Derived tame algebras and Euler-forms

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with an Appendix in collaboration with B. Keller

1 Introduction

There is a long record on the close relation between the representation theory of finite-dimensional algebras with finite global dimension and the respective Euler form, starting with Gabriel's theorem [8], [4]: A hereditary algebra Γ is of finite type if and only if the Euler-form $\boldsymbol{\chi}_\Gamma$ is positive definite. Ringel's book [21] can be seen as an overview of the corresponding tame case. Some more recent references include [2],[18].

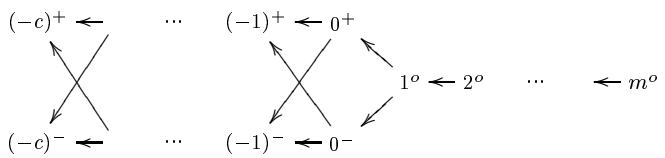
If we are interested in a non-hereditary algebra Γ, we should impose some non-degeneracy condition like "simply connected", in order to avoid trivial effects - the Euler form depends after all only on the Cartan-matrix of Γ. Moreover, it seems to be natural to expect results about the derived category $\mathcal{D}^b(\Gamma)$ rather than on the module category, since the essential properties of $\boldsymbol{\chi}_\Gamma$ are invariant under derived equivalence. We shall show the following theorem:

Theorem *Let* k *be an algebraically closed field and* Γ *a weakly simply connected* k*-algebra, then the following holds:*

(a) Γ *is derived tame if and only if it is either piecewise hereditary, or it is derived equivalent to semi-chain* $\Lambda_{c,m}$ *for* $c \geq 2$ *(see below). In particular, the Euler-form of* Γ *is non-negative in that situation.*

(b) Suppose that the Euler-form χ_Γ *of* Γ *is non-negative, then* Γ *is derived tame if* $\operatorname{corank} \chi_\Gamma \leq 2$ *or if* Γ *is a tree algebra.*

Here, $\Lambda_{c,m}$ denotes for $(c,m) \in (\mathbb{N}_0 \times \mathbb{N}_0) \setminus \{(0,0),(0,1),(1,0)\}$ the poset algebra corresponding to

$\mathcal{Q}_{c,m} :=$ 

which we call also semi-chain algebra.

By definition, Γ is *weakly simply connected*, if it can be constructed inductively by one-point extensions and co-extensions with (finite-dimensional) indecomposable modules, starting from the ground field.

We write $\mathcal{D}^b(\Gamma)$ for the bounded derived category of the category of finite-dimensional Γ-modules, and denote by $\mathbb{N}_0^{(\mathbb{Z})}$ the functions $\mathbb{Z} \longrightarrow \mathbb{N}_0$ which have finite support. Γ is called *derived tame*, if for each $\mathbf{r} = (h^i)_{i\in\mathbb{Z}} \in \mathbb{N}_0^{(\mathbb{Z})}$ the indecomposable objects $X \in \mathcal{D}^b(\Gamma)$ with $(\dim_{\mathrm{k}} H^i(X))_{i\in\mathbb{Z}} = \mathbf{r}$ can be parametrised by a finite number of one-parameter families. This definition is stable under derived equivalence. In our situation of finite global dimension, Γ is derived tame, if and only if the repetitive algebra [15] $\widehat{\Gamma}$ is tame, being this the original definition [17]. See [11] for details.

As the definition of weakly simply connected suggests, our proof of the theorem is based on induction. When Γ is piecewise hereditary, the induction step is already well-known, see 2.3. Otherwise we have to reduce the problem to question about the derived category of some semi-chain $\Lambda_{c,m}$. This is done in section 3 with the technique of derived subspace problems, which is developed in the Appendix.

Finally, we understand the derived categories $\mathcal{D}^b(\Lambda_{c,m})$ well enough since it is by Happel's theorem [12] equivalent to the stable category $\widehat{\Lambda}_{c,m}$-$\underline{\mathrm{mod}}$. Now, $\widehat{\Lambda}_{c,m}$ is quasi-clannish [10], i.e. we have a good description of the indecomposable objects [6] and the homomorphisms between them [9].

Notice that as a special case of the above theorem, a tree algebra is derived tame if and only if the Euler-form is non-negative. This was shown independently and with quite different methods by Th. Brüstle [5].

2 Preliminaries

2.1 (Weakly simply connected algebras) We call a finite-dimensional k-algebra Γ *weakly simply connected (wsc)* if there is a sequence $\mathrm{k} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n = \Gamma$ of algebras and a sequence of indecomposable modules $M_i \in \Gamma_i$-mod, such that $\Gamma_{i+1} \cong \Gamma_i[M_i]$ or $\Gamma_{i+1} \cong [M_i]\Gamma_i$ for $i = 1, \ldots n-1$.

Here $\Gamma[M] = \left(\begin{smallmatrix} \mathrm{k} & 0 \\ M & \Gamma \end{smallmatrix}\right)$ is the one-point extension of Γ with the module $M \in \Gamma$-mod and $[M]\Gamma = \left(\begin{smallmatrix} \mathrm{k} & DM \\ 0 & \Gamma \end{smallmatrix}\right)$ the one-point co-extension by M with $DM = \mathrm{Hom}_{\mathrm{k}}(M, \mathrm{k}) \in$ mod-Γ. See for example [21, 2.5, 4.10] for more details.

Note, that trivially with Γ also the subalgebras Γ_i are wsc. It is easy to see that strongly simply connected algebras [23], and in particular tree algebras, are wsc.

The semi-chain algebras $\Lambda_{c,m}$ are wsc, but not strongly simply connected for $c \geq 1$.

2.2 (Quadratic forms) Let $C \in \mathbb{Z}^{n\times n}$ be invertible (over $\mathbb{Z}$). We define a bilinear form

$$\mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z},\ (\mathbf{v}, \mathbf{w}) \mapsto \langle \mathbf{v}, \mathbf{w}\rangle_C := \mathbf{v}^T C^{-T} \mathbf{w}$$

and a quadratic form $\boldsymbol{\chi}_C \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}, \mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{v}\rangle_C$. Moreover we define a linear transformation

$$\boldsymbol{\phi}_C \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n, \mathbf{v} \mapsto -C^T C^{-1} \mathbf{v}.$$

Finally, for $\mathbf{v} \in \mathbb{Z}^n$ we set $C[\mathbf{v}] := \left(\begin{smallmatrix} C & \mathbf{v} \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{Z}^{(n+1)\times(n+1)}$ and $[\mathbf{v}]C := \left(\begin{smallmatrix} C & 0 \\ \mathbf{v} & 1 \end{smallmatrix}\right) \in \mathbb{Z}^{(n+1)\times(n+1)}$. We write $\boldsymbol{\chi}_C \geq 0$ if $\boldsymbol{\chi}_C(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{Z}^n$. In this situation $\mathrm{rad}\, \boldsymbol{\chi}_C = \{\mathbf{x} \in \mathbb{Z}^n \mid \boldsymbol{\chi}_C(\mathbf{x}) = 0\}$.

Remark: If C is the Cartan-matrix of a triangular algebra Λ, then $\boldsymbol{\chi}_C$ is the Euler-form of Λ, and $\boldsymbol{\phi}_C$ the Coxeter-transformation for Λ. If moreover $M \in \Lambda$-mod with $\mathbf{dim}\, M = \mathbf{v}$ then $C[\mathbf{v}]$ is the Cartan-matrix of the one-point extension $\Lambda[M]$.

The following is an exercise in linear algebra and (differential-) calculus:

Lemma *Let $C \in \mathbb{Z}^{n\times n}$ as above, and $\mathbf{v} \in \mathbb{Z}^n$. suppose that $\boldsymbol{\chi}_C \geq 0$ then we have*

$$\boldsymbol{\chi}_{C[v]} \geq 0 \Longleftrightarrow \begin{cases} \textit{(i)} \langle \mathbf{v}, \operatorname{rad} \boldsymbol{\chi}_C \rangle_C = 0 \\ \textit{(ii)} \exists \mathbf{y} \in \mathbb{Q}^n \textit{ with } (C^{-T} + C^{-1})\mathbf{y} = C^{-1}\mathbf{v} \textit{ and } \boldsymbol{\chi}_C(\mathbf{y}) \leq 1 \end{cases}$$

Note, that condition (i) implies, that there exists a solution $\mathbf{y}$ for the equation in (ii), and the value $\boldsymbol{\chi}_C(\mathbf{y})$ does not depend on the choice of the solution.

Sometimes it is convenient to replace condition (ii) by the following equivalent condition

$$\exists \mathbf{y} \in \mathbb{Q}^n \text{ with } (\boldsymbol{\phi}_C - \mathbb{1})\mathbf{y}' = \mathbf{v} \text{ and } \boldsymbol{\chi}_C(\mathbf{y}') \leq 1$$

(where $\boldsymbol{\phi}_C \mathbf{y}' = \mathbf{y}$).

Proof: Clearly

$$\boldsymbol{\chi}_{C[\mathbf{v}]} \left(\begin{smallmatrix} \mathbf{x} \\ x \end{smallmatrix}\right) = \boldsymbol{\chi}_C(\mathbf{x}) - x\langle \mathbf{v}, \mathbf{x} \rangle_C + x^2,$$

thus $\boldsymbol{\chi}_{C[\mathbf{v}]} \geq 0$ if and only if the following quadratic form is no-negative:

$$\chi' \colon \mathbb{Q}^n \longrightarrow \mathbb{Q}, \mathbf{x} \mapsto \boldsymbol{\chi}_C(\mathbf{x}) - 1/4 \langle \mathbf{v}, \mathbf{x} \rangle_C{}^2$$

This is obviously the case if and only if $\langle \operatorname{rad} \boldsymbol{\chi}_C, \mathbf{v} \rangle_C = 0$, and the minimum of $\boldsymbol{\chi}_C$ on the hyper-plane

$$H := \{\mathbf{y} \in \mathbb{Q}^n \mid \langle \mathbf{v}, \mathbf{y} \rangle_C = \langle \mathbf{v}, C^{-1}\mathbf{v} \rangle_C\}$$

is at least $1/4\langle \mathbf{v}, C^{-1}\mathbf{v} \rangle_C^2$. This gives condition (ii) □

Corollary *Suppose $\boldsymbol{\chi}_C \geq 0$, then*

$$\boldsymbol{\chi}_{C[\mathbf{v}]} \geq 0 \Longleftrightarrow \boldsymbol{\chi}_{C[\boldsymbol{\phi}_C \mathbf{v}]} \geq 0 \Longleftrightarrow \boldsymbol{\chi}_{[\mathbf{v}]C} \geq 0$$

2.3 (Piecewise hereditary algebras) Recall, that by definition [14] a piecewise hereditary algebra Γ is either derived equivalent to a hereditary algebra or to a canonical algebra. Thus the Euler-form $\boldsymbol{\chi}_\Gamma$ of Γ is non-negative if and only if it is derived equivalent to a hereditary algebra of Dynkin or Euclidian type, or if it is derived equivalent to a canonical algebra of tubular type. We have the following well-known result, which can be derived with [3] (or more generally the Appendix) from [16] and [2]:

Proposition *Let Γ be a piecewise hereditary algebra with non-negative Euler-form and $M \in \Gamma$-mod indecomposable. Then the one-point extension $\Gamma[M]$ is derived-tame if and only if the Euler-form of $\Gamma[M]$ is non-negative. In this situation $\Gamma[M]$ is either*

(i) piecewise hereditary

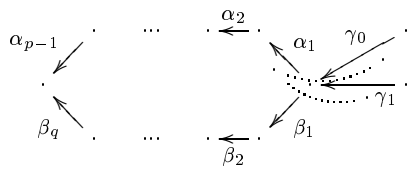
(ii) derived equivalent to $\Lambda_{2,m}$ and Γ is derived equivalent to $\Lambda_{1,m+1}$ (i.e. of type $\widetilde{\mathbb{D}}_{m+4}$).

(iii) derived equivalent to $\Lambda^*_{2,p,q}$ and Γ is derived equivalent to a canonical algebra of type (p,q).

In any of these cases also the trivial extension of $\Gamma[M]$ is tame.

Here, $\Lambda^*_{2,p,q}$ denotes the following clannish (in fact skewed gentle [10]) algebra, which we present as quiver with relations (dotted lines indicate zero-relations, i.e. here we have the only relations $\alpha_1\gamma_0 = \alpha_1\gamma_1$):

$\Lambda^*_{1,p,q}$:



Remark: If (1) Γ is piecewise hereditary of type $\widetilde{\mathbb{A}}_{p,q}$, then Γ is not weakly simply connected by the description of these algebras given in [1].

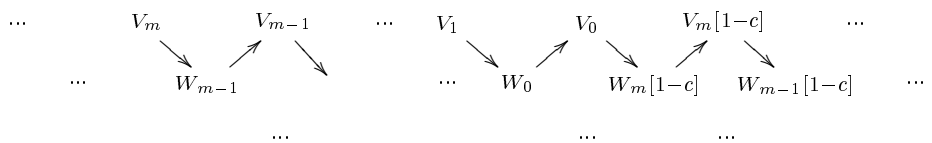
(2) The proposition remains valid, if we replace "one-point extensions" by "one-point coextension" by the above corollary and the fact that $\Gamma[M]$ and $[M]\Gamma$ are derived equivalent.

3 Reduction to Semi-chains

In this section let Γ denote always an algebra which is derived equivalent to some semi-chain $\Lambda_{c,m}$ for $c \geq 1$.

3.1 Proposition *If $c \geq 2$, the Auslander-Reiten quiver of $\mathcal{D}^b(\Gamma)$ has the following structure:*

There are precisely $c-1$ components of type $\mathbb{Z}\mathbb{A}_\infty$; up to shift the mouth of these components has the following shape:



In particular these components are conjugate under shift.

The remaining "discrete" components are of type $\mathbb{Z}\mathbb{A}^\infty_\infty$ and $\mathbb{Z}\mathbb{D}_\infty$; moreover there are infinitely many families $(\mathcal{T}_\mathfrak{m})_{\mathfrak{m}\in\mathrm{Spec}(\mathrm{k}[T,T^{-1}])}$ of homogeneous tubes, and infinitely many families $(\mathcal{T}_\mathfrak{m})_{\mathfrak{m}\in\mathrm{Spec}(\mathrm{k}[T])}$ with $\mathcal{T}_\mathfrak{m}$ a tube of rank 2 for $\mathfrak{m} \in \{(T),(T-1)\}$ and homogeneous otherwise.

Clearly, we have to prove these statements only for $\mathcal{D}^b(\Lambda_{c,m})$, see 4.7.

Remark: For $c = 1$ the description of $\mathcal{D}^b(\Lambda_{1,m})$ as the derived category of a hereditary algebra of type $\widetilde{\mathbb{D}}_{m+3}$ is well-known. In particular, if $m \geq 2$, there is in the Auslander-Reiten quiver up to shift a unique tube of rank $m+1$; we can in particular consider this description as a degenerate case of the above description of the $\mathbb{Z}\mathbb{A}_\infty$-components. If $m = 1$, there are up to shift 3 tubes of rank 2, which can be described similarly.

3.2 Definition If $M \in \mathcal{D}^b(\Gamma)$ is indecomposable, we will say, that *M is in position i* if $[M]$ lies in the i-th layer of a component of type $\mathbb{Z}\mathbb{A}_\infty$. If $c = 1$ (and $m \geq 1$), we shall say, that *M is in position i*, if it lies in the i-the layer of a tube of rank $m+1$ in $\mathcal{D}^b(\Gamma)$. We will say that

- Γ is *controlled*, if $M \in \Gamma$-mod indecomposable and $\mathbf{dim}\, M = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma)$ in position 1 resp. 2 implies that $M \in \mathcal{D}^b(\Gamma)$ is in position 1 resp. 2.
- Γ is 1-*controlled*, if $M \in \Gamma$-mod indecomposable and $\mathbf{dim}\, M = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma)$ in position 1 implies that $M \in \mathcal{D}^b(\Gamma)$ is in position 1.
- Γ is *weakly controlled*, if for $M \in \Gamma$-mod indecomposable and thin (i.e. $(\mathbf{dim}\, M)_i \in \{0,1\}$ for all vertices i) with $\mathbf{dim}\, M = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma)$ in position 1 resp. 2 implies that $M \in \mathcal{D}^b(\Gamma)$ is in position 1 resp. 2.

Remark: It is well-known that Γ is controlled (resp. 1-controlled) for $c = 1$ and $m \geq 2$ (resp. $m = 1$).

In case $c \geq 2$ there exists $Y \in \mathcal{D}^b(\Gamma)$ indecomposable, with $\mathbf{dim}\, Y = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma)$ in position 1, with Y lying in a $\mathbb{Z}\mathbb{A}_\infty^\infty$-component of the Auslander-Reiten quiver of $\mathcal{D}^b(\Gamma)$.

3.3 Proposition *Let $c \geq 2$ and denote by C is the Cartan-matrix of Γ. Suppose $X \in \mathcal{D}^b(\Gamma)$ is indecomposable with $\mathbf{v} := \mathbf{dim}\, X \neq 0$. Then we have:*

$$\boldsymbol{\chi}_{C[\mathbf{v}]} \geq 0 \Longleftrightarrow \exists Y \in \mathcal{D}^b(\Gamma) \text{ in position 1 or 2 with } \mathbf{dim}\, Y = \mathbf{v}.$$

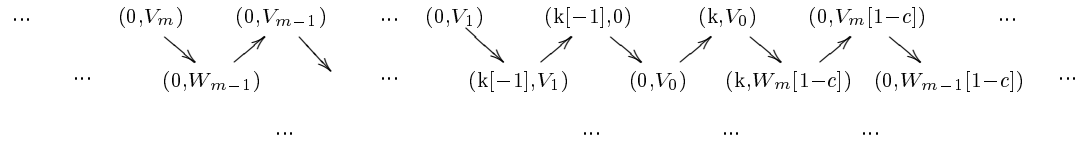
In this situation $\operatorname{corank} \boldsymbol{\chi}_{C[\mathbf{v}]} = \operatorname{corank} \boldsymbol{\chi}_C$ *if Y is in position 1 and* $\operatorname{corank} \boldsymbol{\chi}_{C[\mathbf{v}]} = \operatorname{corank} \boldsymbol{\chi}_C + 1$ *if Y is in position 2.*

Clearly, we have to prove this only for $\Lambda_{c,m}$, see 5.6.

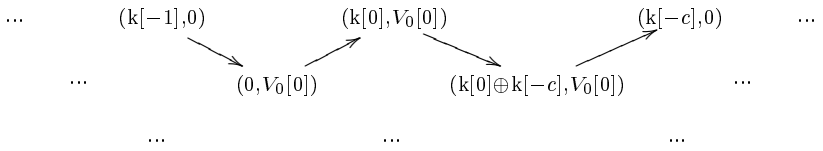
3.4 Proposition *Let $c \geq 2$ and suppose $M \in \Gamma$-mod is not in position 1 or 2, then $\Gamma[M]$ is not derived tame.*

This will follow from the fact, that if $X \in \mathcal{D}^b(\Gamma)$ is not in position 1 or 2, the derived subspace problem $\mathcal{U}(\mathcal{D}^b(\Gamma), X)$ is wild, see Proposition 1 of the Appendix. By Proposition 2 of the Appendix, we may verify this for just $\mathcal{D}^b(\Lambda_{c,m})$, see 4.9.

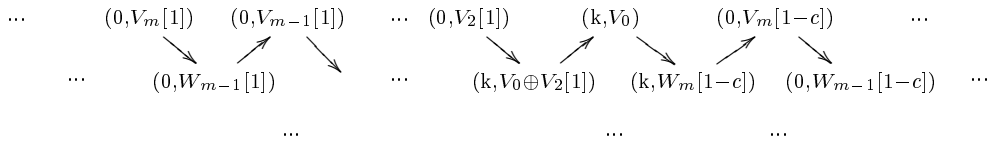
3.5 Proposition *(a) Suppose $V_0 \in \Gamma$-mod $\subset \mathcal{D}^b(\Gamma)$ is in position 1, then $\Gamma[V_0]$ is derived equivalent to $\Lambda_{c,m+1}$. In case $m \geq 1$ the $\mathbb{Z}\mathbb{A}_\infty$ components of $\mathcal{D}^b(\Gamma[V_0])$ have up to shift the following shape.*



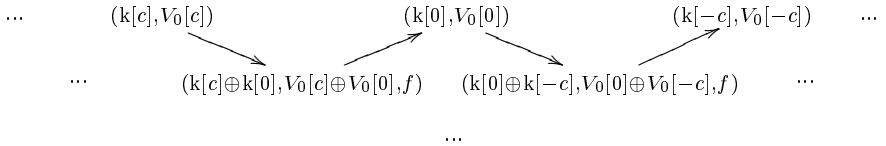
In case $m = 0$ we obtain the following shape:



(b) Suppose $W_0 \in \Gamma$-mod $\subset \mathcal{D}^b(\Gamma)$ is in position 2. Then $m \geq 1$ and $\Gamma[W_0]$ is derived equivalent to $\Lambda_{c+1,m-1}$. In case $m \geq 2$ the $\mathbb{Z}\mathbb{A}_\infty$ components of $\mathcal{D}^b(\Gamma[W_0])$ have up to shift the following shape:



In case $m = 1$, we obtain the following:



with $f := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

We will first verify, that if $X \in \mathcal{D}^b(\Lambda_{c,0})$ is in position 2, then $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{c,m})}(X, X[1-c]) \neq 0$, thus in this situation, for $c \geq 2$, there is no *module* in Γ-mod in position 2. In fact, since here $m = 0$ and X is in position 2, we have $X[1-c] \cong \tau^{-1}X$ by Proposition 3.1. Moreover we have for X in position 2 always $\mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(X, \tau^{-1}X) \neq 0$ by the Example in 4.8.

It follows from 3.1, that given $M \in \Gamma$-mod and $N \in \Lambda_{c,m}$-mod, both in position 1 (resp. 2), then there exists a standard equivalence

$$T \otimes^{\mathbb{L}}_{\Gamma} - \colon \mathcal{D}^b(\Gamma) \longrightarrow \mathcal{D}^b(\Lambda_{c,m})$$

with $T \otimes^{\mathbb{L}}_{\Gamma} M \cong N$. In fact, by Rickard's theorem [19] there exists a standard equivalence $L\colon \mathcal{D}^b(\Gamma) \longrightarrow \mathcal{D}^b(\Lambda_{c,m})$, and $L(M)$ will be in position 1 (resp. 2). Next, shift and

Auslander-Reiten-translation are also standard equivalences, thus by composing L with these, we obtain the desired standard equivalence, since the composition of standard-equivalences is isomorphic to a standard equivalence.

By Proposition 2 of the Appendix, it is now sufficient, to verify the shape of the $\mathbb{Z}\mathbb{A}_\infty$-components stated in (a) and (b) for $\Lambda_{c,m}$ and one module in position 1 resp. 2, this will be done in 4.10.

3.6 Lemma *With the notations and hypothesis of 3.5 the following holds:*

(a) Let $M = (\mathrm{k}, M_0) \in \mathcal{D}^b(\Gamma[V_0])$ *be indecomposable with* $\mathbf{dim}\, M_0 = \mathbf{dim}\, V_0$. *Then* M_0 *is indecomposable.*

(b) Suppose $c = 1$, $m \geq 1$ *and let* $N = (\mathrm{k}, N_0) \in \mathcal{D}^b(\Gamma[W_0])$ *be indecomposable with* $\mathbf{dim}\, N_0 = \mathbf{dim}\, V_0$. *Then* N_0 *decomposable implies* $\mathrm{Hom}_{\mathcal{D}^b(\Gamma[W_0])}(N, N[-1]) \neq 0$.

(c) Suppose $c \geq 1$, $m \geq 2$ *and let* $N = (\mathrm{k}, N_0) \in \mathcal{D}^b(\Gamma[W_0])$ *be indecomposable with* $\mathbf{dim}\, N_0 = \mathbf{dim}\, V_0 + \mathbf{dim}\, V_2[1]$ *then* N_0 *is decomposable.*

Again, by a similar argument as above, we have to show this only for $\Gamma = \Lambda_{c,m}$, see 5.7.

3.7 Corollary *Let* Γ *be derived equivalent to* $\Lambda_{1,m}$ *with* $m \geq 1$. *and* $W_0 \in \Gamma$-mod *indecomposable. If* $W_0 \in \mathcal{D}^b(\Gamma)$ *is in position 2, then* $\Gamma[W_0]$ *is derived equivalent to* $\Lambda_{2,m-1}$ *and* $\Gamma[W_0]$ *is* 1*-controlled.*

Remark: In fact, with some variations of Lemma 3.6 (b), we could show, that in that situation $\Gamma[W_0]$ is controlled, but we have no need for this here.

Proof: We have to show only $\Gamma[W_0]$ is 1-controlled. Thus, let $N = (\mathrm{k}^n, N_0) \in \Gamma[W_0]$-mod be indecomposable with $\mathbf{dim}\, N = (n, \mathbf{dim}\, N_0) = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma[W_0])$ in position 1.

We conclude from Proposition 3.5 (a), that

$$\mathbf{dim}\, N \in \{\pm(0, \dim V_i) \mid 2 \leq i \leq m\} \cup \{(1, \mathbf{dim}\, V_0)\}.$$

If $\mathbf{dim}\, N$ belongs to first set, N_0 is obviously indecomposable. If $\mathbf{dim}\, N = (1, \mathbf{dim}\, V_0)$, we conclude from Lemma 3.6 (b) that N_0 is indecomposable, since N is a *module*. Since Γ is 1-controlled, $N_0 \in \mathcal{D}^b(\Gamma)$ is in position 1, and we conclude from Proposition 3.5 (b) that N is in position 1. □

3.8 Proposition *Let* Γ *be derived equivalent to* $\Lambda_{c,m}$ *for* $c \geq 2$, *and let* $L \in \Gamma$-mod *be indecomposable with* $\boldsymbol{\chi}_{\Gamma[L]} \geq 0$.

(a) Suppose that Γ *is* 1*-controlled and* $\operatorname{corank} \boldsymbol{\chi}_{\Gamma[L]} = c$. *Then* $\Gamma[L]$ *is derived equivalent to* $\Lambda_{c,m+1}$ *and* $\Gamma[L]$ *is* 1*-controlled.*

(b) Suppose that Γ is weakly controlled, and L is thin and cyclic. Then $\Gamma[L]$ is derived equivalent to $\Lambda_{c,l+1}$ or to $\Lambda_{c+1,l-1}$, and in both cases $\Gamma[L]$ is weakly controlled.

Proof: Note, that $\operatorname{corank} \boldsymbol{\chi}_\Gamma = \operatorname{corank} \boldsymbol{\chi}_{\Lambda_{c,m}} = c$.

(a) Since Γ is 1-controlled, we know by Proposition 3.3 that L is in position 1. Thus we may assume $L = V_0$ in the notation of Proposition 3.5. Suppose that $M = (\mathrm{k}^n, M_0) \in \Gamma[V_0]$-mod is indecomposable and $\mathbf{dim}\, M = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma[V_0])$ in position 1. We conclude from Proposition 3.5 (a) that

$$\mathbf{dim}\, M \in \{\pm(0, \mathbf{dim}\, V_i) \mid 1 \leq i \leq m\} \cup \{(1, \mathbf{dim}\, V_0), (1, 0)\}.$$

Now, M_0 is clearly an indecomposable module if $\mathbf{dim}\, M$ belongs to the first set. If $\mathbf{dim}\, M = (1, \mathbf{dim}\, V_0)$, we get M_0 indecomposable by Lemma 3.6 (b). Thus we conclude in any case, that M is in position 1 by Proposition 3.5 (a), since Γ is 1-controlled.

(b) Note: If $N = (\mathrm{k}, N_0)$ is an indecomposable $\Gamma[L]$-module, then N_0 is indecomposable because L is cyclic by hypothesis. Next, since Γ is weakly controlled and L is thin, we conclude from Proposition 3.3, that L is in position 1 or 2.

If L is in position 1, we may assume $L = V_0$ in the notation of Proposition 3.5, and we consider a thin indecomposable $N = (\mathrm{k}^n, N_0) \in \Gamma[V_0]$-mod (in particular $n \in \{0, 1\}$) with $\dim N = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma[V_0])$ in position $i \in \{0, 1\}$. In case $i = 1$ we repeat the argument from (a). In case $i = 2$ we conclude from Proposition 3.5 (a), that

$$\mathbf{dim}\, N \in \{\pm(0, \mathbf{dim}\, W_i) \mid 1 \leq i \leq m-1\} \cup \{(0, \mathbf{dim}\, V_0), (1, \mathbf{dim}\, W_1[1-c]), (1, \mathbf{dim}\, V_1[1]\},$$

if $m \geq 1$. If $m = 0$ we have only $\mathbf{dim}\, M = (0, \mathbf{dim}\, V_0)$ because we are interested in thin modules. Since N_0 is indecomposable by the above remark and moreover thin, we can apply our hypothesis and conclude, that N is in position 1 by Proposition 3.5 (a).

If L is in position 2, we may assume $L = W_0$ in the notation of Proposition 3.5, and we consider a thin indecomposable $N = (\mathrm{k}^n, N_0) \in \Gamma[W_0]$-mod (in particular $n \in \{0, 1\}$) with $\dim N = \mathbf{dim}\, X$ for some $X \in \mathcal{D}^b(\Gamma[W_0])$ in position $i \in \{0, 1\}$. Here, the case i is treated as in the proof of Corollary 3.7, note however, that we have now N_0 indecomposable by the above note. In case $i = 2$ we conclude from Proposition 3.5 (b), that

$$\mathbf{dim}\, N \in \{\pm(0, \mathbf{dim}\, W_i \mid 2 \leq i \leq m-1\} \cup \{(1, \mathbf{dim}\, W_m[1-c]), (1, \mathbf{dim}\, V_0 + \mathbf{dim}\, V_2[1])\},$$

where we may assume $m \geq 2$. In fact, for $m = 1$ we obtain either $\mathbf{dim}\, M = 0$ or $\mathbf{dim}\, M = 2(1, \mathbf{dim}\, V_0)$, which are both impossible in our situation. Next, if $\mathbf{dim}\, N = (1, \mathbf{dim}\, V_0 + \mathbf{dim}\, V_2[1])$ we conclude from Lemma 3.6 (b) that N_0 is decomposable since N is a *module*; this is impossible since we noted above that N_0 is indecomposable. In the remaining cases we can apply our hypothesis and conclude that N is in position 2 by Proposition 3.5 (b). □

3.9 (Proof of the Theorem) Let Γ be weakly simply connected, thus we have a sequence of algebras $\mathrm{k} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n = \Gamma$ and indecomposable modules $M_i \in \Gamma_i$-mod for $1 \leq i < n$ as in the definition.

For (a) suppose that Γ is derived tame, then the Γ_i are also derived tame. We may assume therefore by 2.3 that for some j we have Γ_j derived equivalent to $\Lambda_{2,m(j)}$. Now we conclude by induction (using Proposition 3.4 and Proposition 3.5) that Γ_i is derived equivalent to $\Lambda_{c(i),m(i)}$ for all $i > j$.

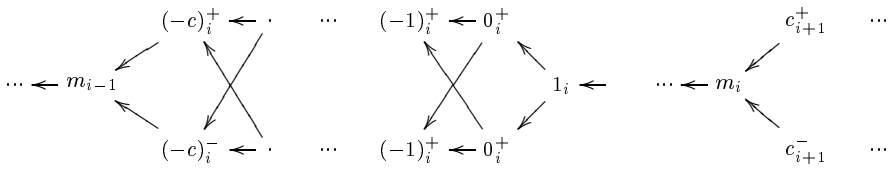
For (b) suppose that $\boldsymbol{\chi}_\Gamma \geq 0$, then we have also $\boldsymbol{\chi}_{\Gamma_i} \geq 0$ for all i. We may assume therefore again by 2.3 that for some j we have Γ_j derived equivalent to $\Lambda_{2,m(j)}$, and this is moreover 1-controlled by Proposition 3.7.

If $\operatorname{corank} \boldsymbol{\chi}_\Gamma \leq 2$ we can now induction, using Proposition 3.8 (a).

If Γ is a tree-algebra, the proof of Proposition 3.8 (b) shows, that Γ_j is actually weakly controlled. Thus we can do induction with Proposition 3.8 (b), showing that Γ_i is derived equivalent to $\Lambda_{c(i),m(i)}$ for $i > j$.

4 The repetitive algebra of a semi-chain

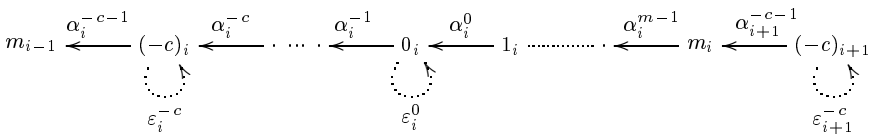
4.1 (Repetitive algebra) The algebras $\Lambda_{c,m}$ are skewed-gentle [10], thus the repetitive algebra $\widehat{\Lambda}_{c,m}$ can be determined similarly to the case of gentle algebras. In this situation it is given as the quotient of the poset-algebra



modulo the following relations for all $i \in \mathbb{Z}$:

$$\begin{aligned} j_{i-1}^- &\prec j_i^+ \quad \text{and } j_{i-1}^+ \prec j_i^- \quad (-c \leq j \leq 0) \\ 0_{i-1}^- &\prec 1_i \quad \text{and } 0_{i-1}^+ \prec 1_i \\ (j-1)_{i-1} &\prec j_i \quad (2 \leq j \leq m) \end{aligned}$$

For our purposes however, the following description as quasi-clannish algebra with the following quiver $\widehat{\mathcal{Q}}(c,m)$ and relations $\mathcal{R}$ is more convenient:



$$
\begin{aligned}
\mathcal{R}^{\mathrm{sp}} &:= \{(\varepsilon_i^j)^2 - \varepsilon_{i,j} \mid i \in \mathbb{Z},\ -c \le j \le 0\} \\
\mathcal{R}^{\mathrm{cl}} &:= \{\alpha_i^{j-1}\alpha_i^j \mid i \in \mathbb{Z},\ -c \le j \le 0\} \\
\mathcal{R}^{\mathrm{long}} &:= \{\alpha_{i-1}^{j-1}\Pi_{i,j} \mid i \in \mathbb{Z},\ 1 \le j \le m\} \\
\mathcal{R}^{\mathrm{sym}} &:= \{(\mathbb{1}_{i-1}^j - \varepsilon_{i-1}^j)\Pi_{i,j}\varepsilon_i^j,\ \varepsilon_{i-1}^j\Pi_{i,j}(\mathbb{1}_i^j - \varepsilon_i^j) \mid i \in \mathbb{Z},\ -c \le j \le 0\} \\
\mathcal{R} &:= \mathcal{R}^{\mathrm{sp}} \cup \mathcal{R}^{\mathrm{cl}} \cup \mathcal{R}^{\mathrm{long}} \cup \mathcal{R}^{\mathrm{sym}}
\end{aligned}
$$

where $\Pi_{i,j}$ denotes the unique shortest path in $\widehat{\mathcal{Q}}(c,m)$ from j_i to j_{i-1} which avoids the relations $\mathcal{R}^{\mathrm{cl}}$.

We define for later use

$$
\begin{aligned}
\widehat{\mathcal{Q}}_0^{\mathrm{sp}}(c,m) &:= \{j_i \mid i \in \mathbb{Z}, -c \le j \le 0\} \\
\widehat{\mathcal{Q}}_1^{\mathrm{sp}}(c,m) &:= \{\varepsilon_i^j \mid i \in \mathbb{Z}, -c \le j \le 0\} \\
\widehat{\mathcal{Q}}_0^{\mathrm{ord}}(c,m) &:= \{j_i \mid i \in \mathbb{Z}, 1 \le j \le m\} \\
\widehat{\mathcal{Q}}_1^{\mathrm{ord}}(c,m) &:= \{\alpha_i^j \mid i \in \mathbb{Z}, -c-1 \le m-1\} \\
\widehat{\mathcal{Q}}_0^{*}(c,m) &:= (\widehat{\mathcal{Q}}_0^{\mathrm{sp}}(c,m) \times \{0\}) \cup (\widehat{\mathcal{Q}}_0^{\mathrm{ord}}(c,m) \times \{-1,+1\})
\end{aligned}
$$

together with an involution $?^*$ of $\widehat{\mathcal{Q}}_0^*(c,m)$ given by $(q,\delta)^* := (q,-\delta)$.

Note, that $(\widehat{\mathcal{Q}}(c,m),\mathcal{R})$ admits an automorphism $\mathcal{N}$, which sends j_j to j_{i+1}.

4.2 (Letters and words) As usual, we have functions which define the starting and terminal point of each arrow $s,t\colon \widehat{\mathcal{Q}}_1^{\mathrm{ord}}(c,m) \longrightarrow \widehat{\mathcal{Q}}_0(c,m)$. For technical reasons We define functions $\mathbf{s},\mathbf{t}\colon \widehat{\mathcal{Q}}_1^{\mathrm{ord}}(c,m) \longrightarrow \widehat{\mathcal{Q}}_0^*(c,m)$ by

$$
\mathbf{s}(\alpha) := \begin{cases} (s(\alpha),+1) & \text{if } s(\alpha) \in \widehat{\mathcal{Q}}_0^{\mathrm{ord}}(c,m) \\ (s(\alpha),\ \ 0) & \text{else} \end{cases}
$$

$$
\mathbf{t}(\alpha) := \begin{cases} (t(\alpha),-1) & \text{if } t(\alpha) \in \widehat{\mathcal{Q}}_0^{\mathrm{ord}}(c,m) \\ (t(\alpha)\ \ 0) & \text{else} \end{cases}
$$

and introduce two types of "letters" which admit formal inverses l^{-1}

type of l	symbol	$\mathbf{s}(l)$	$\mathbf{t}(l)$	l^{-1}
direct	$\underline{\alpha}$ with $\alpha \in \widehat{\mathcal{Q}}_1^{\mathrm{ord}}(c,m)$	$\mathbf{s}(\alpha)$	$\mathbf{t}(\alpha)$	$\underline{\alpha}^{-1}$
inverse	$\underline{\alpha}^{-1}$ with $\alpha \in \widehat{\mathcal{Q}}_1^{\mathrm{ord}}(c,m)$	$\mathbf{t}(\alpha)$	$\mathbf{s}(\alpha)$	$\underline{\alpha}$

Now, a *word* is a sequence $\mathfrak{w} = w_1 w_2 \dots w_n$ of letters with $\mathbf{s}(w_i) = (\mathbf{t}(w_{i+1}))^*$ for $i = 1,\dots,n-1$; moreover we demand, that each subsequence $w_i w_{i+1} \dots w_{i+c+m}$ contains at least one inverse letter and at least one direct letter. We set $\mathbf{s}(\mathfrak{w}) := \mathbf{s}(w_n)$ and $\mathbf{t}(\mathfrak{w}) := \mathbf{t}(w_1)$. The inverse word of $\mathfrak{w}$ is given by $\mathfrak{w}^{-1} := w_n^{-1} w_{n-1}^{-1} \dots w_1^{-1}$. If $\mathfrak{v} := v_1 \cdots v_m$ is an other word, we define the concatenation $\mathfrak{v}\mathfrak{w} := v_1 \dots v_m w_1 \dots w_n$, if this is a word.

Moreover, we need *trivial words* $\mathbb{1}_q^\delta$ for $(q,\delta) \in \widehat{\mathcal{Q}}_0^*(c,m)$ with $\mathbf{t}(\mathbb{1}_q^\delta) = (q,\delta) = (\mathbf{s}(\mathbb{1}_q^\delta))^*$, moreover by definition $(\mathbb{1}_q^\delta)^{-1} := (\mathbb{1}_q^{-\delta})$ and $\mathbb{1}_q^\delta \mathfrak{w} := \mathfrak{w}$ if $(q,\delta) = \mathbf{t}(\mathfrak{w})$, $\mathfrak{v}\,\mathbb{1}_q^\delta := \mathfrak{v}$ if

$(q,\delta) = \mathbf{s}^*(\mathfrak{v})$. For the word $\mathfrak{w} = w_1, \cdots w_n$ and $1 \le i \le j \le n$ we introduce the words $\mathfrak{w}_{[i]} := w_1 \cdots w_i$ while ${}^{[i]}\mathfrak{w}$ and ${}^{[i]}\mathfrak{w}_{[j]}$ are uniquely determined by the conditions $\mathfrak{w} = \mathfrak{w}_{[i]}{}^{[i]}\mathfrak{w}$ and $\mathfrak{w} = \mathfrak{w}_{[i]}({}^{[i]}\mathfrak{w}_{[j]})^{[j]}\mathfrak{w}$.

Finally, we write $\mathfrak{W}(c,m) = \mathfrak{W}$ for the set of all words, including the trivial ones and

$$\mathfrak{W}_{q,\delta} := \{\mathfrak{w} \in \mathfrak{W} \mid \mathfrak{t}(\mathfrak{w}) = (q,\delta)\} \quad \text{for } (q,\delta) \in \widehat{\mathcal{Q}}_0^*(c,m).$$

The translation $\mathcal{N}$ induces an automorphism on $\mathfrak{W}$ which we also denote by $\mathcal{N}$, sending $\mathfrak{W}_{q,\delta}$ bijectively to $\mathfrak{W}_{\mathcal{N}q,\delta}$.

4.3 (Total orders) The sets $\mathfrak{W}_{q,\delta}$ admit a total order $<$ where

$$\mathfrak{v} > \mathfrak{w} \Longleftrightarrow \begin{cases} \mathfrak{v} = \mathfrak{w} l' \mathfrak{v}' & \text{with } l' \text{ an inverse letter, or} \\ \mathfrak{w} = \mathfrak{v} l'' \mathfrak{w}' & \text{with } l'' \text{ a direct letter, or} \\ \mathfrak{v} = \mathfrak{u} l' \mathfrak{v}', \mathfrak{w} = \mathfrak{u} l'' \mathfrak{w}' & \text{with } l' \text{ inverse }, l'' \text{ direct} \end{cases}$$

Each $\mathfrak{W}_{q,\delta}$ has a (unique) maximal word $\mathfrak{p}(q,\delta)$ and a (unique) minimal word $\mathfrak{i}(q,\delta)$: If $\delta \in \{-1,0\}$, then $\mathfrak{p}(q,\delta)$ is the unique word of length $c+m$ in $\mathfrak{W}_{q,\delta}$ formed by inverse letters, while $\mathfrak{p}(q,+1) = \mathbb{1}_q^{+1}$. If $\delta \in \{0,1\}$, then $\mathfrak{i}(q,\delta)$ is the unique word of length $c+m$ in $\mathfrak{W}_{q,\delta}$ formed by direct letters, while $\mathfrak{i}(q,-1) = \mathbb{1}_q^{-1}$. Note, that

$$\begin{aligned} \mathfrak{i}((-c)_i,0)^{-1} &= \mathfrak{p}(m_i,-1), \\ \mathfrak{i}(j_i,0)^{-1} &= \mathfrak{p}((j-1)_{i+1},0) \text{ for } -c+1 \le j \le 0, \\ \mathfrak{i}(1_i,1)^{-1} &= \mathfrak{p}(0_{i+1},0) \\ \mathfrak{i}(j_i,1)^{-1} &= \mathfrak{p}((j-1)_{i+1},-1) \text{ for } 2 \le j \le m \\ \mathfrak{i}(j_i,-1)^{-1} &= \mathfrak{p}(j_i,+1) \text{ for } 2 \le j \le m \end{aligned}$$

i.e. $?^{-1}\colon \{\mathfrak{i}(q,\delta) \mid (q,\delta) \in \mathcal{Q}_0^*(c,m)\} \longrightarrow \{\mathfrak{p}(q,\delta) \mid (q,\delta) \in \mathcal{Q}_0^*(c,m)\}$ is bijective.

Each $\mathfrak{w} \in \mathfrak{W}_{q,\delta} \setminus \{\mathfrak{p}(q,\delta)\}$ has a unique successor

$$\mathfrak{w}_+ \text{ given by } \begin{cases} \mathfrak{w}_+ = \mathfrak{w}\underline{\alpha}^{-1}\mathfrak{i}(\mathfrak{t}^*(\alpha)) & \text{if } \mathfrak{w}\underline{\alpha}^{-1} \text{ is a word for some arrow } \alpha, \\ \mathfrak{w} = \mathfrak{w}_+\underline{\alpha}\mathfrak{p}(\mathbf{s}^*(\alpha)) & \text{else.} \end{cases}$$

Similarly, we define ${}_+\mathfrak{w} := ((\mathfrak{w}^{-1})_+)^{-1}$ if $\mathfrak{w}^{-1} \ne \mathfrak{p}(\mathfrak{t}(\mathfrak{w}^{-1}))$. Note that $\mathfrak{w} \ne \mathfrak{p}(\mathfrak{t}w)$ implies that ${}_+(\mathfrak{w}_+)$ is defined, and moreover in our (symmetric) situation for any word $\mathfrak{w}$ at least one of $\tau\mathfrak{w} := ({}_+\mathfrak{w})_+$ and $\tau\mathfrak{w} := {}_+(\mathfrak{w}_+)$ is defined, and if both are defined, they coincide.

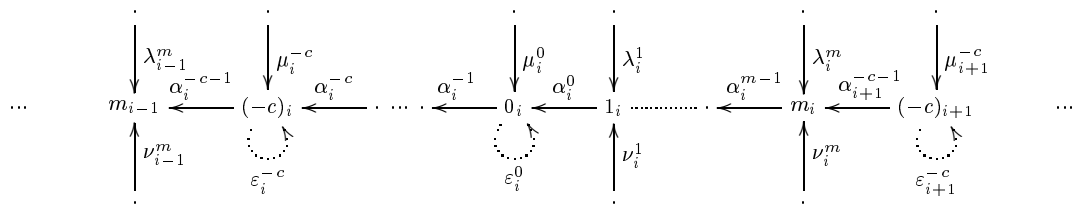
Finally, let $I(c,m) := \{\mathfrak{i}(q,\delta) \mid (q,\delta) \in \widehat{\mathcal{Q}}_0^*(c,m)\}$ and define a bijection $?[1]\colon\ I \longrightarrow I, \mathfrak{i}(q,\delta) \mapsto \mathcal{N}\mathfrak{p}(q,\delta)^{-1}$. For the n-th power ($n \in \mathbb{Z}$) we write $?[n]$.

Lemma *(1) For $\mathfrak{w} \in I(c,m)$ holds $\mathfrak{w}[c-1] = \tau^{m+1}\mathfrak{w}$ and $\mathfrak{w}[2] = \mathcal{N}\tau\mathfrak{w}$.*

(2) For $\mathfrak{w} \in \mathfrak{W}(c,m)$ we have $\mathfrak{p}(\mathfrak{t}_+\mathfrak{w})^{-1}[1] = \mathfrak{i}(\mathfrak{t}(w))$ if ${}_+\mathfrak{w}$ is defined.

Note,that τ restricts to $I(c,m)$.

4.4 (Indecomposable Modules) We have a full inclusion $\widehat{\Lambda}_{c,m}$-mod into the category of representations $\mathsf{R}(\mathcal{C}_{c,m})$ of the clan $\mathcal{C}_{c,m}$. Here $\mathcal{C}_{c,m}$ is determined by the following quiver, which is obtained from the quiver $\widehat{\mathcal{Q}}(c,m)$ by inserting additional arrows in the following way



together with the following ordering of the letters (for all $i \in \mathbb{Z}$):

$$\begin{aligned}(\underline{\alpha}_i^{j-1})^{-1} > \underline{\mu}_i^j > \underline{\alpha}_i^j \text{ for } & -c \le j \le 0\\ (\underline{\alpha}_i^{j-1})^{-1} > \underline{\lambda}_i^j \text{ for } & \quad 1 \le j \le m\\ \underline{\nu}_i^j > \underline{\alpha}_i^j \text{ for } & \quad 1 \le j \le m-1\\ \underline{\nu}_i^j > \underline{\alpha}_{i+1}^{-c-1} & \end{aligned}$$

See [9][1] for the conventions on clans used here.

Let $\mathfrak{S}$ be a set of representatives of the words in $\mathfrak{W}$ under the equivalence relation, which identifies $\mathfrak{w}$ with $\mathfrak{w}^{-1}$. We call the elements of $\mathfrak{S}$ *strings*, and $\mathfrak{w} \in \mathfrak{S}$ is *symmetric* if $\mathfrak{w}^{-1} = \mathfrak{w}$ and asymmetric otherwise. Let $\mathfrak{B}$ be a set of representatives of the words in

$$\mathfrak{B}' := \{\mathfrak{w} \in \mathfrak{W} \mid \mathfrak{w}^2 \in \mathfrak{W},\ \mathfrak{w} \neq \mathfrak{v}^n \text{ for some } \mathfrak{v} \in \mathfrak{W},\ n \ge 2\}$$

under the equivalence relation, which identifies $\mathfrak{w}$ with all $(\mathfrak{w}^{[i]}{}_{[i]}\mathfrak{w})^{\pm 1}$. $\mathfrak{w} \in \mathfrak{B}$ is called *symmetric*, if $\mathfrak{w} = (\mathfrak{w}^{[i]}{}_{[i]}\mathfrak{w})^{-1}$ for some i and *asymmetric* otherwise. Now, let $\mathfrak{X} := \mathfrak{S} \dot\cup \mathfrak{B}$.

Next note, that each $\mathfrak{w} = l_1 l_2 \cdots l_n \in \mathfrak{S}$ gives rise to a unique *inextensible* word (i.e. a string) $\widetilde{\mathfrak{w}} = \mathfrak{l}_1 \mathfrak{l}_2 \cdots \mathfrak{l}_n \mathfrak{l}_{n+1}$ for our clan $\mathcal{C}_{c,m}$ according to the follwing rule:)

$$\mathfrak{l}_1 = \begin{cases} (\underline{\mu}_i^j)^{-1}(\underline{\varepsilon}_i^j)^* l_1 & \text{if } \mathbf{t}(l_1) = (j_i, 0)\\ (\underline{\nu}_i^j)^{-1} l_1 & \text{if } \mathbf{t}(l_1) = (j_i, +1)\\ (\underline{\lambda}_i^j)^{-1} l_1 & \text{if } \mathbf{t}(l_1) = (j_i, -1)\end{cases}$$

$$\mathfrak{l}_h = \begin{cases} \underline{\varepsilon}_{ij}^* l_h & \text{if } \mathbf{t}(l_h) = (j_i, 0)\\ l_h & \text{if } \mathbf{t}(l_h) = (j_i, \pm 1)\end{cases} \quad \text{for } 2 \le h \le n$$

$$\mathfrak{l}_{n+1} = \begin{cases} (\underline{\varepsilon}_i^j)^* \underline{\mu}_i^j & \text{if } \mathbf{s}(l_n) = (j_i, 0)\\ \underline{\nu}_i^j & \text{if } \mathbf{s}(l_n) = (j_i, +1)\\ \underline{\lambda}_i^j & \text{if } \mathbf{s} l_n = (j_i, -1)\end{cases}$$

Similarly, we obtain for each $\mathfrak{w} \in \mathfrak{B}$ a band $\widetilde{\mathfrak{w}}$ for $\mathcal{C}_{c,m}$.

For each $\mathfrak{w} \in \mathfrak{X}$ we obtain a functor $M_{\mathfrak{w}}\colon A_{\mathfrak{w}}$-mod $\longrightarrow \widehat{\Lambda}_{c,m}$-mod, where A_w is k, $\mathrm{k}[x]/(x^2-x)$, $\mathrm{k}[x,x^{-1}]$, $\mathrm{k}\langle x,y\rangle/(x^2-x,y^2-y)$ if $\mathfrak{w}$ is a asymmetric string, symmetric string, asymmetric band, symmetric band, respectively. Here $M_{\mathfrak{w}}$ is obtained from the functor $S_{\widetilde{\mathfrak{w}}}$ as defined in [9][2] by "forgettenig" the maps corresponding to the auxiliary arrows $\lambda_i^j, \mu_i^j, \nu_i^j$ and by extending the maps correspoding to the α_i^j in the obvious way.

Theorem *Let $S_{\mathfrak{w}}$ be a complete system of representatives of the finite dimensional indecomposable $A_{\mathfrak{w}}$-modules, then $\{M_{\mathfrak{w}}(V) \mid \mathfrak{w} \in \mathfrak{X},\ V \in S_{\mathfrak{w}}\}$ is a complete system of representatives of the indecomposable $\widehat{\Lambda}_{c,m}$-modules which are not projective injective.*

Notation. If $\mathfrak{w}$ is a string, we write $M_{\mathfrak{w}}$ for $M_{\mathfrak{w}}(A_{\mathfrak{w}})$. Moreover we set $\Omega_{\mathfrak{w}} := \{+,-\}$ if $\mathfrak{w}$ is symmetric, and $\Omega_{\mathfrak{w}} := \{0\}$ else. Thus we obtain naturally

$$M_{\mathfrak{w}} = \oplus_{\varepsilon\in\Omega_{\mathfrak{w}}} M_{\mathfrak{w}}^{\varepsilon}.$$

If $\mathfrak{w}$ is symmetric, $M_{\mathfrak{w}} = M_{\mathfrak{w}}^{+} \oplus M_{\mathfrak{w}}^{-}$ corresponds to the decomposition of the $A_{\mathfrak{w}}$-module $\mathrm{k}[x]/(x^2-x) = \mathrm{k}[x]/(x) \oplus \mathrm{k}[x]/(x-1)$.

Remark: One may define $M_{\mathfrak{w}}$ for any $\mathfrak{w} \in \mathfrak{W}$ but then $M_{\mathfrak{w}} \cong M_{\mathfrak{w}^{-1}}$, similar results hold for bands.

4.5 (Morphisms) Following [9][5.1], for strings $\mathfrak{v} \geq \mathfrak{w}$ and $\delta \in \Omega_{\mathfrak{v}}, \varepsilon \in \Omega_{\mathfrak{w}}$ we can choose a morphism

$$\varphi_{\mathfrak{v},\mathfrak{w}}^{\delta,\varepsilon}\colon M_{\mathfrak{v}}^{\delta} \longrightarrow M_{\mathfrak{w}}^{\varepsilon}$$

corresponding to the h-line that contains the vertex $(0,0)$. We agree, that $\varphi_{\mathfrak{v},\mathfrak{w}}^{\delta,\varepsilon} := 0$ if $\mathfrak{v} \geq \mathfrak{w}$ does not hold.

Lemma *(a) If $\mathfrak{w}$ is a band, then $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{p}(q,\omega)}, M_{\mathfrak{w}}(V)) = 0$.*

(b) If $\mathfrak{w}$ is a string, then the set $\{\varphi_{\mathfrak{p}(q,\omega),\mathfrak{w}}^{o,\varepsilon}, \varphi_{\mathfrak{p}(q,\omega),\mathfrak{w}^{-1}}^{o,\varepsilon}\} \setminus \{0\}$ is a k-basis of $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{p}(q,\omega)}, M_{\mathfrak{w}}^{\varepsilon})$.

(c) For strings $\mathfrak{w} \leq \mathfrak{v} \leq \mathfrak{p}(q,\omega) = \mathfrak{p}$ we have in the stable category $\widehat{\Lambda}$-$\underline{\mathrm{mod}}$

$$0 \neq \varphi_{\mathfrak{p},\mathfrak{w}}^{0,\varepsilon} = \varphi_{\mathfrak{v},\mathfrak{w}}^{\delta,\varepsilon}\varphi_{\mathfrak{p},\mathfrak{v}}^{0,\delta}$$

Proof: If $m \geq 1$, we may assume $(q,\omega) = (m_0,+1)$ and write $\mathfrak{p} = \mathfrak{p}(q,\omega)$. Then $M_{\mathfrak{p}}$ is simple, and all statements follow directly from the description of $\mathrm{Hom}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{p}}, M_{\mathfrak{w}}(V))$ given in [9]. Just note, that no band-module has $M_{\mathfrak{p}()}$ in the socle, and the only string-modules having $M_{\mathfrak{p}}$ in the socle are those of the form $M_{\mathfrak{w}}^{\varepsilon}$ with $\mathfrak{w} \in \mathfrak{W}_{m_0,+1}$.

In case $m = 0$, we may assume $(q,\omega) = (0_0,0)$. Then we have a short exact sequence

$$\Sigma\colon \quad 0 \longrightarrow M_{\mathfrak{p}(0_0,0)} \longrightarrow P_{(-c)_1^+} \oplus P_{(-c)_1^-} \longrightarrow M_{\mathfrak{p}(-c_1,0)} \longrightarrow 0$$

which allows us to calculate here $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{p}}, M_{\mathfrak{w}}(V)))$ easyly. □

Remark: By duality we have:

- If $\mathfrak{w}$ is a band, then
$$\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{w}}(V), M_{\mathfrak{i}(q,\delta)}) = 0$$
for all (q,δ) and all $V \in A_{\mathfrak{w}}$-mod.

- If $\mathfrak{w}$ is string, we have
$$\dim_{\mathrm{k}} \underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M^{\varepsilon}_{\mathfrak{w}}, M_{\mathfrak{i}(q,\delta)}) = \begin{cases} 0 & \text{if } (q,\delta) \notin \{\mathfrak{tw}, \mathfrak{tw}^{-1}\} \\ 2 & \text{if } \mathfrak{w} \text{ is asymmetric and } (q,\delta) = \mathfrak{tw} = \mathfrak{tw}^{-1}\} \\ 1 & \text{else} \end{cases}$$

4.6 (Auslander-Reiten sequences) With the notation of the previous sections we have the following facts:

- If $\mathfrak{w}$ is a band, the functor $M_{\mathfrak{w}}\colon A_{\mathfrak{w}}$-mod $\longrightarrow \widehat{\Lambda}_{c,m}$-mod preserves Auslander-Reiten sequences.

- If $\mathfrak{w}$ is an asymmetric string not of the form $\mathfrak{p}(q,\delta)$ or $\mathfrak{i}(q,\delta)$, then
$$0 \longrightarrow M_{\tau\mathfrak{w}} \longrightarrow M_{\mathfrak{w}_+} \oplus M_{{}_+\mathfrak{w}} \longrightarrow M_{\mathfrak{w}} \longrightarrow 0$$
is an Auslander-Reiten sequence.

- for each (ordinary) arrow α_i^j , we have Auslander-Reiten sequences
$$0 \longrightarrow M_{\mathfrak{i}(\mathfrak{t}(\alpha_{i,j}))} \longrightarrow M_{\mathfrak{i}(\mathfrak{t}(\alpha_i^j))(\alpha_i^j)^{-1}\mathfrak{p}(\mathfrak{s}(\alpha_i^j))} \longrightarrow M_{\mathfrak{p}(\mathfrak{s}(\alpha_i^j))} \longrightarrow 0$$
where $\mathfrak{i}(\mathfrak{t}(\alpha_i^j)) = \tau\mathfrak{p}(\mathfrak{s}(\alpha_i^j))$.

- For $1 \leq j \leq m$ we have Auslander-Reiten sequences
$$0 \longrightarrow M_{\mathfrak{i}(j_{i-1},1)} \longrightarrow P_{j_i} \oplus M_{{}_+\mathfrak{p}(j_i,-1)} \longrightarrow M_{\mathfrak{p}(j_i,-1)} \longrightarrow 0$$
where $\mathfrak{i}(j_{i-1},1)^{-1} = \tau\mathfrak{p}(j_i,-1)$ and $M_{{}_+\mathfrak{p}(j_i,-1)} = \operatorname{rad} M_{\mathfrak{p}(j_i,-1)}$.

- If $\mathfrak{w}$ is a symmetric string not of the form $\mathfrak{p}(q,0)^{-1}\mathfrak{p}(q,0)$, we have Auslander-Reiten sequences
$$0 \longrightarrow M^{-\delta}_{\mathfrak{w}} \longrightarrow M_{\mathfrak{w}_+} \longrightarrow M^{\delta}_{\mathfrak{w}} \longrightarrow 0$$
for $\delta \in \{-,+\}$.

- Finally, for $\mathfrak{w} = \mathfrak{p}(j_i,0)^{-1}\mathfrak{p}(j_i,0)$ with $-c \leq j \leq 0$ we have Auslander-Reiten sequences
$$M^{\delta}_{\tau\mathfrak{w}} \longrightarrow M_{\mathfrak{w}_+} \oplus P_{j_i^{\delta}} \longrightarrow M^{\varepsilon}_{\mathfrak{w}} \longrightarrow 0$$
where $P_{j_i^{\varepsilon}}$ is the projective $\widehat{\Lambda}_{c,m}$-module corresponding to the vertex i_j^{ε}, and $\tau\mathfrak{w} = (\mathfrak{i}(j_{i-1},0))^{-1}\mathfrak{i}(j_{i-1},0)$.

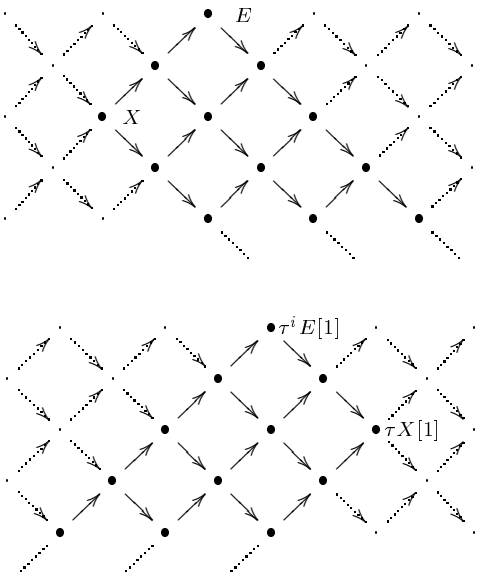
- We have $M_{\mathrm{i}(q,\delta)} \cong \Omega M_{\mathrm{i}(q,\delta)[1]}$, where ΩM is the first Syzygy of M.
- If for a string $\mathfrak{w} \in \mathfrak{W}$ also ${}_+\mathfrak{w}$ is defined, we have in $\widehat{\Lambda}_{c,m}$-$\underline{\mathrm{mod}}$ a distinguished triangle
$$M_{\mathfrak{p}(\mathfrak{t}({}_+\mathfrak{w}))} \longrightarrow M_{{}_+\mathfrak{w}} \longrightarrow M_{\mathfrak{w}} \longrightarrow M_{\mathrm{i}(\mathfrak{t}(\mathfrak{w}))}$$
where we note, that $\mathrm{i}(\mathfrak{t}(\mathfrak{w})) = \mathfrak{p}(\mathfrak{t}({}_+\mathfrak{w}))^{-1}[1]$ by the Lemma in 4.3, thus $M_{\mathrm{i}(\mathfrak{t}(\mathfrak{w}))} \cong M_{\mathfrak{p}(\mathfrak{t}({}_+\mathfrak{w}))}[1]$.

4.7 (Auslander-Reiten components) By Happel's Theorem [13, II.4] we may identify the Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda_{c,m})$ with the stable Auslander-Reiten quiver of the quasi-clannish algebra $\widehat{\Lambda}_{c,m}$. The general description of the components of stable Auslander-Reiten quiver of such an algebra was already given in [10], in particular we know from there already the following: Since for $c \geq 2$ our $\widehat{\Lambda}_{c,m}$ is of non-polynomial growth, there will be only components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$, $\mathbb{Z}\mathbb{D}_\infty$ as well as infinitely many families of homogeneous tubes and infinitely many tubes of rank 2. Moreover there will be a finite number (different from zero) of components of type $\mathbb{Z}\mathbb{A}_\infty$ and possibly a finite number (up to shift) of tubes of rank larger than 2. Thus it remains to give the precise description of the components of type $\mathbb{Z}\mathbb{A}_\infty$ and to discard the existence of "big" tubes. Now, the only candidates for modules lying at the border of $\mathbb{Z}\mathbb{A}_\infty$-components or "big" tubes are the modules of the form $M_{\mathrm{i}(q,\delta)}$, but it follows from the Lemma in 4.3 and our considerations in 4.6, that we have for these modules $\tau^{m+1} M_{\mathrm{i}(q,\delta)} \cong M_{\mathrm{i}(q,\delta)}[c-1]$, i.e. we are precisely in the situation described in 3.1.

4.8 (Auslander-Reiten triangles) We will use frequently the following simple observation: Let
$$\tau X \xrightarrow{f} E \xrightarrow{g} X \xrightarrow{\tau} X[1]$$
be an Auslander-Reiten triangle and $Y \in \mathcal{D}^b(\Gamma)$ an indecomposable object. Then we obtain from the long cohomology-sequence an exact sequence
$$\mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(Y, \tau X) \xrightarrow{(Y,f)} \mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(Y, E) \xrightarrow{(Y,g)} \mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(Y, X).$$
Here, (Y, f) is injective iff $Y \ncong X[-1]$ and (Y, g) is surjective iff $Y \ncong X$. This follows directly from the defining properties of AR-triangles.

Example Let $X \in \mathcal{D}^b(\Lambda_{c,m})$ be in position $i \in \mathbb{N}$ (see 3.2). Then we have $\dim_{\mathrm{k}} \mathrm{Hom}_{\mathcal{D}^b(\Lambda_{c,m})}(X, Y) \in \{0, 1\}$ for all indecomposable $Y \in \mathcal{D}^b(\Lambda_{c,m})$ which lie on a $\mathbb{Z}\mathbb{A}_\infty$-component, and those Y (in a $\mathbb{Z}\mathbb{A}_\infty$-component) with $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{c,m})}(X, Y) \neq 0$

are marked bold in the following diagram (where X is in position 3):



In fact, we derive from 4.5 that the only indecomposables Y in position 1 with $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{c,m})}(X,Y) \neq 0$ are E_X and $\tau^i E_X[1]$. Now we apply inductively the above remark.

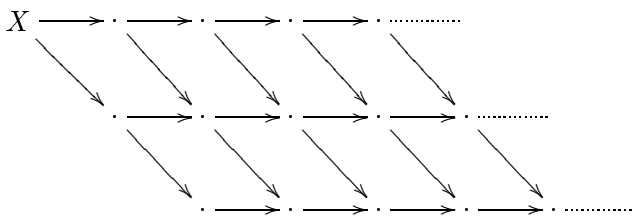
In particular, for $c \geq 3$ the $\mathbb{Z}\mathbb{A}_\infty$-components in $\mathcal{D}^b(\Lambda_{c,m})$ are standard in the sense of [22].

In case $c = 2$ we need a more detailed analysis: Write $X \succ Y$ for two vertices lying in the $\mathbb{Z}\mathbb{A}_\infty$-component of $\mathcal{D}^b(\Lambda_{2,m})$, if there is an oriented path from X to Y in that component. For such X and Y we conclude from our previous calculations that $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(X,Y) \neq 0$ implies either $X \succ Y$ or $Y \succ \tau X[1]$.

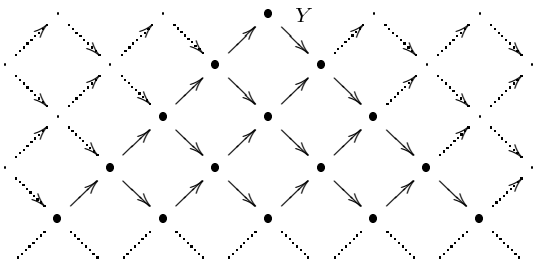
Suppose now $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(X,Y) \neq 0$ with $X \succ Y$ and $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(X,Z) \neq 0$ with $X \succ Z$. If $0 \neq f \in \mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(Z,Y)$ with $Y \succ \tau Z[1]$, then $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(X,f) = 0$, since f factors clearly over some Z' with $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(X,Z') \neq 0$

4.9 Proposition *Let $c \geq 2$ and $X \in \widehat{\Lambda}_{2,m}$-$\underline{\mathrm{mod}}$ indecomposable, not in position 1 or 2. Then the subspace-problem corresponding to $\underline{\mathrm{Hom}}_{\widehat{\Lambda}}(X,-)$ is wild.*

Proof: If X is in position i for $i \geq 3$, it follows from the previous example, that the subspace-problem contains the following wild problem:



If $X = M_{\mathfrak{w}}^{\varepsilon}$ for some string $\mathfrak{w}$ (not lying in an $\mathbb{Z}\mathbb{A}_\infty$-component), we may assume that there is a $\mathbb{Z}\mathbb{A}_\infty$-component $\mathcal{C}$ which contains a unique module Y in position 1 with $\dim_{\mathrm{k}} \underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(X,Y) = 1$, and thus again by the previous example the subspace-problem contains the following wild problem:



In fact, if $\mathcal{C}$ contains two modules Y_1, Y_2 in position 1 with $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(X,Y_i) \neq 0$ or $\dim_{\mathrm{k}} \underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(X,Y) = 2$, we can pass to a full subcategory, which contains the above problem.

If $X = M_{\mathfrak{w}}(V)$ is a band-module, we distinguish two cases: Firstly, we consider the case when $\mathfrak{w}$ is equivalent to one of the (symmetric) bands in the following set:

$$\begin{aligned}\mathfrak{B}_{\mathrm{short}} := \{&\mathfrak{w}_{(c+1)i+j} := (\alpha_i^j)^{-1}\alpha_i^j \mid i \in \mathbb{Z}, -c \leq j \leq -1\} \\ &\cup \{\mathfrak{w}_{(c+1)i} := (\alpha_i^0\alpha_i^1\cdots\alpha_i^{m-1}\alpha_{i+1}^{-c-1})^{-1}\alpha_i^0\alpha_i^1\cdots\alpha_i^{m-1}\alpha_{i+1}^{-c-1} \mid i \in \mathbb{Z}\}\end{aligned}$$

Here, we notice the following facts:

- We have for $\mathfrak{v} \in \mathfrak{B}_{\mathrm{short}}$:

$$\mathrm{Hom}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{v}}(V), M_{\mathfrak{v}}(V')) = 0$$

  if V, V' are non-isomorphic simple $\mathrm{k}\langle x,y\rangle/(x^2-x, y^2-y)$-modules.

- For $k \in \mathbb{Z}$ we have

$$\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{w}_k}(V), M_{\mathfrak{w}_{k+1}}(V') \neq 0$$

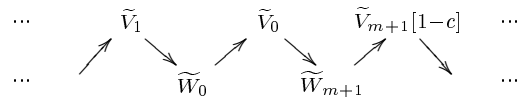
  if V, V' are (any) simple $\mathrm{k}\langle x,y\rangle/(x^2-x, y^2-y)$-module, since $c \geq 2$.

Thus, again $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(X,-)$ gives rise to wild subspace problem in that situation.

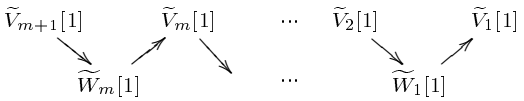
Finally, if $\mathfrak{w}$ is a band, not equivalent to one from $\mathfrak{B}_{\mathrm{short}}$, it is easy to find $\mathfrak{v} \in \mathfrak{B}_{\mathrm{short}}$, such that we get a *surjective* morphism $M_{\mathfrak{w}}(V) \longrightarrow M_{\mathfrak{v}}(V')$ for any simple $\mathrm{k}\langle x,y\rangle/(x^2-x, y^2-y)$-modules V, V'. Thus also in this last situation we found a wild problem in the subspace-category coming from $\underline{\mathrm{Hom}}_{\widehat{\Lambda}_{c,m}}(X,-)$. □

4.10 (1-point extensions in position 1 and 2) We start with the following observation: Let $m \geq 1$ and $V_0 := M_{\mathfrak{p}(m_0,-1)^{-1}}$, thus V_0 is the unique indecomposable $\Lambda_{c,m}$-*module* with $\mathbf{dim}\, V_0 = (1,1,\ldots,1)$. Moreover, V_0 is in position 1 and $\Lambda_{c,m}[V_0] = \Lambda_{c,m+1}$.

Now take the $\Lambda_{c,m+1}$-module $\tilde{V}_0 := \tilde{M}_{\tilde{\mathfrak{p}}((m+1)_0,-1)^{-1}}$ which is in $\mathcal{D}^b(\Lambda_{c,m+1})$ in position 1, and consider the following pieces of $\mathbb{Z}\mathbb{A}_\infty$-components in $\mathcal{D}^b(\Lambda_{c,m+1})$:



and



In $\widehat{\Lambda}_{c,m+1}$-$\underline{\text{mod}}$ we have $\widetilde{V}_1 = \widetilde{M}_{\tilde{\mathfrak{p}}(m_0,-1)^{-1}} = \widetilde{M}_{\tilde{\mathfrak{i}}((-c)_0,0)}$, $\widetilde{W}_0 = \operatorname{rad}\widetilde{V}_0$ and $\widetilde{W}_{m+1}[1-c] = \widetilde{M}_{\mathfrak{w}}$ for $\mathfrak{w} = \tilde{\mathfrak{p}}(\mathfrak{s}\alpha_0^{-c})^{-1}\alpha_0^{-c-1}\tilde{\mathfrak{i}}((-c)_0,0)$.

Next we see $\widetilde{V}_i[1] = \widetilde{M}_{1_{0,-1}^{(m+1-i)}}$ for $1 \le i \le m+1$ and $\widetilde{W}_i[1] = \widetilde{M}_{\alpha_0^{m-1}}$ for $1 \le i \le m$. this allows, to interpret these $\widehat{\Lambda}_{c,m+1}$ as follows in $\mathcal{D}^b(\Lambda_{c,m}[V_0])$:

$$\begin{aligned} \widetilde{V}_0 &= (\mathrm{k}, V_0) & \widetilde{W}_0 &= (0, V_0) \\ \widetilde{V}_1[1] &= (\mathrm{k}, 0) & \widetilde{W}_1[1] &= (\mathrm{k}, V_1[1]) \\ \widetilde{V}_i[1] &= (0, V_{i-1}[1]) & \widetilde{W}_j[1] &= (0, W_{j-1}[1]) \end{aligned}$$

for $2 \le i \le m+1$ and $2 \le j \le m$, since all the objects considered here are in fact $\Lambda_{c,m+1}$-modules.

Finally note, that we have in $\widehat{\Lambda}_{c,m+1}$-mod non-split short exact sequences:

$$0 \longrightarrow \widetilde{X} \longrightarrow \widetilde{W}_{m+1}[1-c] \longrightarrow M_{1_0^{m+1}} \longrightarrow 0$$

and

$$0 \longrightarrow \widetilde{W}_0 \longrightarrow \widetilde{X}[1-c] \longrightarrow \widetilde{V}_{m+1}[1-c] \longrightarrow 0$$

which give naturally rise to triangles in $\mathcal{D}^b(\Lambda_{c,m+1})$. By the above identifications we obtain in the second case

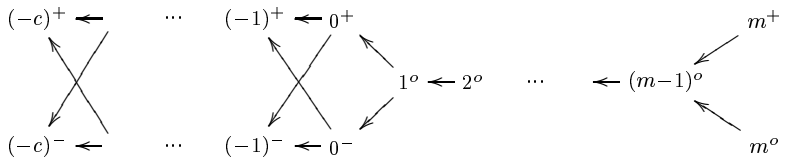
$$(0, V_0) \longrightarrow \widetilde{X} \longrightarrow (0, V_m[1-c]) \longrightarrow (0, V_0[1]),$$

which allows us to identify $\widetilde{X} = (0, W_m[1-c])$, since

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{D}^b(\Lambda_{c,m}[V_0])}((0, V_m[1-c]), (0, V_0)[1]) \\ &\qquad\qquad \cong \operatorname{Hom}_{\mathcal{D}^b(\Lambda_{c,m})}(V_m[1-c]), V_0[1]) \end{aligned}$$

is one-dimensional. All together, $\widetilde{W}_m[1-c] = (\mathrm{k}, W_m[1-c])$. These calculations show Proposition 3.5 (a) in case $m \ge 1$.

Next assume $m \geq 2$ and consider $\Gamma := \Lambda_{c,m}[W_0]$ with $W_0 = \operatorname{rad} V_0$ in position 2 as defined above. This is the poset algebra corresponding to the following quiver:



and is also a skewed-gentle algebra, and the repetitive algebra $\widehat{\Gamma}$ is identified with $\widehat{\Lambda}_{c+1,m-1}$, in particular $\Lambda_{c,m}[W_0]$ is derived equivalent to $\Lambda_{c+1,m-1}$. Now, similar calculations as in the case of position 1 show part (b) of Proposition 3.5.

5 Classes of indecomposables

5.1 (Notation) Let $\mathcal{Q} = \mathcal{Q}_{c,m}$ be the quiver of $\Lambda_{c,m}$. We identify the Grothendieck group $K_0 = K_0(\Lambda_{c,m}\text{-mod}) = K_0(\widehat{\Lambda}_{c,m}\text{-}\underline{\text{mod}})$ with the free abelian group of functions $\mathcal{Q}_0 \longrightarrow \mathbb{Z}$, and $\{\mathbf{e}_{i^\varepsilon} \mid i^\varepsilon \in \mathcal{Q}_0\}$ will be the natural basis of K_0. For $X \in \mathcal{D}^b(\Lambda_{c,m})$ we denote by $\mathbf{dim}\, X \in K_0$ the corresponding class.

C denotes the Cartan matrix, and $\boldsymbol{\chi}$ $K_0 \longrightarrow \mathbb{Z}$ resp. $\boldsymbol{\phi} : K_0 \longrightarrow K_0$ are the Euler form resp. Coxeter transformation. Finally let $?^\wedge$ be the involution of K_0 defined by $\mathbf{v}^\wedge(i^\varepsilon) = \mathbf{v}(i^{-\varepsilon})$.

We will use frequently the following elements of K_0:

$$\begin{aligned}
\mathbf{v}_0 &= \sum_{i^\varepsilon \in \mathcal{Q}_0} \mathbf{e}_{i^\varepsilon} \\
\mathbf{v}_i &= -\mathbf{e}_{(m+1-i)^0} \text{ for } 1 \leq i \leq m \\
\mathbf{w}_i &= \boldsymbol{\phi}\ \mathbf{v}_i + \mathbf{v}_i \text{ for } 0 \leq i \leq m \\
\mathbf{h}_j &= \mathbf{e}_{(-j)^-} + \mathbf{e}_{(-j)^+} + \mathbf{e}_{(-j+1)^-} + \mathbf{e}_{(-j+1)^+} \text{ for } 1 \leq j \leq c
\end{aligned}$$

5.2 Lemma *(a) We have for the Euler-form*

$$2\boldsymbol{\chi}\ (\mathbf{v}) = \sum_{i=-c}^{0} (\mathbf{v}(i^+) - \mathbf{v}(i^-))^2 + \left((\sum_{i=-c}^{0} (-1)^i (\mathbf{v}(i^+) + \mathbf{v}(i^-))) - \mathbf{v}(1^o) \right)^2 + \sum_{i=1}^{m-1} (\mathbf{v}(i^o) - \mathbf{v}((i-1)^o))^2 + \mathbf{v}(m^o)^2$$

in particular, the $\mathbf{h}_i$ *form a basis of* $\operatorname{rad} \boldsymbol{\chi}$.

(b) $\mathbf{v}$ *is orthogonal to the radical, i.e.* $\langle \mathbf{v}, \operatorname{rad} \boldsymbol{\chi}\ \rangle = 0$ *if and only if*

$$\mathbf{v}((-i)^-) + \mathbf{v}((-i)^+) - (\mathbf{v}((-i+1)^-) + \mathbf{v}((-i+1)^+)) = 0$$

for $i = 1, 2, \ldots, c$.

(c) We have for the Coxeter transformation:

$$\begin{aligned}\boldsymbol{\phi}\ \mathbf{v}_i &= \mathbf{v}_{i+1} \quad \textit{for } 0 \le i \le m-1\\ \boldsymbol{\phi}\ \mathbf{v}_m &= (-1)^{c-1}\mathbf{v}_0\\ \boldsymbol{\phi}\ \mathbf{e}_{i^\varepsilon} &= \mathbf{e}^{\wedge}_{i^\varepsilon} - (-1)^{i+c}\mathbf{v}_0 \textit{ for } -c \le i \le 0\end{aligned}$$

This is a straightforward verification.

5.3 Remark: Recall, that for $X \in \mathcal{D}^b(\Lambda_{c,m})$ we have $\mathbf{dim}\, X[1] = -\mathbf{dim}\, X$. If X is indecomposable, we have $\mathbf{dim}\,\tau X = \boldsymbol{\phi}\ \mathbf{dim}\, X$, since $\Lambda_{c,m}$ has finite global dimension [13, I.4.6].

Next, recall from 4.7 that that $M_{\mathrm{i}((-c)_0,0)}$ is in position 1; moreover this is the unique indecomposable $\Lambda_{c,m}$-module with dimension-vector $\mathbf{v}_0$. Thus, if X is in position 1, it follows that $\mathbf{dim}\, X = \sigma(X)\mathbf{v}_{i(X)}$ for some $\sigma(X) \in \{-1,+1\}$ and some $i(X) \in \{0,1,\ldots,m\}$. Consequently, if $Y \in \mathcal{D}^b(\Lambda_{c,m})$ is in position 2, we have $\mathbf{dim}\, X = \sigma(Y)\mathbf{w}_{i(Y)}$. We conclude:

(a) If $\mathfrak{w}$ is an asymmetric string, with $\mathbf{dim}\, M_{\mathrm{i}(\mathbf{s}(\mathfrak{w}))} = \sigma\mathbf{v}_s$ and $\mathbf{dim}\, M_{\mathrm{i}(\mathbf{s}(\mathfrak{w}))} = \tau\mathbf{v}_t$ for some $\sigma,\tau \in \{-1,+1\}$ and $s,t \in \{0,1,\ldots,m\}$, then

$$\mathbf{dim}\, M_{\mathfrak{w}} - \mathbf{dim}\,\tau M_{\mathfrak{w}} = \sigma\mathbf{v}_s + \tau\mathbf{v}_t$$

(b) Similarly, if $\mathfrak{w}$ is a symmetric string with $\mathbf{dim}\, M_{\mathrm{i}(\mathbf{s}(\mathfrak{w}))} = \sigma\mathbf{v}_s$, then

$$\begin{aligned}(\mathbf{dim}\, M^+_{\mathfrak{w}})^{\wedge} &= \mathbf{dim}\, M^-_{\mathfrak{w}}\\ \mathbf{dim}\, M^+_{\mathfrak{w}} - \mathbf{dim}\, M^-_{\mathfrak{w}} &= \pm(\mathbf{e}_{i^+} - \mathbf{e}_{i^-})\\ \mathbf{dim}\, M^+_{\mathfrak{w}} - (\mathbf{dim}\,\tau M^+_{\mathfrak{w}})^{\wedge} &= \sigma\mathbf{v}_s\end{aligned}$$

(c) If $\mathfrak{w}$ is an asymmetric band, then $\tau M_{\mathfrak{w}}(V) \cong M_{\mathfrak{w}}(V)$, and thus $\mathbf{dim}\, M_{\mathfrak{w}}(V) \in \mathrm{rad}\,\boldsymbol{\chi}$ for all $V \in \mathrm{k}[T,T^{-1}]$-mod indecomposable.

(d) If $\mathfrak{w}$ is a symmetric band, then $\tau^2 M_{\mathfrak{w}}(W) \cong M_{\mathfrak{w}}(W)$ for all $W \in \mathrm{k}\langle x,y\rangle/(x^2 - x, y^2 - y)$-mod indecomposable. Thus either $\mathbf{dim}\,\tau M_{\mathfrak{w}}(W) = \mathbf{dim}\, M_{\mathfrak{w}}(W)$ and $\mathbf{dim}\, M_{\mathfrak{w}}(W) \in \mathrm{rad}\,\boldsymbol{\chi}$, or otherwise

$$\begin{aligned}\mathbf{dim}\, M_{\mathfrak{w}}(W) - (\mathbf{dim}\ M_{\mathfrak{w}}(W))^{\wedge} &= \pm(\mathbf{e}_{i^+} - \mathbf{e}_{i^-}) \pm (\mathbf{e}_{j^+} - \mathbf{e}_{j^-})\\ \mathbf{dim}\, M_{\mathfrak{w}}(W) - (\mathbf{dim}\,\tau M_{\mathfrak{w}}(W))^{\wedge} &= 0\end{aligned}$$

for some $i,j = -c,\ldots,-1,0$.

5.4 (Solutions) If we set $\mathbf{r}_i := \mathbf{e}_{(-i)^+} - \mathbf{e}_{(-i)^+}$ for $0 \le i \le c$ we find with Lemma 5.2 easyly the following facts:

- $\{\mathbf{h}_1,\ldots,\mathbf{h}_c\}$ is z $\mathbb{Z}$-basis for $\ker(\mathbb{1} - \boldsymbol{\phi}\ ) = \mathrm{rad}\,\boldsymbol{\chi}$.
- $\{\mathbf{h}_1,\ldots,\mathbf{h}_c\} \cup \{\mathbf{r}_0,\ldots,\mathbf{r}_c\}$ is a $\mathbb{Q}$-basis for $\ker(\mathbb{1}^{\wedge} - \boldsymbol{\phi}\ ) \otimes \mathbb{Q}$.
- $\{\mathbf{v}_0,\ldots,\mathbf{v}_m\} \cup \{\mathbf{r}_0,\ldots,\mathbf{r}_c\}$ is a $\mathbb{Q}$-basis for $\mathrm{Im}(\mathbb{1} - \boldsymbol{\phi}\ ) \otimes \mathbb{Q}$.

- $\{\mathbf{v}_0, \dots, \mathbf{v}_m\}$ is a $\mathbb{Q}$-basis for $\mathrm{Im}(\mathbb{1}^\wedge - \boldsymbol{\phi}\ ) \otimes \mathbb{Q}$.

In fact, if we set

$$\begin{aligned}
\mathbf{x}_0 &:= (-1)^c/2\ (\mathbf{e}_{0^+} + \mathbf{e}_{0^-}),\\
\mathbf{x}_i &:= -1/2\ (\mathbf{e}_{0^+} + \mathbf{e}_{0^-}) - \sum_{k=1}^{m+1-i} \mathbf{e}_{k^0} \text{ for } 1 \le i \le m,\\
\mathbf{x}_{m+1} &:= (-1)^{c-1}\mathbf{x} \ \text{ (for later convenience)}\\
\mathbf{y}_{0,j^\varepsilon} &:= (-1)^{c+j}\mathbf{e}_{j^\varepsilon} \text{ for } -c \le j \le m,\\
\mathbf{y}_{i,j^\varepsilon} &:= (-1)^{j+1}\mathbf{e}_{j^\varepsilon} + \sum_{k=i}^{m} \mathbf{v}_k \text{ for } -c \le j \le 0, \quad 1 \le i \le m
\end{aligned}$$

and note $(\mathbb{1} - \boldsymbol{\phi}\ )\mathbf{x}_i = \mathbf{v}_i$ and $(\mathbb{1}^\wedge - \boldsymbol{\phi}\ )\mathbf{y}_{i,j^\varepsilon} = \mathbf{v}_i$.

With this we derive from 5.3:

5.5 Corollary *(a) Let $\mathfrak{w}$ be a string as in Remark 5.3 (a), then we have* $\mathbf{dim}\, M_\mathfrak{w} - \sigma\mathbf{x}_s - \tau\mathbf{x}_t \in \mathrm{rad}\,\boldsymbol{\chi}$.

(b) If $\mathfrak{w}$ is a symmetric string, we have similarly $\mathbf{dim}\, M^+_\mathfrak{w} - \sigma\mathbf{y}_{s,j^\varepsilon} \in \mathrm{rad}\,\boldsymbol{\chi}$ *for some $-c \le j \le 0$, $\varepsilon \in \{+,-\}$ and* $\mathbf{dim}\, M^-_\mathfrak{w} = (\mathbf{dim}\, M^+_\mathfrak{w})^\wedge$.

(c) If $\mathfrak{w}$ is an asymmetric band, $\mathbf{dim}\, M_\mathfrak{w}(V) \in \mathrm{rad}\,\boldsymbol{\chi}$ *for all (finite dimensional) $\mathrm{k}[T,T^{-1}]$-modules V.*

(d) If $\mathfrak{w}$ is a symmetric band, then

$$\mathbf{dim}\, M_\mathfrak{w}(W) - \sigma\mathbf{e}_s + \tau\mathbf{e}_t \in \mathrm{rad}\,\boldsymbol{\chi}$$
$$\text{for some } \sigma,\tau \in \{-1,1\} \text{ and } s,t \in \{(-c)^-, (-c)^+, \dots, 0^+\}$$

Example Consider for $m \ge 1$ the words in $\mathfrak{W}_{m,-1}$. Then $V_0 := M_{\mathfrak{p}(m_0,-1)}$ is the unique indecomposable $\Lambda_{c,m}$-*module* with dimension-vector $\mathbf{v}_0$, and $M_{\mathfrak{i}(m,-1)}$ is the unique simple $\Lambda_{c,m}$-module with dimension-vector $-\mathbf{v}_1$. We conclude, that for a string $\mathfrak{w} \in \mathfrak{W}_{m,-1}$ we have

$$\mathbf{dim}\, M_\mathfrak{w} - (\sigma\mathbf{x}_s - \mathbf{x}_1) \in \mathrm{rad}\,\boldsymbol{\chi}$$

for σ, s depending only on $\mathbf{s}(\mathfrak{w})$. If $\mathfrak{w}$ is symmetric, we have

$$\mathbf{dim}\, M^\delta_\mathfrak{w} - \mathbf{y}_{1,j^\varepsilon} \in \mathrm{rad}\,\boldsymbol{\chi}$$

for some $j^\varepsilon \in \{(-c)^-, \dots, 0^+\}$. In particular, $\mathbf{v}_0 - (\mathbf{x}_0 - \mathbf{x}_1) \in \mathrm{rad}\,\boldsymbol{\chi}$ and $-\mathbf{v}_1 - (\mathbf{x}_2 - \mathbf{x}_1) = 0$.

5.6 Proposition *For $c \ge 2$ let $M \in \widehat{\Lambda}_{c,m}$-$\underline{\mathrm{mod}}$ be indecomposable with $\mathbf{x} := \mathbf{dim}\, M \ne 0$. If $m \ge 1$, then*

$$\boldsymbol{\chi}_{C[\mathbf{x}]} \ge 0 \iff \mathbf{x} \in \{\pm\mathbf{v}_i \mid 0 \le 1 \le m\} \cup \{\pm\mathbf{w}_i \mid 0 \le 1 \le m\}.$$

If $m = 0$, then

$$\boldsymbol{\chi}_{C[\mathbf{x}]} \ge 0 \iff \mathbf{x} \in \{+\mathbf{v}_0, -\mathbf{v}_0\}$$

Proof: We leave the case $m = 0$ as an exercise, thus suppose $m \geq 1$. Let first $\boldsymbol{\chi}_{C[\mathbf{x}]} \geq 0$, then by the Lemma in 2.2 $\langle \mathbf{x}, \operatorname{rad} \boldsymbol{\chi} \rangle = 0$. By Corollary 5.5 and Lemma 5.2 (b) $M \cong M_{\mathfrak{w}}$ for an asymmetric string, or $\mathbf{x} \in \operatorname{rad} \boldsymbol{\chi}$.

(1) If $\mathbf{x} \in \operatorname{rad} \boldsymbol{\chi}$, then $\langle \mathbf{x}, \operatorname{rad} \boldsymbol{\chi} \rangle$ implies c odd and $\mathbf{x} = r(\mathbf{h}_1 + \mathbf{h}_3 + \cdots + \mathbf{h}_c)$. Thus $\mathbf{x} = r(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_m) = r(\mathbb{1} - \boldsymbol{\phi})(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 - \cdots + \mathbf{x}_m)$, and $2\boldsymbol{\chi}\ (r(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 - \cdots + \mathbf{x}_m)) = r^2(m+1)$ (see Lemma 2.2) implies $m = 1$ and $r = \pm 1$, this means $\mathbf{x} = \pm \mathbf{w}_0$.

(2) If $M \cong M_{\mathfrak{w}}$ for an asymmetric string, we can assume by the calculation in (1), that up to a sign, $\mathbf{x} = \mathbf{x}_i - \mathbf{x}_j$ for $1 \leq i < j \leq m$ or c even and $\mathbf{x} = -\mathbf{x}_i - \mathbf{x}_j + \mathbf{h}_2 + \mathbf{h}_4 + \cdots + \mathbf{h}_c$ for $1 \leq i \leq j \leq m+1$. Here, we set for our convenience $\mathbf{x}_{m+1} := (-1)^{c+1}\mathbf{x}_0$. In the first case, we find $\mathbf{x}_i - \mathbf{x}_j = -(\mathbb{1} - \boldsymbol{\phi})(\mathbf{x}_i + \mathbf{x}_{i+1} + \cdots + \mathbf{x}_{j-1})$, but $2\boldsymbol{\chi}\ (\mathbf{x}_{i+1} + \mathbf{x}_{i+2} + \cdots + \mathbf{x}_i) = j - i$. Again, by the Lemma in 2.2, this is only possible for $j - i \in \{1, 2\}$. Now observe that $\mathbf{x}_i - \mathbf{x}_{i+1} = \mathbf{v}_i$ for $i = 1, \ldots, m$, and $\mathbf{x}_i - \mathbf{x}_{i+2} = \mathbf{w}_j$ for $i = 1, \ldots, m-1$.

In the second case, we find $\mathbf{x} = (\mathbb{1} - \boldsymbol{\phi})\mathbf{x}_{i,j}$ for

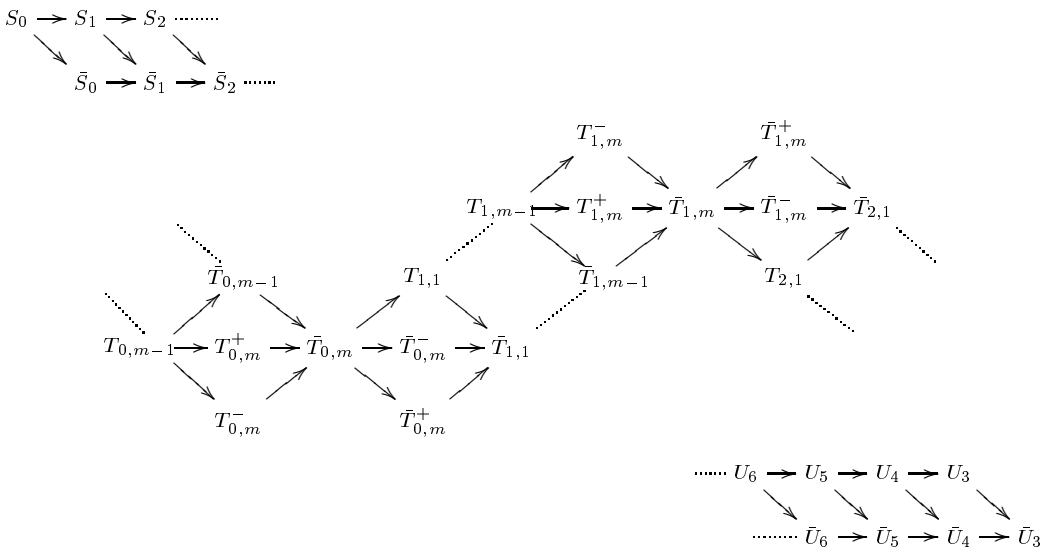
$$\mathbf{x}_{i,j} := (\mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_{i-1} - (\mathbf{x}_j + \cdots + \mathbf{x}_m))$$

and $2\boldsymbol{\chi}\ (\mathbf{x}_{i,j}) = m + 1 - j + i$, thus again by the Lemma in 2.2, $j - i \in \{m-1, m\}$ are the only possibilities. Now c even implies $\mathbf{x} \in \{\mathbf{v}_0, \mathbf{w}_0, \mathbf{w}_m\}$. From these calculations also the other direction is clear. □

5.7 (Proof of Lemma 3.6) . (a) Let $\mathfrak{p} = \mathfrak{p}(m_0, -1)$ if $m \geq 1$ and $\mathfrak{p} = \mathfrak{p}(0_0, 0)$ if $m = 0$. By the description of $\underline{\operatorname{Hom}}_{\widehat{\Lambda}_{c,m}}(M_{\mathfrak{p}}, -)$ given by the Lemma in 4.5 M_0 decomposable is only possible for $M_0 = M_{\mathfrak{w}}$, where $\mathfrak{w} > \mathfrak{p}$ is a symmetric string. But then it follows from Corollary 5.5 (b), that $\mathbf{dim}\, M_{\mathfrak{w}}$ has at least one component in $2\mathbb{Z}$, this is impossible by the definition of $dv_0 = \mathbf{dim}\, V_0$.

(b) Since we are interested only in indecomposables of the form $N = (\mathrm{k}, N_0)$ we have to study only the functor $\operatorname{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(W_0, -)\colon \mathcal{D}^b(\Lambda_{2,m}) \longrightarrow \mathrm{k}\text{-mod}$. The corresponding subspace-problem is a symmetric version of the $\mathbb{D}_n$ -pattern considered in [20], see

also [7].



Here, $S_0 = W_0$, $\bar{S}_0 = V_0$ and $U_3 = V_2[1]$. In fact it is not hard to work out the corresponding dimension-vectors:

$$\begin{array}{lllll}
\mathbf{dim}\, S_{k(m+1)+l} & = +\mathbf{x}_{m+1-l} & -\mathbf{x}_2 & +(k+1)\mathbf{h} & (0 \le l \le m) \\
\mathbf{dim}\, \bar{S}_{k(m+1)+l} & = +\mathbf{x}_{m+1-l} & -\mathbf{x}_1 & +(k+1)\mathbf{h} & (0 \le l \le m) \\
\mathbf{dim}\, T_{i,j} & = -\mathbf{x}_{m+2-j} & -\mathbf{x}_2 & -i\mathbf{h} & (1 \le j \le m-1) \\
\mathbf{dim}\, \bar{T}_{i,j} & = -\mathbf{x}_{m+2-j} & -\mathbf{x}_1 & -i\mathbf{h} & (1 \le j \le m) \\
\mathbf{dim}\, T^{\varepsilon}_{2i,m} & = & -\mathbf{y}_{1,0^\varepsilon} & -i\mathbf{h} & \\
\mathbf{dim}\, \bar{T}^{\varepsilon}_{2i,m} & = & -\mathbf{y}_{2,0^\varepsilon} & -i\mathbf{h} & \\
\mathbf{dim}\, T^{\varepsilon}_{2i+1,m} & = & -\mathbf{y}_{1,(-1)^\varepsilon} & -i\mathbf{h} & \\
\mathbf{dim}\, \bar{T}^{\varepsilon}_{2i+1,m} & = & -\mathbf{y}_{2,(-1)^\varepsilon} & -(i+1)\mathbf{h} & \\
\mathbf{dim}\, U_{k(m+1)+l} & = +\mathbf{x}_l & -\mathbf{x}_2 & -k\mathbf{h} & (1 \le l \le m+1) \\
\mathbf{dim}\, \bar{U}_{k(m+1)+l} & = +\mathbf{x}_l & -\mathbf{x}_1 & -k\mathbf{h} & (1 \le l \le m+1)
\end{array}$$

Recall, that we have $\mathbf{x}_{m+1} = \mathbf{x}_0$, and we have set moreover $\mathbf{h} := \mathbf{h}_{-1}$.

Thus we find only the following possibilities if we want N_0 decomposable with $\mathbf{dim}\, N_0 = \mathbf{v}_0$:

$$\begin{array}{lllll}
(1) & N_0 = N' \oplus N'' = \bar{S}_{k(m+1)} & \oplus U_{k(m+1)+2} & & (k \ge 1) \\
(2) & = \bar{S}_{k(m+1)+m-1} & \oplus U_{k(m+1)+m+1} & & (k \ge 0)
\end{array}$$

We have to show, that in any of these cases $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m}[W_0])}(N, N_0) \neq 0$, i.e. under any derived equivalence N will not be sent to a module. To this end it is sufficient to find a

morphism

$$0 \neq f \in \mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(N', N''[-1]) \text{ with } \mathrm{Hom}_{\mathcal{D}^b(\Lambda_{2,m})}(W_0, N')f = 0$$

since the reduction functor to the derived subspace-problem (see Appendix) is full. We will see this in $\widehat{\Lambda}_{2,m}$-$\underline{\mathrm{mod}}$. There we have:

(1) $\bar{S}_{k(m+1)} = M_{\mathfrak{s}_k}$ and $U_{k(m+1)+2}[-1] = M_{\mathfrak{u}_k}$ We determine now $\mathfrak{s}_k$ by the fact, that it is the $k(m+1)$-th successor of $\mathfrak{p}(m_0, -1)$, while $\mathfrak{u}_k$ is the $(k(m+1)-1)$-th predecessor of $\mathfrak{i}((m-2)_0, +1)$ (thus $M_{\mathfrak{i}((m-2)_0,+1)} \cong V_2 = U_3[-1]$). We find:

$$\begin{aligned}\mathfrak{s}_k &= (\alpha_0^{-1}(\alpha_0^{-1})^{-1})^k \alpha_0^{-1} \cdots \alpha_0^{m-2}\alpha_0^{m-1} \\ \mathfrak{u}_k &= \alpha_{-1}^{m-1}\alpha_0^{-2}(\alpha_0^{-1}(\alpha_0^{-1})^{-1})^k \cdots \alpha_0^{m-3}\alpha_0^{m-2}\end{aligned}$$

Since finally $W_0 = M_{\mathfrak{w}_0}$ with $\mathfrak{w}_0 = \alpha_0^{-1} \cdots \alpha_0^{m-3}\alpha_0^{m-2}$ we can take f as the composition of maps correponding to the following chain of relations:

$$\mathfrak{s}_k > (\alpha_0^{-1}(\alpha_0^{-1})^{-1})^{k-1}\alpha_0^{-1} > (\alpha_0^{-1}(\alpha_0^{-1})^{-1})^{k-1}\alpha_0^{-1}\alpha_0^0 \cdots \alpha_0^{m-2} >' (\mathfrak{u}_k),$$

where we write $\mathfrak{w} >' \mathfrak{v}$ if $\mathfrak{w}^{-1} > \mathfrak{v}^{-1}$.

(2) Similarly, $\bar{S}_{k(m+1)+m-1} = M_{\mathfrak{s}'_k}$ and $U_{k(m+1)+m+1} = M_{\mathfrak{u}'_k}$ with

$$\begin{aligned}\mathfrak{s}'_k &= \alpha_{-1}^{m-1}\alpha_0^{-2}\alpha_0^{-1}((\alpha_0^{-1})^{-1}\alpha_0^{-1})^{k+1}\alpha_0^0 \cdots \alpha_0^{m-1} \\ \mathfrak{u}'_k &= \alpha_{-1}^{m-1}\alpha_0^{-2}\alpha_0^{-1}((\alpha_0^{-1})^{-1}\alpha_0^{-1})^{k+1}\end{aligned}$$

Here we can take f as composition of maps correspondin to the following chain of relations:

$$\mathfrak{s}'_k > \alpha_{-1}^{m-1}\alpha_0^{-2}\alpha_0^{-1}((\alpha_0^{-1})^{-1}\alpha_0^{-1})^k >' \alpha_0^{-1}((\alpha_0^{-1})^{-1}\alpha_0^{-1})^k >' \mathfrak{u}'_k.$$

Alternatively one could note, that the pairs of modules which we have to consider, lie in a stable tube of rank $m+1$ which is generalized standard. There one can find the required morphisms by Auslander-Reiten theory.

(c) $\mathbf{dim}\, V_0 + \mathbf{dim}\, V_2[1] = \mathbf{v}_0 - \mathbf{v}_2 = \mathbf{x}_0 - \mathbf{x}_1 + \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{h}$ with $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_3 + \cdots + \mathbf{h}_c$ if c is odd, and $\mathbf{h} = \mathbf{h}_2 + \mathbf{h}_4 + \cdots + \mathbf{h}_c$ if c is even. But by Corollary 5.5, there is no indecomposable object in $\mathcal{D}^b(\Lambda_{c,m})$ with that dimension-vector.

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