorname{Mod}\Gamma) \to \mathcal{D}^b(\operatorname{Mod}\Lambda)$ *be an equivalence of triangulated categories. Then there exists a constant* $g \in \mathbb{N}$ *and a function* $\gamma\colon \mathbb{N} \to \mathbb{N}$ *such that for all* $X \in \mathcal{D}^b(\operatorname{Mod}\Gamma)$

$$|\,\mathbf{h\text{-}endol}\, G(X)| \le \gamma(|\,\mathbf{h\text{-}endol}\, X|) \quad \textit{and} \quad \|\,\mathbf{h\text{-}endol}\, G(X)\| \le g\,\|\,\mathbf{h\text{-}endol}\, X\|.$$

Proof. Let $X \in \mathcal{D}^b(\operatorname{Mod}\Gamma)$ and $\mathbf{h\text{-}endol}\, X = \mathbf{n}$. We shall produce an estimate for $\mathbf{h\text{-}endol}\, Y$ where $Y = G(X)$. We put $T = G(\Gamma)$ and observe that

$$[\Sigma^{-i}T, Y] = [\Sigma^{-i}\Gamma, X] = \sum_{j\in\Gamma}[\Sigma^{-i}P_j, X] \le \|\mathbf{n}\|$$

for each i. Applying Lemma 3.1, we get $[\Sigma^{-i}\Lambda, Y] = 0$ for $|i| > |\mathbf{n}| + w(\Lambda, T)$ and

$$\sum_{j\in\Lambda}[\Sigma^{-i}P_j, Y] = [\Sigma^{-i}\Lambda, Y] \le 2^{d(\Lambda,T)}\|\mathbf{n}\|$$

for all $i \in \mathbb{Z}$. Defining $g = 2^{d(\Lambda,T)}$ and $\gamma(n) = n + w(\Lambda, T)$, the assertion follows. $\square$

Happel's functor. Consider the Happel functor $F\colon \mathcal{D}^b(\operatorname{Mod}\Lambda) \to \underline{\operatorname{Mod}}\,\widehat{\Lambda}$. We wish to compare the endolength of a complex X with the endolength of the module $F(X)$. This requires a series of lemmas. We start with a simple observation.

Lemma 3.3. *Let* X *be a module without any non-zero injective submodule, and let* S *be a simple module. Then* $\operatorname{Hom}(S, X) = \overline{\operatorname{Hom}}(S, X)$*, and therefore*

$$\operatorname{length}_{\operatorname{End}(X)} \operatorname{Hom}(S, X) = \operatorname{length}_{\overline{\operatorname{End}}(X)} \overline{\operatorname{Hom}}(S, X).$$

Proof. Any non-zero map $S \to X$ which factors through some injective module, also factors through the injective envelope $E(S)$ of S, and induces therefore a monomorphism $E(S) \to X$. This is impossible by our assumption on X. Thus $\operatorname{Hom}(S, X) = \overline{\operatorname{Hom}}(S, X)$. $\square$

Lemma 3.4. *Let* $X \in \mathcal{D}^b(\operatorname{Mod}\Lambda)$ *with* $H^i(X) = 0$ *for* $|i| > n$*. Then* $F(X)_i = 0$ *for* $|i| > 2(n+1)$*.*

Proof. Let $X \in \mathcal{D}^b(\operatorname{Mod}\Lambda)$ and $H^i(X) = 0$ for $|i| > n$. The complex $\Sigma^{n+1}X$ is isomorphic to a complex Y with $Y^i = 0$ for $i < -2(n+1)$ and $i > 0$. The construction of F implies $F(Y)_i = 0$ for $i < -2(n+1)$ and $i > 0$ (cf. II.4 in [10]). We have $F(X) \cong \Omega^{n+1}F(Y)$, and therefore $F(X)_i = 0$ for $|i| > 2(n+1)$ by Lemma 2.1. $\square$

Lemma 3.5. *Suppose that* Λ *has global dimension* d*. Let* $X \in \mathcal{D}^b(\operatorname{Mod}\Lambda)$ *with* $F(X)_i = 0$ *for* $|i| > n$*. Then* $H^i(X) = 0$ *for* $|i| > (n+1)(d+1)$*.*

Proof. Let $X \in \mathcal{D}^b(\operatorname{Mod}\Lambda)$. Then

$$H^i(X) \cong \operatorname{Hom}(\Lambda, \Sigma^i X) \cong \operatorname{Hom}(\Lambda, \Omega^{-i}F(X)).$$

If $F(X)_i = 0$ for $|i| > n$, then $(\Omega^{-i}F(X))_0 = 0$ for $|i| > (n+1)(d+1)$ by Lemma 2.1. Thus $H^i(X) = 0$ for $|i| > (n+1)(d+1)$. $\square$

Let us recall that for every $\widehat{\Lambda}$-module $X = (X_n)_{n\in\mathbb{Z}}$ the endolength $\mathbf{endol}\, X = (\mathbf{endol}\, X_n)_{n\in\mathbb{Z}}$ is a family of vectors in $\mathbb{N}^{(\Lambda)}$.

Lemma 3.6. *There exists a function $\alpha\colon \mathbb{N}\to\mathbb{N}$ such that for all $X\in\mathcal{D}^b(\operatorname{Mod}\Lambda)$*

$$|\,\mathbf{r\text{-}endol}\,F(X)|\le \alpha(|\mathbf{n}|)\quad\textit{and}\quad \|\,\mathbf{r\text{-}endol}\,F(X)\|\le\alpha(|\mathbf{n}|)\|\mathbf{n}\|$$

where $\mathbf{n}=\mathbf{h\text{-}endol}\,X$.

Proof. Let $X\in\mathcal{D}^b(\operatorname{Mod}\Lambda)$ and $\mathbf{h\text{-}endol}\,X=\mathbf{n}$. We define

$$\widehat{\Lambda}_n=\{(i,j)\in\Lambda\times\mathbb{Z}\mid |j|\le 2(n+1)\}\subseteq\widehat{\Lambda}$$

where $n=|\mathbf{n}|$. It follows from Lemma 3.4 that $F(X)_j=0$ for $|j|>2(n+1)$. Therefore $[S_i,F(X)]=0$ for every $i\in\widehat{\Lambda}\backslash\widehat{\Lambda}_n$. Applying Lemma 3.1, we get $[S_i,F(X)]\le 2^{d(S_i,\Lambda)}\|\mathbf{n}\|$ for all $i\in\widehat{\Lambda}$ since

$$[\Sigma^j\Lambda,F(X)]=[\Sigma^j\Lambda,X]\le\|\mathbf{n}\|$$

for all $j\in\mathbb{Z}$. Now let c_i be the maximal Jordan-Hölder multiplicity of the simple S_i in some indecomposable projective $\widehat{\Lambda}$-module. Using Lemma 3.3, we get for all $j\in\widehat{\Lambda}$ that

$$[P_j,F(X)^{\mathrm{red}}]\le\sum_{i\in\widehat{\Lambda}_n}c_i2^{d(S_i,\Lambda)}\|\mathbf{n}\|.$$

The estimate follows if we take

$$\alpha(n)=\max\{2(n+1),|\Lambda|\sum_{i\in\widehat{\Lambda}_n}c_i2^{d(S_i,\Lambda)}\}$$

where $|\Lambda|$ denotes the number of objects of Λ. □

Lemma 3.7. *Suppose that Λ has finite global dimension. Then there exists a function $\beta\colon\mathbb{N}\to\mathbb{N}$ such that for all $X\in\mathcal{D}^b(\operatorname{Mod}\Lambda)$*

$$|\,\mathbf{h\text{-}endol}\,X|\le\beta(|\mathbf{m}|)\quad\textit{and}\quad\|\,\mathbf{h\text{-}endol}\,X\|\le\beta(|\mathbf{m}|)\|\mathbf{m}\|$$

where $\mathbf{m}=\mathbf{r\text{-}endol}\,F(X)$.

Proof. Let $X\in\mathcal{D}^b(\operatorname{Mod}\Lambda)$ and $\mathbf{h\text{-}endol}\,X=\mathbf{n}$. We put $Y=F(X)^{\mathrm{red}}$ and $\mathbf{endol}\,Y=\mathbf{m}$. Thus

$$[S_i,Y]\le[P_i,Y]=\|\mathbf{m}\|$$

for all $i\in\widehat{\Lambda}$. Now define

$$\widehat{\Lambda}_m=\{(i,j)\in\Lambda\times\mathbb{Z}\mid |j|\le m\}\subseteq\widehat{\Lambda}$$

where $m=|\mathbf{m}|$ and denote by E the injective envelope of $\coprod_{i\in\widehat{\Lambda}_m}S_i$. We have $E_i=0$ for almost all i, say for $|i|>r$. It follows that $Y_i=0$ for $|i|>r$ since Y is a submodule of a product of copies of E. Therefore $H^i(X)=0$ for $|i|>(r+1)(d+1)$ by Lemma 3.5, where d denotes the global dimension of Λ. We get for each $i\in\mathbb{Z}$

$$\sum_{j\in\Lambda}(\mathbf{n}_i)_j=[\Sigma^{-i}\Lambda,X]=[\Sigma^{-i}\Lambda,F(X)]=[\Sigma^{-i}\Lambda,Y]$$

and have therefore

$$\sum_{j\in\Lambda}(\mathbf{n}_i)_j\le\sum_{h\in\widehat{\Lambda}}c_h(\Sigma^{-i}\Lambda)[S_h,Y]\le\sum_{h\in\widehat{\Lambda}}c_h(\Sigma^{-i}\Lambda)\|\mathbf{m}\|$$

where $c_h(\Sigma^{-i}\Lambda)$ denotes the Jordan-Hölder multiplicity of the simple S_h in $\Sigma^{-i}\Lambda$. Defining

$$\beta(m)=\max\{(r+1)(d+1),\sum_{h\in\widehat{\Lambda}}c_h(\Sigma^{-i}\Lambda)\mid |i|\le(r+1)(d+1)\},$$

the assertion follows. □

Remark 3.8. Lemma 3.2, 3.6, and 3.7 have analogues where endolength is replaced by dimension; the proofs are identical. Note that [22] has similar estimates for dimensions. However, the proofs given here are different from those in [22] because we need to cover the endolength as well.

4. Transport of families

In this section we discuss the transport of families which arise in the definition of derived tameness. There are two types of functors which we need to study: derived equivalences and Happel's functor. The main results in this section are Propositions 4.8 and 4.9. It is shown that these functors preserve families provided one restricts a given family to some appropriate open subset.

Conventions. Let Λ be a finite locally bounded k-category over some algebraically closed field k. We write $\otimes$ for $\otimes_k$. We denote by R always a finitely generated commutative k-algebra which is a domain, and $K = K(R)$ denotes the quotient field of R. In the sequel we will consider R-Λ-bimodules, which we *always assume finitely generated and free as R-modules*. Thus we view every R-Λ-bimodule as a contravariant functor from Λ into the category of finitely generated free R-modules. Given an R-Λ-bimodule M and $h \in R$, we write M_h for the localized R_h-Λ-bimodule $R_h \otimes_R M$. The same notation applies to complexes of bimodules. For a R-$\widehat{\Lambda}$-bimodule M we always assume $M(x) = 0$ for all but finitely many $x \in \widehat{\Lambda}$.

If M is a finite dimensional $K \otimes \Lambda$-module, then there exists $h \in R$ and an R_h-Λ-sub-bimodule $\tilde{M}$ of M, such that $K \otimes_{R_h} \tilde{M} = M$. We call such a sub-bimodule an R_h-*lattice* of M. A similar result holds for a *bounded* complex of finite dimensional $K \otimes \Lambda$-modules X, i.e. for some $h \in R$ we will find an R_h-lattice $\tilde{X}$ with $K \otimes_{R_h} \tilde{X} = X$. Sometimes we need to localize in several steps and obtain, for example, $(R_h)_g$. In order to avoid clumsy notion we suppress the g and suppose rather that h was chosen adequately.

We usually identify $\operatorname{Spec}(R)(k)$, the k-rational points of $\operatorname{Spec}(R)$, with the isoclasses of simple R-modules, since we are working over an algebraically closed field.

We have for each dimension vector $\mathbf{d} \in \mathbb{N}^{(\widehat{\Lambda})}$ the affine variety $\operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k)$ of $\widehat{\Lambda}$-*module structures* of dimension $\mathbf{d}$, since $\widehat{\Lambda}$ is a locally bounded category. On $\operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}$ acts naturally (by conjugation) the algebraic group $\operatorname{Gl}_{\mathbf{d}}(k) := \prod_{x\in\widehat{\Lambda}} \operatorname{Gl}_{\mathbf{d}(x)}(k)$, i.e. the $\operatorname{Gl}_{\mathbf{d}}(k)$-orbits correspond to the isoclasses of $\widehat{\Lambda}$-modules of dimension $\mathbf{d}$. If $x \in \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}$ we denote by M_x the corresponding $\widehat{\Lambda}$-module with $\mathbf{dim}\, M_x = \mathbf{d}$. See [2] for a systematic discussion of that concept.

Lemma 4.1. *Let M be an R-Λ-bimodule.*

(1) *If N is an R-Λ-bimodule, and $\varphi\colon K \otimes_R M \to K \otimes_R N$ is a morphism of $K \otimes \Lambda$-modules, then there exists $h \in R$ such that φ induces a morphism $\tilde{\varphi}\colon M_h \to N_h$ and $\operatorname{rank}_k(S \otimes_{R_h} \tilde{\varphi}) = \operatorname{rank}_K(\varphi)$ for all simple R_h-modules S.*
(2) *If $K \otimes_R M \cong K \otimes N$ for some $N \in \operatorname{mod}\Lambda$, then for some $h \in R$ we have $S \otimes_{R_h} M_h \cong N$ for all simple R_h-modules S. In particular, if $K \otimes_R M$ is an injective (resp. projective) $K \otimes \Lambda$-module, then $S \otimes_{R_h} M_h$ is a projective (resp. injective) Λ-module for all simple R_h-modules S.*
(3) *Let X be a bounded complex of $K \otimes \Lambda$-modules and Y be a bounded complex of R-Λ-bimodules. If $\pi\colon X \to K \otimes_R Y$ is a quasi-isomorphism, then there exists $h \in R$ and an R_h-lattice $\tilde{X}$ of X such that π induces a morphism $\tilde{\pi}\colon \tilde{X} \to Y_h$*

and $S \otimes_{R_h} \tilde{\pi}$ is a quasi-isomorphism of complexes of Λ-modules for all simple R_h-modules S.

Proof. (1) follows basically from the fact that the function on $\operatorname{Spec}(R_h)(k)$ defined by $[S] \mapsto \operatorname{rank}_k(S \otimes_{R_h} \tilde{\varphi})$ is lower semicontinuous. (2) follows from (1). For (3), note first that we can find for some $h \in R$ an R_h-lattice $\tilde{X}$ of X, such that π induces a morphism $\tilde{\pi}\colon \tilde{X} \to Y_h$. Now, the functions $S \mapsto \dim_k H^i(C(S \otimes_{R_h} \tilde{\pi}))$ (where C is the mapping cone) are upper semicontinuous functions on $\operatorname{Spec}(R_h)(k)$, thus we may assume after some further localization that they are constant. □

We have the following immediate consequence.

Corollary 4.2. *If M is an R-$\widehat{\Lambda}$-bimodule and S' a simple R-module, then there exists $h \in R$ with $S'_h \neq 0$, and a R_h-$\widehat{\Lambda}$-bimodule N such that $S \otimes_{R_h} N \cong \Omega(S \otimes_{R_h} M_h)$ in $\underline{\operatorname{mod}}\, \widehat{\Lambda}$ for all simple R_h-modules S. Moreover $K \otimes_{R_h} N \cong \Omega(K \otimes_R M)$ in $\underline{\operatorname{mod}}\, K \otimes \widehat{\Lambda}$. A similar results holds for Ω^{-1}.*

Proof. From [10, II.4.1] we find a projective-injective $\widehat{\Lambda}$-module P, together with a morphism of R-$\widehat{\Lambda}$-bimodules $p\colon R \otimes P \to M$, such that $S \otimes_R p$ is surjective for all simple R-modules S. Choosing an appropriate basis, we find $h \in R$ and a R_h-lattice N of $\operatorname{Ker}(K \otimes_R p)$ with the required properties. □

Truncation. We consider the left truncation $P^{\geq j}$ of a complex P:

$$(P^{\geq j})^i := \begin{cases} P^i & \text{if } i > j \\ P^j/(\operatorname{Im} d_P^{j-1}) & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

with the obvious restriction of the differentials. We have a natural map of complexes $\gamma_P^{\geq j}\colon P \to P^{\geq j}$ which is a quasi-isomorphism if $H^i(P) = 0$ for $i \leq j$. If $\pi\colon P \to Y$ is a morphism of complexes with $Y^i = 0$ for $i \leq l$, we get for each $m \leq l$ a factorization

$$\pi = \pi^{\geq m} \circ \gamma^{\geq m},$$

where $\pi^{\geq m}\colon P^{\geq m} \to Y^{\geq m} = Y$ is a quasi-isomorphism if π is a quasi-isomorphism.

Lemma 4.3. *Let X be a bounded complex of R-Λ-bimodules with $X^i = 0$ for $i \leq l$. Then there exists for each $m \leq l$ an element $h \in R$ and a morphism of bounded complexes of R_h-Λ-bimodules $\rho^{\geq m}\colon Q \to X_h$ such that*

- *$S \otimes_{R_h} Q^i$ is a projective Λ-module for all $i > m$ and $Q^i = 0$ for $i < m$, and*
- *$S \otimes_{R_h} \rho$ is a quasi-isomorphism*

for all simple R_h-modules S.

Proof. Consider a quasi-isomorphism

$$\pi\colon P \longrightarrow K \otimes_R X$$

of complexes of $K \otimes \Lambda$-modules with P a right bounded complex of projective modules. As above we obtain for $m \leq l$ a quasi-isomorphism

$$\pi^{\geq m}\colon P^{\geq m} \longrightarrow K \otimes_R X.$$

By Lemma 4.1 (3), we find $h \in R$ and a morphism of complexes of R_h-Λ-bimodules

$$\rho^{\geq m}\colon Q \longrightarrow X_h$$

such that $K \otimes_{R_h} \rho^{\geq m} = \pi^{\geq m}$, and $S \otimes_{R_h} \rho^{\geq m}$ is a quasi-isomorphism for each simple R_h-module S. In particular $K \otimes_{R_h} Q = P^{\geq m}$. We conclude from Lemma 4.1 (2) that Q has the desired properties. □

Reduced modules. We study upper-semicontinuity properties of **r-dim** for $\widehat{\Lambda}$-modules, and see that on an appropriate open subset we can replace a family of $\widehat{\Lambda}$-modules by a reduced family of modules.

Lemma 4.4. *Let M be a R-$\widehat{\Lambda}$-bimodule, then the following holds:*

(1) *Suppose that $S' \otimes_R M$ has a projective-injective direct summand P for some simple R-module S'. Then there exists for some $h \in R$ with $S'_h \neq 0$ a R_h-$\widehat{\Lambda}$-bimodule L such that*

$$S \otimes_{R_h} M_h \cong (S \otimes_{R_h} L) \oplus P$$

for all simple R_h-modules S.

(2) *The function*

$$\mathbf{r}\colon \operatorname{Spec}(R)(k) \longrightarrow \mathbb{N}^{(\widehat{\Lambda})}, \quad [S] \mapsto \mathbf{r\text{-}dim}(S \otimes_R M)$$

is componentwise upper-semicontinuous.

Proof. We need only to show (1), since this implies clearly (2). Since P is projective, we have a vector-bundle $\pi\colon E \to \operatorname{Spec}(R)$ with $\pi^{-1}([R/\mathfrak{m}]) = \operatorname{Hom}_{\widehat{\Lambda}}(P, (R/\mathfrak{m}) \otimes_R M)$ for all maximal R-ideals $\mathfrak{m}$. By our hypothesis we find $i \in \pi^{-1}([S'])$ which is a (split) monomorphism. Since being injective is an “open property”, and π is an open map, we find, after replacing possibly $\operatorname{Spec}(R)$ by an open affine neighbourhood $\operatorname{Spec}(R_h)$ of $[S']$, a section σ of π with $\sigma([S])$ a monomorphism for all simple R_h-modules S. In other words, we have a morphism of R_h-$\widehat{\Lambda}$-bimodules $s\colon R_h \otimes P \to L'$, with the property that $S \otimes_{R_h} s$ is a monomorphism for all simple R_h-modules S. Since P is injective, we may assume $L' = L \oplus R_h \otimes P$, using that $K \otimes_R s\colon K \otimes P \to K \otimes_{R_h} L'$ splits as a $K \otimes \widehat{\Lambda}$-morphism. □

Corollary 4.5. *Let M be a R-$\widehat{\Lambda}$-bimodule S, and $n \in \mathbb{Z}$, then the following holds:*

(1) *There exists $h \in R$ and a R_h-$\widehat{\Lambda}$-bimodule N, such that*
   - *$S \otimes_{R_h} N$ is a reduced $\widehat{\Lambda}$-modules for all simple R_h-module S.*
   - *$S \otimes_{R_h} N \cong \Omega^n(S \otimes_{R_h} M_h)$ for all simple R_h-modules S.*

(2) *The map $\operatorname{Spec}(R)(k) \to \mathbb{N}^{(\widehat{\Lambda})}$ which sends $[S]$ to $\mathbf{r\text{-}dim}(\Omega^n(S \otimes_{R_h} M))$ is componentwise upper-semicontinuous. Similarly, for $\mathbf{d} \in \mathbb{N}^{(\mathbb{Z})}$,*

$$\operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k) \longrightarrow \mathbb{N}^{(\mathbb{Z})}, \quad x \mapsto \mathbf{r\text{-}dim}(\Omega^n M_x)$$

is an upper-semicontinuous function.

Proof. We find by Corollary 4.2 some $h \in R$ and a R_h-$\widehat{\Lambda}$-bimodule N with $S \otimes_{R_h} N \cong \Omega^n(S \otimes_{R_h} M_h)$ in $\underline{\operatorname{mod}}\, \widehat{\Lambda}$ for all simple R_g-modules S.

If $T \otimes_{R_h} N$ is not reduced for some simple R_h-module T we find after some localization by Lemma 4.4 (1) an equivalent family of smaller R_h-rank. This process must stop by reasons of dimension, showing (1).

In order to show (2), we use again Corollary 4.2. For any chosen simple R-module S' we may assume $S'_h \neq 0$. Thus, Lemma 4.4 (2) shows, that $\mathbf{r\text{-}dim}\, \Omega^n(- \otimes_{R_h} M) = \mathbf{r\text{-}dim}(- \otimes_{R_h} N)$ is upper-semicontinuous on an open neighbourhood of $[S'] \in \operatorname{Spec} R(k)$.

This shows the first statement of (2). The second follows from that, by working through the irreducible components of $\mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}$. □

Remark. In the preceding proof of (1) we are tempted to split off directly the injective summands of $K \otimes_R M$. However, this might not be sufficient. For instance, $K \otimes_R M$ indecomposable does *not* imply that $S \otimes_R M$ is indecomposable for a simple R-module S, since K is not necessarily algebraically closed; see for example [7].

The image of the Happel functor. Denote by F the Happel functor $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \to \underline{\mathrm{mod}}\,\widehat{\Lambda}$. We say that $M \in \mathrm{mod}\,\widehat{\Lambda}$ belongs to the *image* of F and write $M \in \mathrm{Im}(F)$ if for some $X \in \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ we have $M \cong F(X)$ in $\underline{\mathrm{mod}}\,\widehat{\Lambda}$.

Lemma 4.6. *Let $M \in \mathrm{mod}\,\widehat{\Lambda}$. Then $M \in \mathrm{Im}(F)$ if and only if there exists $m \in \mathbb{N}$ with $(\mathbf{r\text{-}dim}\,\Omega^m M)_{-i} = 0 = (\mathbf{r\text{-}dim}\,\Omega^{-m} M)_i$ for all $i > 0$.*

Proof. We denote by Σ the usual shift in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$. Let first $X \in \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ with $X^i = 0$ for $|i| > m$. Then $(\mathbf{r\text{-}dim}\,F(\Sigma^m X))_{-i} = 0 = (\mathbf{r\text{-}dim}\,F(\Sigma^{-m} X))_i$ for $i > 0$ by the basic properties of the Happel functor. But $F(\Sigma^m X) \cong \Omega^{-m} F(X)$ and $F(\Sigma^{-m} X) \cong \Omega^m F(X)$ in $\underline{\mathrm{mod}}\,\widehat{\Lambda}$.

For the other direction let $M \in \mathrm{mod}\,\widehat{\Lambda}$ and suppose there exists $m \in \mathbb{N}$ with $(\mathbf{r\text{-}dim}\,\Omega^m M)_{-i} = 0 = (\mathbf{r\text{-}dim}\,\Omega^{-m} M)_i$ for all $i > 0$. Let $N = \Omega^m M$ and assume that N is reduced. Thus $N_i = 0$ for $i < 0$ and $(\mathbf{r\text{-}dim}\,\Omega^{-n} N)_i = 0$ for $i > 0$ where $n = 2m$. Clearly, it is sufficient to prove $N \in \mathrm{Im}(F)$, and we do this by induction on n. The case $n = 0$ is clear. For $n > 0$ we find in $\underline{\mathrm{mod}}\,\widehat{\Lambda}$ a distinguished triangle

$$N' \longrightarrow N \longrightarrow N'' \longrightarrow \Omega^{-1} N'$$

with $(N')_i = 0$ for $i \leq 0$ and $(N'')_i = 0$ for $i > 0$, coming from the corresponding natural short exact sequence in $\mathrm{mod}\,\widehat{\Lambda}$. We have also $(\mathbf{r\text{-}dim}\,\Omega^{-n} N')_i = 0$ for $i > 0$. Thus $(\Omega^{-1} N')_i = 0$ for $i < 0$ and $(\mathbf{r\text{-}dim}\,\Omega^{-(n-1)} \Omega^{-1} N')_i = 0$ for $i > 0$. By induction we get $\Omega^{-1} N' \in \mathrm{Im}(F)$. The case $n = 0$ implies $N'' \in \mathrm{Im}(F)$, and we conclude that $N \in \mathrm{Im}(F)$. □

Corollary 4.7. *Let $\mathbf{d} \in \mathbb{N}^{(\widehat{\Lambda})}$. Then $B(\mathbf{d}) = \{x \in \mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k) \mid M_x \in \mathrm{Im}(F)\}$ is an open subset of $\mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k)$ (in the Zariski topology).*

Proof. Fix $\mathbf{d} = (\mathbf{d}_i)_{i \in \mathbb{Z}}$ and suppose that $\mathbf{d}_i = 0$ for $|i| > w$. It follows from Lemma 2.1 that $\mathbf{r\text{-}dim}(\Omega^n M)_{-i} = 0 = \mathbf{r\text{-}dim}(\Omega^{-n} M)_i$ for all $i > w$ and $n \geq 0$, if $M \in \underline{\mathrm{mod}}\,\widehat{\Lambda}$ has $\mathbf{r\text{-}dim}\,M = \mathbf{d}$.

By Corollary 4.5 (2)

$$B_i = \{x \in \mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k) \mid \mathbf{r\text{-}dim}(\Omega^i M_x)_{-l} = 0 = \mathbf{r\text{-}dim}(\Omega^{-i} M_x)_l \text{ for } 1 \leq l \leq w\}$$

is an open subset of $\mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k)$. With the above observation we conclude from Lemma 4.6, that $B(\mathbf{d}) = \bigcup_{i \geq 0} B_i$. □

Remark. (1) It is easy to see that we have in fact $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ for the sets in the above proof. Thus by Hilbert's Basissatz we have $B(\mathbf{d}) = B_m$ for some m (depending on $\mathbf{d}$ and Λ). In particular, $\underline{\mathrm{Hom}}_{\widehat{\Lambda}}(\Lambda, \Omega^{-i} M_x) = 0$ for all $x \in B(\mathbf{d})$ if $|i| > m$ (use $\underline{\mathrm{Hom}}_{\widehat{\Lambda}}(\Lambda, \Omega^{-i} M) \cong \underline{\mathrm{Hom}}_{\widehat{\Lambda}}(\Omega^{\pm 1} \Lambda, \Omega^{-i \pm 1} M)$).

(2) For $X \in \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ we have $H^i(X) \cong \underline{\mathrm{Hom}}_{\widehat{\Lambda}}(\Lambda, \Omega^{-i} F(X)) \cong \mathrm{Ext}^1_{\widehat{\Lambda}}(\Omega^{i-1} \Lambda, F(X))$. Thus it follows from (1), that for given $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$ the set

$$B(\mathbf{d}, \mathbf{h}) = \{x \in \mathrm{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k) \mid M_x \cong F(X) \text{ for some } X \in \mathcal{D}^b(\mathrm{mod}\,\Lambda) \text{ with } \mathbf{h\text{-}dim}\,X = \mathbf{h}\}$$

is locally closed. Here we use that the map $d_N\colon \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}} \to \mathbb{N}, x \mapsto \dim \operatorname{Ext}_{\widehat{\Lambda}}(N, M_x)$ is upper semicontinuous.

Transport of families. We are now in a position to prove the main results of this section.

Proposition 4.8. *Let Λ and Γ be finite dimensional k-algebras and T a bounded complex of Λ-Γ-bimodules. Suppose that X is a bounded complex of R-Λ-bimodules such that there exists $l \in \mathbb{Z}$ with $H^i((S\otimes_R X)\otimes^{\mathbb{L}}_{\Lambda} T) = 0$ for all simple R-modules S if $i \leq l$. Then there exists $h \in R$ and a bounded complex Y of R_h-Γ-bimodules such that*

$$(S \otimes_{R_h} X_h) \otimes^{\mathbb{L}}_{\Lambda} T \cong S \otimes_{R_h} Y \quad \textit{in } \mathcal{D}^b(\operatorname{mod}\Gamma)$$

for all simple R_h-modules S.

Proof. Let $t \in \mathbb{N}$ such that $T^i = 0$ for $|i| > t$ and $j \in \mathbb{Z}$ such $X^i = 0$ for $i \leq j$. Consider $\rho^{\geq m}\colon Q \to X_h$ as in Lemma 4.3 with $m = \min\{j, l-t\}$. For a simple R_h-modules S choose a complex of projective Λ-modules P_S such that ${P_S}^{\geq m} = S \otimes_{R_h} Q$ and $(S \otimes_{R_h} \rho^{\geq m}) \circ \gamma^{\geq m}_{P_S}$ is a quasi-isomorphism. We have by definition

$$(S \otimes_{R_h} X_h) \otimes^{\mathbb{L}}_{\Lambda} T \cong P_S \otimes_{\Lambda} T \cong ((S \otimes_{R_h} X_h) \otimes_{\Lambda} T)^{\geq m} \quad \text{in } \mathcal{D}^b(\operatorname{mod}\Gamma).$$

Now we find (after some further localization) a bounded complex of R_h-Γ-bimodules Y with

$$S \otimes_{R_h} Y \cong ((S \otimes_{R_h} X_h) \otimes_{\Lambda} T)^{\geq m} \quad \text{(as complexes)}$$

for all simple R_h-modules S. ☐

Proposition 4.9. *Let $F\colon \mathcal{D}^b(\operatorname{mod}\Lambda) \to \underline{\operatorname{mod}}\,\widehat{\Lambda}$ be the Happel functor.*

1. *If X is a bounded complex of R-Λ-bimodules, then there exists $h \in R$ and an R_h-$\widehat{\Lambda}$-bimodule M such that*
   - *$F(S \otimes_{R_h} X_h) \cong S \otimes_{R_h} M$ in $\underline{\operatorname{mod}}\,\widehat{\Lambda}$, and*
   - *$S \otimes_{R_h} M$ is reduced*

   for all simple R_h-modules S.
2. *If N is an R-$\widehat{\Lambda}$-bimodule, then precisely one of the following two situations occurs:*
   - *There exist $h \in R$ and a bounded complex of R_h-Λ-bimodules Y such that $F(S \otimes_{R_h} Y) \cong S \otimes_{R_h} N_h$ for all simple R_h-modules S.*
   - *$S \otimes_R X \notin \operatorname{Im}(F)$ for all simple R-modules S.*

Proof. (1) Suppose $X^i = 0$ for $i > r$ and take $m = \max\{0, r\}$. By the dual assertion of Lemma 4.3, there exists for some $h \in R$ a morphism of bounded complexes of R_h-Λ-bimodules

$$\rho\colon X_h \longrightarrow Q$$

such that for all simple R_h-modules S we have a quasi-isomorphism $S \otimes_{R_h} \rho$ and $S \otimes_{R_h} Q^i$ is an injective Λ-module for $i < m$ and $Q^i = 0$ for $i > m$.

Now consider Q as a complex of R_h-$\widehat{\Lambda}$ modules via the natural embedding. Applying again the dual of Lemma 4.3, there exists (after some further localization) a morphism of bounded complexes of R_h-$\widehat{\Lambda}$-bimodules

$$\sigma\colon Q_h \longrightarrow L$$

such that for all simple R_h-modules S we have a quasi-isomorphism $S \otimes_{R_h} \sigma$ of complexes of $\widehat{\Lambda}$-modules and $S \otimes_{R_h} L^i$ an injective $\widehat{\Lambda}$ module for $i < m$ and $L^i = 0$ for $i > m$. Now we can find (after some further localization) by Corollary 4.2 an R_h-$\widehat{\Lambda}$-bimodule M such that

$$\Omega^m(S \otimes_{R_h} L^m) \cong S \otimes_{R_h} M$$

for all simple R_h-modules S. By the description of the Happel functor in [13] our claim follows. Finally, by Corollary 4.5 (1), we may assume that $S \otimes_{R_h} M$ is always reduced.

Note that from our construction also follows $\bar{F}(K \otimes_R X) \cong K \otimes_{R_h} M$, if we write

$$\bar{F} \colon \mathcal{D}^b(\mathrm{mod}(K \otimes \Lambda)) \longrightarrow \underline{\mathrm{mod}}\ (K \otimes \widehat{\Lambda})$$

for the Happel functor for the extended algebra $K \otimes \Lambda$. Just apply to the whole construction the functor $K \otimes_R -$.

(2) By Corollary 4.7 we may assume that for some $h \in R$ we have $S \otimes_{R_h} M \in \mathrm{Im}(F)$ for all simple R_h-modules S. Moreover we may assume that $S \otimes_{R_h} M$ is reduced for all simple R_h-modules S. By the remark after Corollary 4.7 there exists $m \in \mathbb{N}$ such that $(\mathbf{r}\text{-}\mathbf{dim}\, \Omega^m(S \otimes_{R_h} M_h))_{-i} = 0 = (\mathbf{r}\text{-}\mathbf{dim}\, \Omega^{-m}(S \otimes_{R_h} M_h))_i$ for all $i > 0$. We conclude from Corollary 4.2 and 4.5, that there exist R_h-$\widehat{\Lambda}$-bimodules L and N, such that

- $S \otimes_{R_h} L$ and $S \otimes_{R_h} N$ are reduced for all simple R_h-modules S;
- $S \otimes_{R_h} L \cong \Omega^m S \otimes_{R_h} M_h$ and $S \otimes_{R_h} N \cong \Omega^{-m} S \otimes_{R_h} M_h$ in $\underline{\mathrm{mod}}\ \widehat{\Lambda}$ for all simple R_h-modules S;
- $K \otimes_{R_h} L \cong \Omega^m K \otimes_{R_h} M$ and $K \otimes_{R_h} N \cong \Omega^{-m} K \otimes_{R_h} M$ in $\underline{\mathrm{mod}}\ \widehat{\Lambda} \otimes K$.

In particular $(\mathbf{r}\text{-}\mathbf{dim}(\Omega^m K \otimes_{R_h} M))_{-i} = 0 = (\mathbf{r}\text{-}\mathbf{dim}(\Omega^{-m} K \otimes_{R_h} M))_i$ for all $i > 0$, thus $K \otimes_R M \in \mathrm{Im}\, \bar{F}$ by Lemma 4.6. Thus there exists a bounded complex X in $\mathcal{D}^b(\mathrm{mod}(K \otimes \Lambda))$ with $\bar{F}(X) \cong K \otimes_R N$ in $\underline{\mathrm{mod}}\ (K \otimes \widehat{\Lambda})$. For some $h \in R$ we find an R_h-lattice Y of X, and by (1) we may assume that we have an R_h-$\widehat{\Lambda}$-bimodule M such that $S \otimes_{R_h} M$ is reduced and $F(S \otimes_{R_h} Y) \cong S \otimes_{R_h} M$ for all simple R_h-modules S. By the last remark in the proof of (1), we conclude

$$K \otimes_{R_h} M \cong \bar{F}(X) \cong K \otimes_R N \quad \text{ in } \underline{\mathrm{mod}}\ \widehat{\Lambda}.$$

The first and the last module are reduced and therefore isomorphic in $\mathrm{mod}\, \widehat{\Lambda}$. Thus our claim follows from Lemma 4.1 (1). ☐

5. Standard Properties

In this section we prove our main results about derived tame algebras. These results are easy consequences of the results in the previous section and the estimates from Section 3.

Theorem 5.1. *Let Λ and Γ be finite dimensional algebras over an algebraically closed field k. If Λ and Γ are derived equivalent, then Γ is derived tame if and only if Λ is derived tame.*

Proof. By symmetry, it is sufficient to show that Λ derived tame implies Γ derived tame. Let T be a complex of Γ-Λ-bimodules which induces an equivalence of triangulated categories $G := - \otimes^{\mathbb{L}}_{\Lambda} T \colon \mathcal{D}^b(\mathrm{mod}\, \Lambda) \to \mathcal{D}^b(\mathrm{mod}\, \Gamma)$. Such a complex exists by Rickard's theorem [20]. We can assume that T is a *bounded* complex. In fact, the construction (see for example [14, 8.3]) gives a right-bounded complex T with bounded cohomology, and if $H^i(T) = 0$ for $|i| > w$, the left-truncation $T^{\geq n}$ (with $n < -w$) is quasi-isomorphic to T as a complex of bimodules. Using [14, 8.1.4], it is easy to see that $- \otimes^{\mathbb{L}}_{\Lambda} (T^{\geq n})$ induces also an equivalence of the derived categories.

Now fix a homology dimension $\mathbf{h} \in \mathbb{N}^{(\Gamma\times\mathbb{Z})}$. Each object $Y \in \mathcal{D}^b(\operatorname{mod}\Gamma)$ with $\mathbf{h}\text{-}\mathbf{dim}\, Y = \mathbf{h}$ is isomorphic to some $G(X)$ with $|\,\mathbf{h}\text{-}\mathbf{dim}\, X| \leq \gamma(|\mathbf{h}|)$ and $\|\,\mathbf{h}\text{-}\mathbf{dim}\, X\| \leq g\|\mathbf{h}\|$ by Lemma 3.2, see Remark 3.8. Thus, by our hypothesis, there is for some localization $R = k[t]_f$ a finite number of complexes $X_1, \dots, X_n$ of R-Λ-bimodules such that all but finitely many indecomposables $Y \in \mathcal{D}^b(\operatorname{mod}\Gamma)$ with $\mathbf{h}\text{-}\mathbf{dim}\, Y = \mathbf{h}$ are isomorphic to $G(S \otimes_R X_i)$ for some i and some simple R-module S. By Proposition 4.8, we conclude that Γ is derived tame. □

Theorem 5.2. *Let Λ and Γ be finite dimensional algebras over an algebraically closed field k. Suppose there exists an affine (commutative) k-algebra R of (Krull)-dimension ≥ 2 and a bounded complex of R-Λ-bimodules C such that*

- *$S \otimes_R C$ is indecomposable for each simple R-module S, and*
- *$S \otimes_R C \cong S' \otimes_R C$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ implies $S \cong S'$ for any two simple R-modules S and S'.*

Then Λ is not derived tame.

Proof. Let M be a reduced R_h-$\widehat{\Lambda}$ bimodule with $F(S \otimes_{R_h} C_h) \cong S \otimes_{R_h} M$ in $\underline{\operatorname{mod}}\,\widehat{\Lambda}$ for all simple R_h-modules S as in Proposition 4.9 (1), and let $\mathbf{d} := \mathbf{dim}_K K \otimes_{R_h} M$. Moreover, let

$$\mathcal{F} := \{x \in \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k) \mid M_x \text{ indecomposable and } M_x \cong F(X) \text{ in } \underline{\operatorname{mod}}\,\widehat{\Lambda} \\ \text{for some } X \in \mathcal{D}^b(\operatorname{mod}\Lambda) \text{ with } \mathbf{h}\text{-}\mathbf{dim}\, X = \mathbf{d}\},$$

where M_x denotes the $\widehat{\Lambda}$-module corresponding to a point $x \in \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}$. Now assume that Λ is derived tame. We conclude from Proposition 4.9 (1), that there exists $h \in k[t]$ and a $k[t]_h$-$\widehat{\Lambda}$-bimodules $M_1, \dots, M_n$ such that $N \in \operatorname{mod}\widehat{\Lambda}$ indecomposable with $\mathbf{dim}\, N = \mathbf{d}$ and $N \cong F(X)$ for some $X \in \mathcal{D}^b(\operatorname{mod}\Lambda)$ with $\mathbf{h}\text{-}\mathbf{dim}\, X = \mathbf{h}$ implies $N \cong S \otimes_{k[t]_h} M_i$ in $\operatorname{mod}\widehat{\Lambda}$ for some simple $k[t]_h$-module S and some i. Thus, if we denote by $m_i\colon \operatorname{Spec}(k[t]_h)(k) \to \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k)$ the morphism induced by $- \otimes_{k[t]_h} M_i$, we have

$$\mathcal{F} = \bigcup_{i=1}^{n} \operatorname{Gl}_{\mathbf{d}}(k) \operatorname{Im}(m_i)$$

In particular $\mathcal{F}$ is constructible, and the number of parameters (in the sense of [15, 3.4])

$$\mu(\mathcal{F}) := \max\{(\dim\{x \in \mathcal{F} \mid \dim \operatorname{Gl}_{\mathbf{d}} x = i\}) - i \mid i \in \mathbb{N}_0\}$$

is at most 1. The same is trivially true for any constructible subset of $\mathcal{F}$. On the other hand, tensoring with M induces an injective morphism $m\colon \operatorname{Spec}(R_h)(k) \to \operatorname{mod}^{\mathbf{d}}_{\widehat{\Lambda}}(k)$ such that $\operatorname{Im}(m)$ intersects each $\operatorname{Gl}_{\mathbf{d}}$ orbit in at most one point. By construction $\mathcal{F}' = \operatorname{Gl}_{\mathbf{d}}(\operatorname{Im}(m)) \subseteq \mathcal{F}$, moreover $\mathcal{F}'$ is constructible and $\mu(\mathcal{F}') \geq 2$ by a standard argument. Thus Λ is not derived tame. □

Theorem 5.3. *Let Λ be a finite dimensional algebra over an algebraically closed field*

1. *If the repetitive algebra $\widehat{\Lambda}$ is tame, then Λ is derived tame.*
2. *Suppose that Λ is of finite global dimension. Then Λ derived tame implies $\widehat{\Lambda}$ tame.*

Proof. We use the Happel functor F. If $\widehat{\Lambda}$ is tame, let us fix a homology dimension vector $\mathbf{h} \in \mathbb{N}^{(\Lambda\times\mathbb{Z})}$. By Lemma 3.6 (see Remark 3.8) and our hypothesis, there exist $k[t]$-$\widehat{\Lambda}$-bimodules $M_1, \dots, M_n$ such that for all but finitely many indecomposable $X \in \mathcal{D}^b(\operatorname{mod}\Lambda)$ with $\mathbf{h}\text{-}\mathbf{dim}\, X = \mathbf{h}$ we have $F(X) \cong S \otimes_{k[t]} M_i$ in $\underline{\operatorname{mod}}\,\widehat{\Lambda}$ for some i and some simple $k[t]$-module S. By Proposition 4.9 (2), there exists for some localization $R = k[t]_f$ a finite number of bounded complexes $Y_1, \dots, Y_n$ of R-Λ-bimodules such that $F(S \otimes_R Y_i) \cong S \otimes_R (M_i)_f$ in $\underline{\mathrm{mod}}\, \widehat{\Lambda}$ for all i and all simple R-modules S. We conclude that Λ is derived tame since F reflects isomorphisms.

If Λ has finite global dimension F is an equivalence. Suppose now, that Λ is derived tame. Fix a dimension $\mathbf{d} \in \mathbb{N}^{(\mathbb{Z})}$. By Lemma 3.7 (see Remark 3.8), each indecomposable non-projective $\widehat{\Lambda}$-module M with $\mathbf{dim}\, M = \mathbf{d}$ is isomorphic (in $\underline{\mathrm{mod}}\, \widehat{\Lambda}$) to some $F(X)$ for some $X \in \mathcal{D}^b(\mathrm{mod}\, \Lambda)$ with $|\, \mathbf{h\text{-}dim}\, X| \le \beta(|\mathbf{d}|)$ and $\|\, \mathbf{h\text{-}dim}\, X\| \le \beta(|\mathbf{d}|)\|\mathbf{d}\|$. Thus, by our hypothesis, there is for some localization $R := k[t]_f$ a finite number of complexes $X_1, \dots, X_n$ of R-Λ-bimodules such that all but finitely many indecomposable $M \in \mathrm{mod}\, \widehat{\Lambda}$ with $\mathbf{dim}\, M = \mathbf{d}$ are isomorphic in $\underline{\mathrm{mod}}\, \widehat{\Lambda}$ to $F(S \otimes_R X_i)$ for some i and some simple R module S. Using Proposition 4.9 (1), we find appropriate bimodules $M_1, \dots, M_n$ such that all but finitely many indecomposables in $\mathrm{mod}\, \widehat{\Lambda}$ with dimension-vector $\mathbf{d}$ are isomorphic to some $S \otimes_R M_i$. Thus $\widehat{\Lambda}$ is tame.

□

6. Generic complexes

In this section we discuss some basic properties of endofinite and generic complexes.

Endofinite complexes. We present a characterization of endofinite complexes in terms of perfect complexes.

Lemma 6.1. *A complex X in $\mathcal{D}^b(\mathrm{Mod}\, \Lambda)$ is endofinite if and only if for every perfect complex $C \in \mathcal{D}^b(\mathrm{Mod}\, \Lambda)$ the length of $\mathrm{Hom}(C, X)$ over $\mathrm{End}(X)$ is finite.*

Proof. We identify Λ with the stalk complex having $\Lambda = \coprod_{i \in \Lambda} P_i$ in degree 0. Thus $H^i(X) = \mathrm{Hom}(\Sigma^{-i}\Lambda, X)$ for all i and $\langle \Lambda \rangle = \mathcal{K}^b(\mathrm{proj}\, \Lambda)$. The assertion is now an immediate consequence of Lemma 3.1. □

The characterization in Lemma 6.1 shows that the definition of an endofinite complex in $\mathcal{D}^b(\mathrm{Mod}\, \Lambda)$ coincides with the definition given in [16] in terms of small objects in the category $\mathcal{D}(\mathrm{Mod}\, \Lambda)$ of unbounded complexes. This follows from the fact that the small objects in $\mathcal{D}(\mathrm{Mod}\, \Lambda)$ are precisely the perfect complexes (cf. [19]). Recall that an object X is *small* if the representable functor $\mathrm{Hom}(X, -)$ preserves arbitrary coproducts. This observation has an interesting consequence.

Proposition 6.2. *Every endofinite complex X has a decomposition $X = \coprod_{i \in I} X_i$ into indecomposable objects with local endomorphism rings which is essentially unique.*

Proof. See Theorem 1.2 in [16]. □

Derived equivalence. It has been shown in Theorem 5.1 that a derived equivalence preserves the derived representation type. We present now the analogous result for the representation type which is defined in terms of generic complexes. We need the following lemma.

Lemma 6.3. *Let $G\colon \mathcal{D}^b(\mathrm{Mod}\, \Gamma) \to \mathcal{D}^b(\mathrm{Mod}\, \Lambda)$ be an equivalence of triangulated categories, and let X be an object in $\mathcal{D}^b(\mathrm{Mod}\, \Gamma)$.*

1. *X is endofinite if and only if $G(X)$ is endofinite.*
2. *X is generic if and only if $G(X)$ is generic.*

Proof. The equivalence G restricts to equivalences

$$\mathcal{K}^b(\operatorname{proj}\Gamma) \xrightarrow{\sim} \mathcal{K}^b(\operatorname{proj}\Lambda) \quad \text{and} \quad \mathcal{D}^b(\operatorname{mod}\Gamma) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{mod}\Lambda)$$

(cf. [19]). The assertion of the lemma follows from this observation and Lemma 6.1. □

Proposition 6.4. *Let $G\colon \mathcal{D}^b(\operatorname{Mod}\Gamma) \to \mathcal{D}^b(\operatorname{Mod}\Lambda)$ be an equivalence of triangulated categories. Then Γ is generically derived tame if and only if Λ is generically derived tame.*

Proof. It follows from Lemma 6.3 that G induces a bijection between the isomorphism classes of generic complexes in $\mathcal{D}^b(\operatorname{Mod}\Gamma)$ and $\mathcal{D}^b(\operatorname{Mod}\Lambda)$. Now fix a family $\mathbf{n} = (\mathbf{n}_i)_{i\in\mathbb{Z}}$ of vectors in $\mathbb{N}^{(\Lambda)}$. We apply Lemma 3.2. If Λ is generically derived tame, there are only finitely many generic complexes Y in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ such that $|\,\mathbf{h\text{-}endol}\,Y| \leq \gamma(|\mathbf{n}|)$ and $\|\,\mathbf{h\text{-}endol}\,Y\| \leq g\,\|\mathbf{n}\|$. It follows that $\mathcal{D}^b(\operatorname{Mod}\Gamma)$ has only finitely many generic complexes X with $\mathbf{h\text{-}endol}\,X = \mathbf{n}$. Thus Γ is generically derived tame. □

7. Generic modules

Let Λ be a locally bounded k-category. We study the representation type of Λ in terms of generic modules. This is based on work of Crawley-Boevey for finite dimensional algebras [4]. Recall that an endofinite module X is *generic* if X is indecomposable and not finitely presented.

Socle endolength. In order to compute the endolength of a module, it is often useful to work with the socle endolength. Let us introduce this concept. Given two Λ-modules X and Y, we define

$$[X,Y] = \operatorname{length}_{\operatorname{End}(Y)} \operatorname{Hom}(X,Y).$$

For instance, Yoneda's lemma implies that

$$\mathbf{endol}\,X = ([P_i,X])_{i\in\Lambda}.$$

The canonical epimorphism $P_i \to S_i$ induces an isomorphism $\operatorname{Hom}(S_i,X) \cong (\operatorname{soc} X)(i)$ of $\operatorname{End}(X)$-modules and we call therefore

$$\mathbf{s\text{-}endol}\,X = ([S_i,X])_{i\in\Lambda}$$

the *socle endolength* of X. Let us compare $\mathbf{s\text{-}endol}\,X$ and $\mathbf{endol}\,X$. To this end denote for each pair $i,j\in\Lambda$ by $c_{ij} = c_i(P_j)$ the Jordan-Hölder multiplicity of the simple S_i in P_j. Note that the matrix $(c_{ij})_{i,j\in\Lambda}$ has only finitely many non-zero entries in each row and each column since Λ is locally bounded.

Lemma 7.1. *Let X be a Λ-module and $j\in\Lambda$. Then*

$$[S_j,X] \leq [P_j,X] \quad \textit{and} \quad [P_j,X] \leq \sum_{i\in\Lambda} c_{ij}[S_i,X].$$

In particular, X is endofinite if and only if $[S_i,X]$ is finite for all $i\in\Lambda$.

Proof. The first inequality is trivial, and the second one follows by induction on the composition length of P_j. □

We get two inequalities for the endolength of a module X. The first one is

$$\mathbf{s\text{-}endol}\,X \leq \mathbf{endol}\,X$$

and the second involves the matrix $\mathbf{C} = (c_{ij})_{i,j\in\Lambda}$.

Lemma 7.2. *The assignment* $\mathbf{n} \mapsto \mathbf{nC}$ *induces a map* $\sigma\colon \mathbb{N}^{(\Lambda)} \to \mathbb{N}^{(\Lambda)}$ *such that*

$$\mathbf{endol}\, X \leq \sigma(\mathbf{s\text{-}endol}\, X) \quad \textit{for all } X \in \operatorname{Mod} \Lambda.$$

Proof. We have $\mathbf{nC} \in \mathbb{N}^{(\Lambda)}$ for $\mathbf{n} \in \mathbb{N}^{(\Lambda)}$ since $\mathbf{C}$ is row-finite and column-finite. The rest follows from Lemma 7.1. □

Generic tameness. Crawley-Boevey's definition of a generically tame algebra [4] has an obvious analogue for locally bounded categories.

Definition 7.3. A locally bounded category Λ is called *generically tame* if for all $\mathbf{n} \in \mathbb{N}^{(\Lambda)}$ there are only finitely many isomorphism classes of generic Λ-modules X such that $\mathbf{endol}\, X = \mathbf{n}$.

Proposition 7.4. *Let* Λ *be a locally bounded category. Then the following conditions are equivalent:*

(1) Λ *is generically tame.*
(2) *For all* $\mathbf{n} \in \mathbb{N}^{(\Lambda)}$ *there are only finitely many isomorphism classes of generic* Λ*-modules* X *such that* $\mathbf{s\text{-}endol}\, X = \mathbf{n}$*.*
(3) *Every finite full subcategory of* Λ *is generically tame.*

Proof. (1) $\Rightarrow$ (2) Fix a vector $\mathbf{n} \in \mathbb{N}^{(\Lambda)}$. Using Lemma 7.2, it follows that there are only finitely many generic Λ-modules X with $\mathbf{s\text{-}endol}\, X = \mathbf{n}$ if Λ is generically tame. Thus (2) holds.

(2) $\Rightarrow$ (3) Let Γ be a finite full subcategory of Λ. The restriction functor $R\colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ has a fully faithful right adjoint $Q\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ which is defined by $Q(X)(i) = \operatorname{Hom}(R(P_i), X)$. Note that $R \circ Q \cong \operatorname{Id}$. Now fix a vector $\mathbf{n} = (n_i)_{i\in\Gamma}$ and let X be a generic Γ-module with $\mathbf{endol}\, X = \mathbf{n}$. We use the adjointness isomorphism

$$\operatorname{Hom}(S_i, Q(X)) \cong \operatorname{Hom}(R(S_i), X).$$

For $i \in \Lambda \setminus \Gamma$ we get

$$[S_i, Q(X)] = [R(S_i), X] = 0$$

since $R(S_i) = 0$. For $i \in \Gamma$ we get

$$[S_i, Q(X)] = [R(S_i), X] = [S_i, X] \leq [P_i, X] = n_i$$

since $R(S_i) = S_i$. This shows that $Q(X)$ is generic. In fact, $Q(X)$ is endofinite by Lemma 7.1, but not finitely presented since $X \cong R(Q(X))$ is not finitely presented. Assuming (2), it follows that there are only finitely many generic Γ-modules X such that $\mathbf{endol}\, X = \mathbf{n}$. Therefore Γ is generically tame.

(3) $\Rightarrow$ (1) Fix a vector $\mathbf{n} \in \mathbb{N}^{(\Lambda)}$ and consider the finite full subcategory $\Gamma = \{i \in \Lambda \mid n_i \neq 0\}$. Clearly, Λ has only finitely generic modules X with $\mathbf{endol}\, X = \mathbf{n}$ if Γ is generically tame. It follows that Λ is generically tame. □

In [4], Crawley-Boevey has shown that a finite dimensional algebra Λ over an algebraically closed field has tame representation type (in the sense of Drozd) if and only if Λ is generically tame. This result carries over to arbitrary locally bounded categories. To this end recall that Dowbor and Skowroński have shown that the tame representation type of a locally bounded category is characterized by the fact that every finite full subcategory is of tame representation type [5].

Corollary 7.5. *A locally bounded category* Λ *over an algebraically closed field has tame representation type if and only if* Λ *is generically tame.*

The repetitive category. It has been shown in Theorem 5.3 that the derived representation type of a locally bounded category Λ coincides with the representation type of the repetitive category $\widehat{\Lambda}$. We present now the analogous result for the representation type which is defined in terms of generic complexes and modules.

Theorem 7.6. *Let Λ be a finite locally bounded category.*

(1) *If $\widehat{\Lambda}$ is generically tame, then Λ is generically derived tame*
(2) *Suppose that Λ has finite global dimension. If Λ is generically derived tame, then $\widehat{\Lambda}$ is generically tame.*

Proof. We use Happel's functor $F\colon \mathcal{D}^b(\operatorname{Mod}\Lambda) \to \underline{\operatorname{Mod}}\,\widehat{\Lambda}$ which restricts to a functor $\mathcal{D}^b(\operatorname{mod}\Lambda) \to \underline{\operatorname{mod}}\,\widehat{\Lambda}$. It follows from Lemma 3.6 that $F(X)$ is endofinite for every endofinite complex X. Thus F induces an injective map from the set of isomorphism classes of generic complexes in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ into the set of isomorphism classes of generic $\widehat{\Lambda}$-modules X such that the support $\{i \in \widehat{\Lambda} \mid X(i) \neq 0\}$ is finite. Moreover, Lemma 3.7 shows that this map is bijective if Λ has finite global dimension. The assertion of (1) follows from the estimate given in Lemma 3.6, and (2) follows from the estimate in Lemma 3.7. ☐

8. Refinements of tame representation type

For a finite dimensional algebra there are two refinements of tame representation type: domestic type and polynomial growth. In this section we discuss analogous refinements of the derived representation type.

Definitions. Given a family $\mathbf{n} = (\mathbf{n}_i)_{i\in\mathbb{Z}}$ of vectors $\mathbf{n}_i \in \mathbb{N}^{(\Lambda)}$ with $\mathbf{n}_i = \mathbf{0}$ for almost all i, we write

$$\|\mathbf{n}\| = \max\{\sum_{j\in\Lambda}(\mathbf{n}_i)_j \mid i \in \mathbb{Z}\}.$$

For each $w \in \mathbb{N}$ let $\mathcal{D}^b_w(\operatorname{Mod}\Lambda)$ denote the full subcategory formed by the objects X in $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ such that $H^i(X) = 0$ for all $|i| > w$.

Definition 8.1. Let Λ be a finite locally bounded category.

(1) Λ is derived tame of *domestic type* if for all $w \in \mathbb{N}$ there exists $N \in \mathbb{N}$ (depending on w) such that for all $n \in \mathbb{N}$ there are N bounded complexes of R-Λ-bimodules (see Definition 1.1) such that all but finitely many isomorphism classes of indecomposable objects X in $\mathcal{D}^b_w(\operatorname{mod}\Lambda)$ with $\|\,\mathbf{h\text{-}dim}\,X\| \leq n$ arise by tensoring with a simple R-module.
(2) Λ is derived tame of *polynomial growth* if for all $w \in \mathbb{N}$ there exist $d, k \in \mathbb{N}$ (depending on w) such that for all $n \in \mathbb{N}$ there are dn^k bounded complexes of R-Λ-bimodules (see Definition 1.1) such that all but finitely many isomorphism classes of indecomposable objects X in $\mathcal{D}^b_w(\operatorname{mod}\Lambda)$ with $\|\,\mathbf{h\text{-}dim}\,X\| \leq n$ arise by tensoring with a simple R-module.

Definition 8.2. Let Λ be a finite locally bounded category.

(1) Λ is generically derived tame of *domestic type* if for all $w \in \mathbb{N}$ there are only finitely many isomorphism classes of generic complexes in $\mathcal{D}^b_w(\operatorname{Mod}\Lambda)$.
(2) Λ is generically derived tame of *polynomial growth* if for all $w \in \mathbb{N}$ there exist $d, k \in \mathbb{N}$ (depending on w) such that for all $n \in \mathbb{N}$ there are at most dn^k isomorphism classes of generic complexes X in $\mathcal{D}^b_w(\operatorname{Mod}\Lambda)$ with $\|\,\mathbf{h\text{-}endol}\,X\| \leq n$.

Derived equivalence. It has already been shown that a derived equivalence preserves (generic) derived tameness. A refinement of the proof shows that domestic type and polynomial growth are preserved as well.

Proposition 8.3. *Let Λ and Γ be finite locally bounded categories over an algebraically closed field and suppose that Λ and Γ are derived equivalent.*

(1) Λ *is (generically) derived tame of domestic type if and only if* Γ *is (generically) derived tame of domestic type.*
(2) Λ *is (generically) derived tame of polynomial growth if and only if* Γ *is (generically) derived tame of polynomial growth.*

Proof. Adapt the proof of Theorem 5.1 and Proposition 6.4, using Lemma 3.2. □

The repetitive category. We have shown that a locally bounded category Λ of finite global dimension is (generically) derived tame if and only if the repetitive category $\widehat{\Lambda}$ is (generically) tame. Next we prove the analogue for domestic type and polynomial growth. Let us give the definition of domestic type and polynomial growth for a locally bounded category in terms of generic modules.

Definition 8.4. Let Λ be a locally bounded category and denote for each finite set S of objects in Λ by $\operatorname{Mod}_S \Lambda$ the full subcategory formed by the objects X in $\operatorname{Mod}\Lambda$ such that $X(i) = 0$ for all $i \notin S$.

(1) Λ is generically tame of *domestic type* if for all finite $S \subseteq \Lambda$ there are only finitely many isomorphism classes of generic modules in $\operatorname{Mod}_S \Lambda$.
(2) Λ is generically tame of *polynomial growth* if for all finite $S \subseteq \Lambda$ there exist $d, k \in \mathbb{N}$ (depending on S) such that for all $n \in \mathbb{N}$ there are at most dn^k isomorphism classes of generic modules X in $\operatorname{Mod}_S \Lambda$ with $\|\,\mathbf{endol}\, X\| \leq n$.

Note that a locally bounded category Λ over an algebraically closed field is tame of domestic type (resp. of polynomial growth) if and only if Λ is generically tame of domestic type (resp. of polynomial growth). This is a variant of Corollary 7.5.

Proposition 8.5. *Let Λ be a finite locally bounded category over an algebraically closed field and suppose that Λ is of finite global dimension.*

(1) Λ *is (generically) derived tame of domestic type if and only if* $\widehat{\Lambda}$ *is (generically) tame of domestic type.*
(2) Λ *is (generically) derived tame of polynomial growth if and only if* $\widehat{\Lambda}$ *is (generically) tame of polynomial growth.*

Proof. Adapt the proof of Theorem 5.3 and Theorem 7.6, using Lemma 3.6 and 3.7. □

Examples. Tame hereditary algebras are derived tame of domestic type, while canonical algebras of tubular type are derived tame of polynomial growth. Skewed gentle algebras [9], which are not piecewise hereditary are either derived of discrete type, or derived tame, but not of polynomial growth. As we have seen, the same applies to algebras which are derived equivalent to one of these classes.

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